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Physically Motivated and Rigorous Formulation of Magnetic Helicity and its Temporal Evolution

A Progress Report

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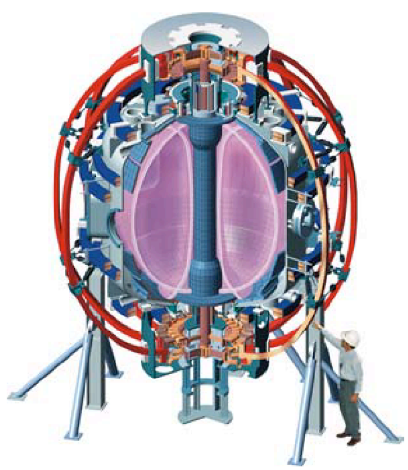

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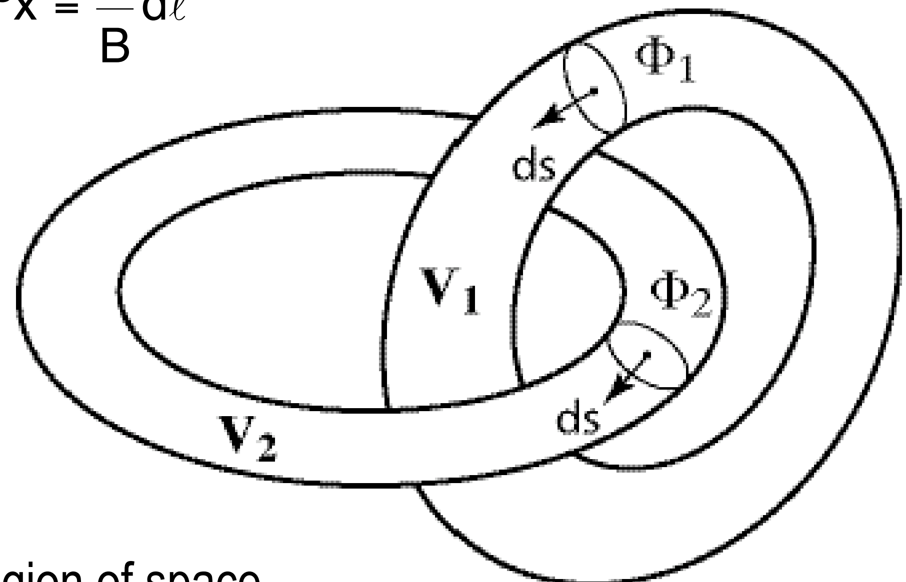
MAGNETIC HELICITY

Magnetic helicity K is a measure of the flux linkage of a magnetic field B.

Moffatt's example of magnetic helicity … linked flux loops:

$$K \equiv 2\Phi_1\Phi_2 = \int_{V=V_1+V_2} \mathbf{A}\cdot\mathbf{B}\,d^3x, \quad d^3x = \frac{\Phi}{B}d\ell$$



Linkage of a vector field is a **global topological** property.

Must integrate over **all** field-containing space.

Local or differential magnetic helicity appears to have no logically justified physical interpretation.

One usually is interested in only a limited region of space.

Region of interest is usually linked by and/or connected with outside fluxes. Becomes gauge dependent, nonphysical.

What to do?

RELATIVE Helicity Has Physical Meaning

Following Berger & Field, **J. Fluid Mech.** 147 **(1984) 133**, **Relative Helicity** K_{rel} is the **difference** between the helicities of two fields each existing in principle in **all** of space**:** the physical field B and a chosen reference field $\mathbf{B}_{ref}$.

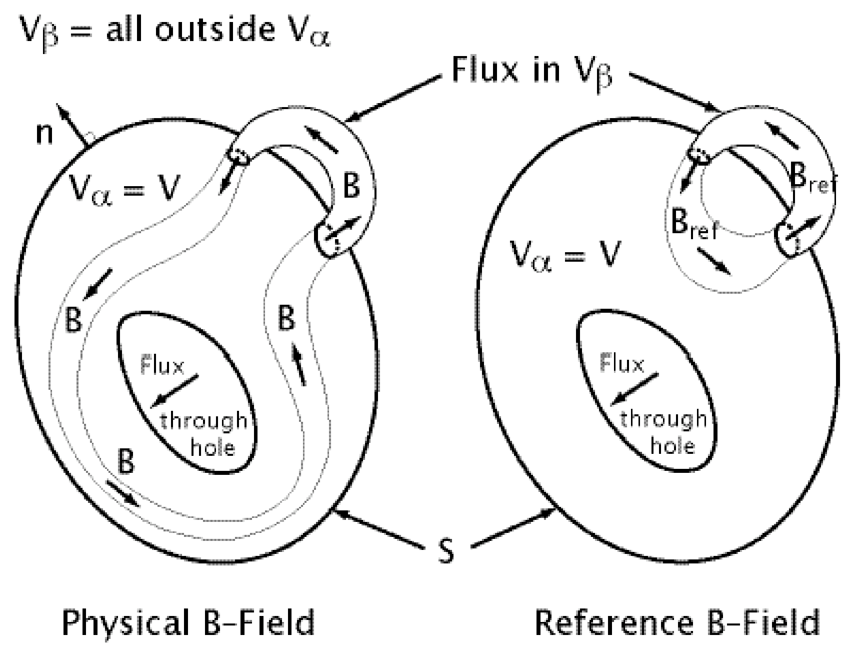
Let $\mathbf{B}_{ref}$ = **B** outside of the volume V = V_α of interest; i.e., throughout V_β.

Then, because the field geometries of $\mathbf{B}_{ref}$ and **B** outside V are equal, the Relative Helicity

$$K_{rel} = \int_{V_\infty} \mathbf{A}\cdot\mathbf{B}\, d^3x - \int_{V_\infty} \mathbf{A}_{ref}\cdot\mathbf{B}_{ref}\, d^3x$$

$$\mathbf{B} = \nabla\times\mathbf{A}, \qquad \mathbf{B}_{ref} = \nabla\times\mathbf{A}_{ref}$$

is the difference of their linkage in the volume of interest V_α.



How to Define and Use Relative Helicity Has Not Been Made Totally Clear

- **Berger-Field** did not extend their ideas to toroidally connected volumes, and some of their proofs were restricted to $\nabla \times \mathbf{B} = 0$ and/or to the Coulomb gauge, $\nabla \cdot \mathbf{A} = 0$, and to fixed boundary.
  - Many authors follow Bevir-Gray and add a simple external flux linkage term, but then the toroidal surface must be a magnetic surface.
  - Alternatively, the torus is cut, yielding a simply connected volume with a cut condition.
  - Extension of Berger-Field requires either restriction on gauge or extra terms, e.g. **Finn & Antonsen** **Comments Plas. Phys. Controlled Fusion** 9 **(1985) 111**.
  - Moving boundary can change (a) linkage within V(t) and (b) connection to outside.
- **Boozer** **Phys. Fl.** 29 **(1986) 4123, Phys. Fl. B** 5 **(1993) 2271**, treated moving and deforming, toroidally and simply connected boundaries, but not with a relative helicity.
- **Moses, Gerwin, Schoenberg,** **Phys. Plas.** 8 **(2001) 4839**, give a simple helicity, easy to apply, but requires cut for torus, and they did only fixed boundary case.

- Need to formulate a physically meaningful, mathematically rigorous magnetic helicity and rules for its use, for at least

toroidally and/or simply connected,
arbitrarily penetrated,
moving boundary

RELATIVE MAGNETIC HELICITY FOR SIMPLY AND TOROIDALLY CONNECTED VOLUMES

Begin with BF relative helicity over all space, $V_\infty = V_\alpha + V_\beta$, with $\mathbf{B}_{ref} = \mathbf{B}$ throughout V_β:

$$K_{rel} = \int_{V_\infty} \mathbf{A}\cdot\mathbf{B}\, d^3x - \int_{V_\infty} \mathbf{A}_{ref}\cdot\mathbf{B}_{ref}\, d^3x$$

$$= \int_{V_\alpha = V} \left(\mathbf{A}\cdot\mathbf{B} - \mathbf{A}_{ref}\cdot\mathbf{B}_{ref}\right) d^3x + \int_{V_\beta} \left(\mathbf{A} - \mathbf{A}_{ref}\right)\cdot\mathbf{B}\, d^3x$$

Since $\mathbf{B}_{ref} = \mathbf{B}$ in V_β, let $\mathbf{A} - \mathbf{A}_{ref} = \nabla f_\beta$ in V_β, with f_β **globally well-defined**, single-valued in V_β.

Require that all sources be within V_∞ so $\mathbf{B}$ does not contribute to integrals at ∞.

Let S be the closed surface that encloses V and $\mathbf{n}$ its outward unit normal vector. Then,

$$K_{rel} = \int_{V} \left(\mathbf{A}\cdot\mathbf{B} - \mathbf{A}_{ref}\cdot\mathbf{B}_{ref}\right) d^3x - \int_{S} f_\beta \mathbf{B}\cdot\mathbf{n}\, d^2x$$

The surface integral is commonly overlooked or else eliminated by restrictive choice(s).

This K_{rel} is gauge independent, but last integral over **outside** of surface is inconvenient.

RELATIVE MAGNETIC HELICITY
Add Some Physics

Actual field **B** and reference field $\mathbf{B}_{ref}$ must both be physically realizable in V_∞, including across surface S.

Then, the following field components must each be separately continuous across S:

$$\mathbf{B}\cdot\mathbf{n},\quad \mathbf{A}\times\mathbf{n},\quad \nabla g\times\mathbf{n},\quad g;\qquad \mathbf{B}_{ref}\cdot\mathbf{n},\quad \mathbf{A}_{ref}\times\mathbf{n},\quad \nabla g_{ref}\times\mathbf{n},\quad g_{ref}$$

Continuity of normal **B** and of tangential **A** means no flux sheet hidden in S.

∇g is any globally well-defined gauge function. Its continuity means no charge double layer (e.g., a plasma sheath) hidden in S.

And then $\nabla f_\alpha \times \mathbf{n} = \nabla f_\beta \times \mathbf{n}$ across S.

Because of continuity we can drop subscripts α and β on S.

Expression for Relative Helicity still looks the same,

$$K_{rel} = \int_V (\mathbf{A}\cdot\mathbf{B} - \mathbf{A}_{ref}\cdot\mathbf{B}_{ref})\,d^3x - \int_S f\,\mathbf{B}\cdot\mathbf{n}\,d^2x$$

except now **f** must satisfy $\Rightarrow$ $\nabla f \times \mathbf{n} = (\mathbf{A} - \mathbf{A}_{ref}) \times \mathbf{n}$ on S.

[Remember to add $g - g_{ref}$ + (any constant) to f if gauge(s) changed.]

RELATIVE MAGNETIC HELICITY
Additional Forms

K_{rel} is independent of gauges ∇g and ∇g_{ref} added to **A** and $\mathbf{A}_{ref}$ and depends only on quantities in V and on S, i.e., the region of interest.

Under the same conditions, K_{rel} can be written in other forms:

$$K_{rel} = \int_V (A \cdot B - A_{ref} \cdot B_{ref})\, d^3x - \int_S fB \cdot n\, d^2x$$

$$= \int_V (A \cdot B - A_{ref} \cdot B_{ref})\, d^3x + \int_S A \times A_{ref} \cdot n\, d^2x$$

$$= \int_V (A + A_{ref}) \cdot (B - B_{ref})\, d^3x$$

Last line is Finn & Antonsen's K_{rel}. Derivation is clearer here than in their 1985 paper.

It has no surface integrals, just an integral over the volume of interest V,
or can integrate over any volume that fully includes V, since $\mathbf{B} - \mathbf{B}_{ref} = 0$ outside V.

The FA (last line) form is, in my experience so far, the easiest to work with, despite the four terms in its integrand.

Decompose B into "CLOSED" and "OPEN" Components

Definitions of open and closed…

Let $\mathbf{B} = \mathbf{B}_{cl} + \mathbf{B}_{op}$ and $\mathbf{B}_{ref} = \mathbf{B}_{ref,cl} + \mathbf{B}_{ref,op}$ with $\nabla\cdot\mathbf{B}_{cl} = \nabla\cdot\mathbf{B}_{op} = \nabla\cdot\mathbf{B}_{ref,cl} = \nabla\cdot\mathbf{B}_{ref,op} = 0$.

Require **closed** components to satisfy $\mathbf{B}_{cl}\cdot\mathbf{n} = \mathbf{B}_{ref,cl}\cdot\mathbf{n} = 0$ on S

Require **open** components to satisfy $\mathbf{B}_{op}\cdot\mathbf{n} = \mathbf{B}_{ref,op}\cdot\mathbf{n} \neq 0$ on S

Also let there be corresponding vector potentials such that

$\mathbf{B}_{cl} = \nabla \times \mathbf{A}_{cl}$ $\qquad$ $\mathbf{B}_{op} = \nabla \times \mathbf{A}_{op}$

$\mathbf{B}_{ref,cl} = \nabla \times \mathbf{A}_{ref,cl}$ $\qquad$ $\mathbf{B}_{ref,op} = \nabla \times \mathbf{A}_{ref,op}$

Closed B is fully contained within V and does not penetrate S; S is a magnetic surface of $\mathbf{B}_{cl}$.

Open B penetrates S and connects with the outside.

This decomposition is **not unique**.

Then,

$$
\begin{aligned}
K_{rel} = & \int_V \left(\mathbf{A}_{cl}\cdot\mathbf{B}_{cl} + \mathbf{A}_{op}\cdot\mathbf{B}_{op} + \mathbf{A}_{cl}\cdot\mathbf{B}_{op} + \mathbf{A}_{op}\cdot\mathbf{B}_{cl}\right) d^3x \\
& - \int_V \left(\mathbf{A}_{ref,cl}\cdot\mathbf{B}_{ref,cl} + \mathbf{A}_{ref,op}\cdot\mathbf{B}_{ref,op} + \mathbf{A}_{ref,cl}\cdot\mathbf{B}_{ref,op} + \mathbf{A}_{ref,op}\cdot\mathbf{B}_{ref,cl}\right) d^3x \\
& - \int_S \left(f_{cl}\,\mathbf{B}_{cl} + f_{op}\,\mathbf{B}_{op} + f_{cl}\,\mathbf{B}_{op} + f_{op}\,\mathbf{B}_{cl}\right)\cdot\mathbf{n}\, d^2x
\end{aligned}
$$

(BF form shown; similarly for other forms)

… and Simplify It …

The K_{rel} expressions with open-closed decomposition simplify greatly if we require that

$B_{op} = B_{ref,op}$ everywhere in V.

This choice also makes the decomposition unique. Many terms cancel.

All forms of K_{rel} simplify to the same forms, which are equivalent:

$$K_{rel} = \int_V \left[A_{cl} \cdot B_{cl} - A_{ref,cl} \cdot B_{ref,cl} + 2\left(B_{cl} - B_{ref,cl}\right) \cdot A_{op} \right] d^3x$$

$$= \int_V \left[A_{cl} \cdot B_{cl} - A_{ref,cl} \cdot B_{ref,cl} + 2\left(A_{cl} - A_{ref,cl}\right) \cdot B_{op} \right] d^3x$$

$$= \int_V \left[\left(A_{cl} + A_{ref,cl}\right) \cdot \left(B_{cl} - B_{ref,cl}\right) + 2\left(B_{cl} - B_{ref,cl}\right) \cdot A_{op} \right] d^3x$$

$$= \int_V \left[\left(A + A_{ref}\right) \cdot \left(B_{cl} - B_{ref,cl}\right) \right] d^3x \quad \text{etc.}$$

Interpretation:

Relative helicity in toroidally and simply connected volumes can be reduced to the relative helicity of just a **closed** field component, plus (if S is penetrated by B) a cross linkage between closed and open field components.

Special Case Yields Further Simplification

So far we have neither restricted gauges nor chosen $\mathbf{B}_{ref}$.

Now consider the special case where:

- $\mathbf{B}_{op}$ is current-free (a common choice):
  - Then let $\mathbf{B}_{op} = \mathbf{B}_{ref,op} = \nabla\chi$ in V, where χ is a well-defined scalar potential in V.
  - "Vacuum" **B** has lowest energy in V with given boundary conditions.
- $\mathbf{A}_{op}$ and $\mathbf{A}_{ref,op}$ gauges chosen so that each has the same "electric charge":
  - Then $\nabla\bullet(\mathbf{A}_{op} - \mathbf{A}_{ref,op}) = 0$ in V and $(\mathbf{A}_{op} - \mathbf{A}_{ref,op})\bullet\mathbf{n} = 0$ on S.

Then, K_{rel} simplifies to depend only on closed-field components,

$$K_{rel} = \int_V \left(\mathbf{A}_{cl} \cdot \mathbf{B}_{cl} - \mathbf{A}_{ref,cl} \cdot \mathbf{B}_{ref,cl}\right) d^3x$$

Even more special, $\nabla\bullet\mathbf{A} = 0$ in V and $\mathbf{A}\bullet\mathbf{n} = 0$ on S for all components is allowed.

- Moses et al. Gauge choice
- Then all electric charges are accounted for in $\nabla^2\phi$, not in **A**.

Time Derivative of K Yields Helicity Evolution/Conservation Eqn.

Differentiate K_{rel} with respect to time (less algebra with FA form):

$$\frac{dK_{rel}}{dt} = \int_{V(t)} \frac{\partial}{\partial t}\left[(A + A_{ref}) \cdot (B - B_{ref})\right] d^3x + \int_{S(t)} (A + A_{ref}) \cdot (B - B_{ref}) (U \cdot n) d^2x$$

Here **U**(**x**,t) is the local **normal** component of the velocity of S(t) at **x**. **x** is the coordinate vector in a "fixed" frame. Let prime (') denote a value measured at point **x** in an inertial frame moving with the instantaneous local **U**(**x**,t).

Using $\mathbf{E} = -\partial \mathbf{A}/\partial t - \nabla\phi$, where ϕ is a scalar electric potential, and $\mathbf{E}' = \mathbf{E} + \mathbf{U}\times\mathbf{B}$, yields

$$\frac{dK_{rel}}{dt} = -2\int_{V(t)} (E \cdot B - E_{ref} \cdot B_{ref}) d^3x - \int_{S(t)} (E' - E'_{ref}) \times (A + A_{ref}) \cdot n \, d^2x$$

However, $(\mathbf{E}' - \mathbf{E}'_{ref}) \times \mathbf{n} = 0$ in frame moving with S(t), so the final result is just

$$\frac{dK_{rel}}{dt} = -2\int_{V(t)} (E \cdot B - E_{ref} \cdot B_{ref}) d^3x$$

The **parallel** (to **B**) component of **E** changes magnetic helicity (flux linkage).

… Add Physics …

Use Braginskii two-fluid "Ohm's law" to express **E·B** (actual fields) as plasma physics.
Manipulate reference field $\mathbf{E}_{ref}\cdot\mathbf{B}_{ref}$ integral to show some possibilities. Then,

$$\frac{dK_{rel}}{dt} = -\int_{V(t)} 2\left(\eta_{||}J - \frac{k}{e}T_e \nabla \ln n_e\right)\cdot B\, d^3x \;-\int_{S(t)} 2\left(\phi' - 1.7\frac{k}{e}T_e\right) B\cdot n\, d^2x$$

Ohm — density gradient — electric potential — thermo electric

$$-\int_{S(t)} \left(A \times \frac{\partial A'}{\partial t}\right)\cdot n\, d^2x \;-\frac{d}{dt}\int_{V(t)} A_{ref}\cdot B_{ref}\, d^3x$$

Change of surface fluxes — Change of Reference helicity

Transformations between fixed and moving frames $\phi' = \phi - U\cdot A \qquad \frac{\partial A'}{\partial t} = \frac{dA}{dt} = \frac{\partial A}{\partial t} + U\cdot\nabla A$

No Hall terms in helicity evolution.

DISCUSSION

- When S(t) is a magnetic surface, helicity can be changed by: plasma resistivity, density gradient potential; changing external flux linkages; and changing reference field's helicity.
- Static electric and thermoelectric potentials transport helicity across S(t) if **B** penetrates the surface.
- $E_{||}$ changes magnetic helicity; $E_{\perp}$ convects magnetic flux.

- Need to understand better:
  - How to interpret helicity evolution when reference field changes with changing geometry.
  - Develop linked external fluxes contribution; should give a natural unique definition of surface-averaged linkage when **B** penetrates S(t).