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The effect of an anisotropic pressure of thermal particles on resistive wall mode stability

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The effect of an anisotropic pressure of thermal particles on resistive wall mode stability in tokamak fusion plasmas is derived through kinetic theory and assessed through calculation with the MISK code [B. Hu *et al.*, Phys. Plasmas **12**, 057301 (2005)]. The fluid anisotropy is treated as a small perturbation on the plasma equilibrium and modeled with a bi-Maxwellian distribution function. A complete stability treatment without an assumption of high frequency mode rotation leads to anisotropic kinetic terms in the dispersion relation in addition to anisotropy corrections to the fluid terms. With the density and the average pressure kept constant, when thermal particles have a higher temperature perpendicular to the magnetic field than parallel, the fluid pressure-driven ballooning destabilization term is reduced. Additionally, the stabilizing kinetic effects of the trapped thermal ions can be enhanced. Together these two effects can lead to a modest increase in resistive wall mode stability. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4901568]

I. INTRODUCTION

Tokamak fusion plasmas require a high ratio of plasma stored energy to magnetic confining field energy (characterized by the quantity β_N) in order to efficiently generate energy. In order to reach these conditions without disruption of the plasma current due to the growth of MHD kink-ballooning instabilities, these modes must be stabilized. The presence of a resistive wall around the plasma can slow the growth of these modes down to the time scale of penetration of the magnetic perturbations through the wall, converting the mode into a resistive wall mode (RWM). However, the RWM can also disrupt the plasma when β_N is above the so-called no-wall limit unless it is itself stabilized by passive or active means.[1]

It has long been recognized that anisotropy of the plasma pressure with respect to the direction of the magnetic field can play a role in plasma stability. Consideration of anisotropy goes back as far as Refs. 2 and 3, and some other prominent examples include Refs. 4–6. One possible approach is to consider the perturbed perpendicular and parallel pressures from Chew-Goldberger-Low (CGL)[7] theory. It will be demonstrated here, however, that using CGL theory is akin to an assumption of high frequency modes, which is not applicable to the RWM. Instead, kinetic theory, in which the perturbed pressures are rigorously solved from a perturbed distribution function, will be employed. Kinetic theory, recently expanded to be relevant to low frequency modes such as the RWM,[8] has been successfully compared to experimental instability[9–12] in the National Spherical Torus Experiment[13] (NSTX) with calculations from the MISK code.[14] In particular, the importance of resonances between the plasma rotation and the motions of thermal particles was elucidated. Here, we will expand the treatment of those thermal particles to include anisotropy, such as might arise when Maxwellian electrons are modified by electron cyclotron resonance heating (ECRH)[15] or ions by ion cyclotron resonance heating (ICRH).[16] Note that the effect of anisotropy of energetic particles has also been considered,[11,16–19] and indeed may be experimentally relevant and important to plasma stability, but it is not the subject of the present paper.

In Sec. II, we outline the Energy Principle approach to RWM stability calculations with a perturbative approach. In order to use such an approach, we must then demonstrate in Sec. III that anisotropy of the pressure represents a small perturbation on the equilibrium. In Sec. IV, an anisotropic perturbed pressure tensor is used to determine a general equation for the anisotropic corrections to the fluid δW term. Equations for the kinetic effects depend on the distribution function of the particles chosen; in Sec. V these are derived for a specific case: a bi-Maxwellian distribution of thermal particles. In Sec. VI, we return to the fluid anisotropic correction specifically for the pressure-driven ballooning destabilization term and incorporate this distribution function. Calculations with the MISK code are carried out using these derived expressions in Sec. VII for an analytical Solov'ev equilibrium to test the effect of thermal particle anisotropy, and finally conclusions are drawn.

II. STABILITY CALCULATION THROUGH AN ENERGY PRINCIPLE APPROACH

Pressure anisotropy leads to a modified Energy Principle expression for the complex mode frequency, ω, normalized

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by the wall time constant τ_w, where $\omega = \omega_r + i\gamma$ comprised the growth rate, γ, and real mode rotation frequency, ω_r

$$-i\omega\tau_w = -\frac{\delta W_V^\infty + \delta W_F + \delta W_A + \delta W_K}{\delta W_V^b + \delta W_F + \delta W_A + \delta W_K}. \quad (1)$$

Here, δW_V^∞ and δW_V^b are the usual changes in vacuum potential energy without a wall and with an ideal wall, respectively. δW_F is the usual isotropic fluid term, while δW_A is an anisotropic fluid correction and δW_K is the kinetic term, which also must be modified by anisotropy. In Ref. 9, stability diagrams were described, which show contours of constant normalized growth rate on plots of $Im(\delta W_K)$ vs. $Re(\delta W_K)$. The anisotropy term modifies those diagrams by adding $-\delta W_A$ to the offset a of the $\gamma\tau_w = 0$ curve that defines the unstable region, as shown in the example in Fig. 1.

Now, in order to solve for the δW terms, we use a plasma force balance $\rho(d\mathbf{v}/dt) = \mathbf{j}\times\mathbf{B} - \nabla\cdot\mathbb{P}$. This leads to expressions for the plasma equilibrium, $\mathbf{j}\times\mathbf{B} = \nabla\cdot\mathbb{P}$, when $\mathbf{v}$ is constant, and for the change in potential energy of the plasma due to a small displacement $\boldsymbol{\xi}_\perp$

$$\delta W = \frac{1}{2}\int \xi_\perp^* \cdot \left[\mathbf{j}_0 \times \tilde{\mathbf{B}} + \tilde{\mathbf{j}} \times \mathbf{B}_0 - \nabla\cdot\tilde{\mathbb{P}}\right] d\mathbf{V}, \quad (2)$$

when it is not. Here, x_0 are equilibrium quantities, $\tilde{x}$ are perturbed quantities, $\mathbf{j}$ is the plasma current, $\mathbf{B}$ is the magnetic field, ρ is the density, $\mathbf{v}$ is the velocity, $\mathbb{P}$ is the pressure tensor, and $\mathbf{V}$ is the plasma volume. In the perturbative approach to stability calculations, it is assumed that the RWM eigenfunction, $\boldsymbol{\xi}_\perp$, is unchanged by both the kinetic effects that come in through $\tilde{\mathbb{P}}$ and, now, the anisotropy of the equilibrium as well.

The well-known problem of closure of the set of equations now requires us to make specification for the equilibrium and perturbed pressures. First, however, we will examine the effect of anisotropy on the plasma equilibrium to assure the applicability of the perturbative approach to the problem.

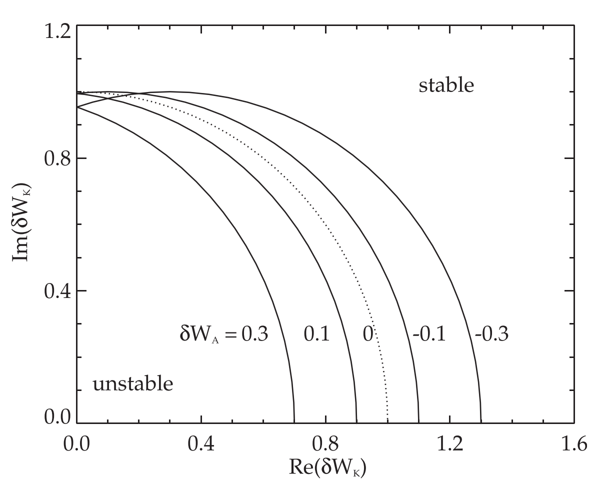

FIG. 1. Example of a stability diagram, showing contours of $\gamma\tau_w = 0$ with $\delta W_\infty = -1$ and $\delta W_b = 1$ in arbitrary units, modified by anisotropy. Positive δW_A shifts the unstable region to the left, while negative δW_A shifts it to the right.

III. EQUILIBRIUM

Anisotropy of the plasma pressure can affect the plasma equilibrium.[20–30] In the present work, however, we will use the isotropic equilibrium as a basis for stability calculations. Therefore, we must now demonstrate that the anisotropy is a small perturbation on the isotropic equilibrium if the degree of anisotropy is small, and discuss the implications.

We will consider a pressure tensor with separate components in the directions parallel and perpendicular to the magnetic field

$$\mathbb{P} = p_\parallel \hat{\mathbf{b}}\hat{\mathbf{b}} + p_\perp(\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}), \quad (3)$$

where $\hat{\mathbf{I}}$ is the identity tensor and $\hat{\mathbf{b}} = \mathbf{B}/B$.

Let us now also define an anisotropy parameter[5,20,25,31–33]

$$\sigma = 1 + \frac{\mu_0(p_\perp - p_\parallel)}{B^2} = 1 + \frac{2\mu_0 p}{B^2}\left(\frac{1}{2}\left(\frac{p_\perp - p_\parallel}{p}\right)\right), \quad (4)$$

which we note is unity when $p_\perp = p_\parallel$. Note that for a given normalized pressure difference $(p_\perp - p_\parallel)/p$, $|\sigma - 1|$ is larger for higher beta plasmas.

Then using Ampere's law and the anisotropic pressure tensor from Eq. (3), we have

$$\frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} = \nabla\cdot(p_\parallel\hat{\mathbf{b}}\hat{\mathbf{b}} + p_\perp(\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}})). \quad (5)$$

Now using the magnetic curvature, $\boldsymbol{\kappa} = \hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}$, and

$$\nabla\cdot(\hat{\mathbf{b}}\hat{\mathbf{b}}) = \hat{\mathbf{b}}\nabla_\parallel + \boldsymbol{\kappa} + \hat{\mathbf{b}}(\nabla\cdot\hat{\mathbf{b}}), \quad (6)$$

$$\nabla\cdot(\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) = \nabla_\perp - \boldsymbol{\kappa} - \hat{\mathbf{b}}(\nabla\cdot\hat{\mathbf{b}}), \quad (7)$$

we have

$$-\nabla\frac{\mathbf{B}^2}{2\mu_0} + \left(\boldsymbol{\kappa} + \hat{\mathbf{b}}(\nabla\cdot\hat{\mathbf{b}})\right)\frac{B^2}{\mu_0} = \nabla_\perp p_\perp + \hat{\mathbf{b}}\nabla_\parallel p_\parallel + (\boldsymbol{\kappa} + \hat{\mathbf{b}}(\nabla\cdot\hat{\mathbf{b}}))(p_\parallel - p_\perp). \quad (8)$$

In the perpendicular direction, the equilibrium is[5,32]

$$\nabla_\perp\left(\sigma\frac{\mathbf{B}^2}{2\mu_0} + p_{\rm avg}\right) = \sigma\boldsymbol{\kappa}\frac{B^2}{\mu_0}, \quad (9)$$

where we have defined $p_{\rm avg} = (p_\parallel + p_\perp)/2$. In the isotropic case, the equilibrium relation reduces to $\nabla_\perp\left(\frac{\mathbf{B}^2}{2\mu_0} + p\right) = \boldsymbol{\kappa}\frac{B^2}{\mu_0}$, and Eq. (9) can be shown to be equal to the isotropic equilibrium in zeroth order, with a first order anisotropic correction in the parameter $\sigma - 1$. For more detail on the anisotropic correction to the equilibrium and its impact on δW see Appendix C. In the present paper, however, since $p_{\rm avg}$ is unchanged (a condition we will impose later), and σ is not much different from unity, we will use the perturbative approach to the problem (using the isotropic equilibrium unchanged).

IV. USE OF THE ANISOTROPIC PERTURBED PRESSURE TENSOR FOR STABILITY CALCULATIONS

In order to obtain $\tilde{\mathbb{P}}$ for use in Eq. (2), we now linearize Eq. (3), remembering that $\hat{\mathbf{b}}$ can also be perturbed.[5] Therefore,

$$\tilde{\mathbb{P}} = \tilde{p}_\parallel \hat{\mathbf{b}}\hat{\mathbf{b}} + \tilde{p}_\perp(\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) + (p_\parallel - p_\perp)B^{-2}(\tilde{\mathbf{B}}\mathbf{B} + \mathbf{B}\tilde{\mathbf{B}}). \quad (10)$$

At this point, the problem naturally separates into fluid and kinetic approaches. In the fluid approach, the perturbed pressures are given in terms of macroscopic quantities. There are two common fluid approximations. The first is to assume the equilibrium pressure and the perturbed pressure are isotropic so, $\nabla \cdot \tilde{\mathbb{P}} = \nabla \tilde{p}$, which results in a fluid compressibility term, $\frac{1}{2}\int \gamma p |\nabla \cdot \boldsymbol{\xi}_\perp|^2 d\mathbf{V}$. Then the adiabatic equation is used to find $\tilde{p}$. In the second common fluid approach, two adiabatic equations are used to find the two Chew-Golberger-Low (CGL)[7] perturbed pressures, $\tilde{p}_\perp$ and $\tilde{p}_\parallel$. This method, which is applicable in the high frequency limit, is outlined in Appendix A, and could result in the calculation of a δW_{CGL} term.[34,35]

In the kinetic approach,[5,36–39] $\tilde{p}_\perp$ and $\tilde{p}_\parallel$ are defined by using the perturbed distribution function $\tilde{f}$. First, we write

$$\tilde{\mathbb{P}} = \sum_j m_j \int \mathbf{v}\mathbf{v}\left(\tilde{f}_j + \frac{\partial f_j}{\partial B}\boldsymbol{\xi}_\perp \cdot \nabla \mathbf{B} + \frac{\partial f_j}{\partial \Phi}\boldsymbol{\xi}_\perp \cdot \nabla \Phi\right) d^3\mathbf{v}, \quad (11)$$

and then use for $\tilde{f}_j$ the solution of the linearized Vlasov equation (ignoring perturbed potential and therefore electrostatic effects)

$$\tilde{f}_j = -\boldsymbol{\xi}_\perp \cdot \nabla f_j + i m_j\left(\omega\frac{\partial f_j}{\partial \varepsilon} - n\frac{\partial f_j}{\partial P_\phi}\right)(\mathbf{v}\cdot\boldsymbol{\xi}_\perp + \tilde{s}_j) - \frac{m_j}{B}\frac{\partial f_j}{\partial \mu}\left(-i\omega \boldsymbol{\xi}_\perp \cdot \mathbf{v}_\perp + \frac{\mu}{m_j}\tilde{\mathbf{B}}_\parallel + \frac{v_\parallel}{B}\mathbf{v}_\perp \cdot \tilde{\mathbf{B}}\right). \quad (12)$$

Here, P_ϕ is the toroidal canonical momentum, n is the toroidal mode number, μ is the magnetic moment, and j denotes the type of particle that is being considered (ions or electrons). The quantity $\tilde{s}_j$ represents the integral along the unperturbed orbits and is essentially the term that gives rise to kinetic effects in the problem (see, for example, Ref. 35).

Generally, it is assumed that the equilibrium pressure is isotropic ($p_\parallel = p_\perp$), even in the kinetic approach. Here, we extend that approach so that in Eq. (10) the final term (perturbation of the direction of the magnetic field in an anisotropic equilibrium pressure plasma[5]) is not zero, and $\partial f_j/\partial B$ is also not zero in Eq. (11). After carrying through much algebra and neglecting finite Larmor radius effects, we arrive at the expression

$$\begin{aligned}\tilde{\mathbb{P}} = \hat{\mathbf{b}}\hat{\mathbf{b}}\Bigg[&-\boldsymbol{\xi}_\perp \cdot \nabla p_\parallel - (\nabla\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp)B\frac{\partial p_\parallel}{\partial B} \\ &+ \sum_j m_j \int v_\parallel^2\left[i m_j\left(\omega\frac{\partial f_j}{\partial \varepsilon} - n\frac{\partial f_j}{\partial P_\phi}\right)\tilde{s}_j\right]d^3\mathbf{v}\Bigg] \\ &+ (\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}})\Bigg[-\boldsymbol{\xi}_\perp \cdot \nabla p_\perp - (\nabla\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp)B\frac{\partial p_\perp}{\partial B} \\ &+ \sum_j m_j \int \frac{1}{2}v_\perp^2\left[i m_j\left(\omega\frac{\partial f_j}{\partial \varepsilon} - n\frac{\partial f_j}{\partial P_\phi}\right)\tilde{s}_j\right]d^3\mathbf{v}\Bigg] \\ &+ \frac{1}{B}\left(\hat{\mathbf{b}}\tilde{\mathbf{B}}_\perp + \tilde{\mathbf{B}}_\perp\hat{\mathbf{b}}\right)(p_\parallel - p_\perp). \end{aligned} \quad (13)$$

Finally, this represents a form of the perturbed pressure tensor that we can use to evaluate δW, from Eq. (2).

It is useful when doing so to separate out the various modes of instability, for example, Eq. (39) in Ref. 40, Eq. (58) in Ref. 5, or Eq. (11) in Refs. 41 and 42. Then the various terms of the potential energy can be seen to be contributions from stabilizing shear Alfvén waves, fast magneto-acoustic (compressional Alfvén) waves, and the two terms that can drive instability by current driven kink or pressure driven ballooning modes. Finally, using an alternative form for $\mathbf{j}_0 \times \tilde{\mathbf{B}} + \tilde{\mathbf{j}} \times \mathbf{B}_0$, $\boldsymbol{\xi}_\perp^* \cdot \nabla \cdot \hat{\mathbf{b}}\hat{\mathbf{b}} = \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*$, and $\boldsymbol{\xi}_\perp^* \cdot \nabla \cdot (\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) = -(\nabla\cdot\boldsymbol{\xi}_\perp^* + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*)$ from Eqs. (6) and (7), a fair bit of algebraic manipulation, and splitting δW into isotropic fluid, anisotropic fluid, and kinetic parts, we have

$$\delta W_F = \frac{1}{2}\int\left\{\left(\underbrace{-\frac{|\tilde{\boldsymbol{B}}_\perp|^2}{\mu_0}}_{\substack{\text{shear Alfvén}\\ \text{(mag. bending)}}} \underbrace{- \frac{B^2}{\mu_0}|\nabla\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}|^2}_{\substack{\text{fast magneto-acoustic}\\ \text{(mag. compression)}}} + \underbrace{j_\parallel\left(\boldsymbol{\xi}_\perp^* \times \hat{\boldsymbol{b}}\right)\cdot\tilde{\boldsymbol{B}}_\perp}_{\text{current-driven kink}}\right) + \underbrace{2\left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right)\left(\boldsymbol{\xi}_\perp\cdot\nabla p_{\rm avg}\right)}_{\substack{\text{pressure-driven}\\ \text{ballooning}}}\right\}d\mathbf{V}, \quad (14)$$

$$\delta W_A = \frac{1}{2}\int\left\{(\sigma - 1)\left(\overbrace{-\frac{|\tilde{\boldsymbol{B}}_\perp|^2}{\mu_0}}^{\substack{\text{shear Alfvén}\\ \text{(mag. bending)}}} \overbrace{- \frac{B^2}{\mu_0}|\nabla\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}|^2}^{\substack{\text{fast magneto-acoustic}\\ \text{(mag. compression)}}} \overbrace{+ j_\parallel\left(\boldsymbol{\xi}_\perp^* \times \hat{\boldsymbol{b}}\right)\cdot\tilde{\boldsymbol{B}}_\perp}^{\text{current-driven kink}}\right) \overbrace{- 2B|\nabla\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp|^2\frac{\partial p_{\rm avg}}{\partial B}}^{\substack{\text{pressure-driven}\\ \text{ballooning}}}\right\}d\mathbf{V}, \quad (15)$$

and

$$\delta W_K = \frac{1}{2}\sum_j \int\int \frac{1}{2}m_j v^2\left(\frac{v_\perp^2}{v^2}\nabla\cdot\boldsymbol{\xi}_\perp^* + \left(\frac{v_\perp^2}{v^2} - 2\frac{v_\parallel^2}{v^2}\right)\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right)\left[i m_j\left(\omega\frac{\partial f_j}{\partial \varepsilon} - n\frac{\partial f_j}{\partial P_\phi}\right)\tilde{s}_j\right]d^3\mathbf{v}\,d\mathbf{V}. \quad (16)$$

Similar results have been previously derived in, for example, Refs. 4, and 43–46. One can easily see that if the equilibrium pressure is isotropic, δW_A is zero. Anisotropy of the perturbed pressure $\mathbb{P}$ is implicit in δW_K through its derivation from Eq. (10). Anisotropy of the equilibrium pressure is now included in two ways: δW_A modification of the fluid term, and inclusion of anisotropic pressure in δW_K through the distribution function f_j.

The above equation for δW_F is solved by various numerical codes (with $p_{\rm avg} = p$, a flux function). For example, the `PEST` code[47] solves for δW_F, in the form of Eq. (17) of Ref. 48, and uses the `VACUUM` code[49] to solve for δW_V. We wish to keep the definition of δW_F unchanged from its historical form for isotropic plasmas. Therefore in the following, we will consider $p_{\rm avg}$ unchanged from the isotropic pressure, as well as being a flux function. The correction due to anisotropy will come entirely from the δW_A term. The assumption of $p_{\rm avg}$ = constant, however, means that the *total* pressure, $p = \frac{1}{3}p_\parallel + \frac{2}{3}p_\perp$, can change as scans in $T_\parallel/T_\perp$ are performed, which will impact the kinetic effects.

The last term of Eq. (15) represents a modification of the pressure-driven ballooning destabilization term, which we will call δW_{A2}. This term has a different anisotropy correction than the others because of its explicit dependence on the pressure. It will be separately, and rigorously, calculated and discussed further in Sec. VI. Additionally, we note that the $\sigma - 1$ corrections to the shear Alfvén, magnetic compression, and kink destabilization terms will necessarily be small due to the restriction of $\sigma \approx 1$ imposed by equilibrium considerations in Sec. III.

The fluid terms ($\delta W_F + \delta W_A$) should be self-adjoint and therefore strictly real.[50] In particular, δW_{A2} in Eq. (15) is obviously real, as are the first two terms (the shear Alfvén and the magnetic compression terms) of δW_A in Eq. (15) and δW_F in Eq. (14). When the equilibrium pressure is isotropic ($\sigma = 1$ and $p_{\rm avg} = p$), one can show that the imaginary parts of the last two terms of δW_F (the kink and ballooning destabilization terms) cancel. With anisotropy that property is no longer obvious, but a lengthly manipulation can be used to show that indeed $\delta W_F + \delta W_A$ is still self-adjoint (see Appendix B).

Finally, to date, the kinetic term δW_K has been generally calculated for thermal particles with the distribution function f_j being for isotropic, Maxwellian particles. Here, we generalize that treatment to include a bi-Maxwellian f_j (which has, incidentally, previously been used to describe energetic ions heated by ICRH[16,19]).

V. KINETIC EFFECTS WITH ANISOTROPIC PRESSURE

An expression for δW_K that shows explicitly the dependence on the distribution function of the particles considered can be derived from Eq. (16) (ignoring electrostatic effects) and written[11,17]

$$\delta W_K = \sum_j \sum_{l=-\infty}^{\infty} 2\sqrt{2}\pi^2 \iiint \left[|\langle HT_j \rangle|^2 \lambda_{j,l} \frac{f_j}{T_j} \right] \frac{\hat{\tau}}{m_j^{\frac{3}{2}} B} |\chi| \varepsilon^{\frac{1}{2}} d\varepsilon d\chi d\Psi. \tag{17}$$

Here, ε is energy, $\chi = v_\parallel / v$ is the pitch angle, Ψ is the magnetic flux, H and $\hat{\tau}$ are given by Eqs. (12) and (13) of Ref. 51, and we have defined the frequency resonance fraction, $\lambda_{j,l}$, as

$$\lambda_{j,l} = \frac{\frac{T_j}{f_j}\left((\omega - n\omega_E) \frac{\partial f_j}{\partial \varepsilon} - \frac{n}{Z_j e} \frac{\partial f_j}{\partial \Psi} \right)}{n\langle \omega_D^j \rangle + (l + \alpha n q)\omega_b^j - i\nu_{\rm eff}^j + n\omega_E - \omega}. \tag{18}$$

Here, ω_E is the $E \times B$ frequency, $\langle \omega_D \rangle$ is the bounce-averaged precession drift frequency, l is the bounce harmonic, $\alpha = 0$ for trapped particles or $\alpha = 1$ for circulating particles, ω_b is the bounce frequency, and $\nu_{\rm eff}$ is the effective collision frequency. Note that the T_j/f_j factor cancels out in Eq. (17).

Clearly, the kinetic effects depend on the particle distribution function through its derivatives with respect to Ψ and energy.[11,17] We will see in Sec. VI that the derivative $\partial f_j / \partial \chi$ enters into the fluid anisotropy term as well.

In order to be consistent with the assumption of different pressures in the parallel and perpendicular directions, one should use a bi-Maxwellian distribution, which has pressure anisotropy due to different temperatures parallel and perpendicular to the magnetic field, in δW_K. The Maxwellian distribution is really just a special case of the more general bi-Maxwellian, with $T_{j\parallel} = T_{j\perp}$, so the bi-Maxwellian form

$$f_j^{bM}(\varepsilon, \Psi, \chi) = n_j \left(\frac{m_j}{2\pi} \right)^{\frac{3}{2}} \frac{1}{T_{j\perp} T_{j\parallel}^{\frac{1}{2}}} e^{-\varepsilon\chi^2/T_{j\parallel}} e^{-\varepsilon\left(1-\chi^2\right)/T_{j\perp}}, \tag{19}$$

can be used in general. Here, j denotes the particle type that is being considered (ions or electrons) and bM indicates bi-Maxwellian, ε is energy, $\chi = v_\parallel / v$ is the pitch angle, and Ψ is the magnetic flux. The density, $n_j(\Psi)$, and the two temperatures, $T_{j\parallel}(\Psi)$ and $T_{j\perp}(\Psi)$, are each assumed to be flux functions, so that the two pressures $p_\parallel$ and $p_\perp$ are as well. The pressures are given by $p_\parallel = \sum_j \int m_j v_\parallel^2 f_j d^3\mathbf{v} = \sum_j n_j T_{j\parallel}$ and $p_\perp = \sum_j \int \frac{1}{2} m_j v_\perp^2 f_j d^3\mathbf{v} = \sum_j n_j T_{j\perp}$. One can, of course, recover the Maxwellian, isotropic solution when $T_{j\parallel} = T_{j\perp}$.

We can see from Eq. (19) that

$$\frac{\partial f_j^{bM}}{\partial \varepsilon} = -\frac{f_j^{bM}}{T_j} \left(\frac{T_j}{T_{j\parallel}} \right), \tag{20}$$

and $\partial f_j / \partial \Psi$ takes the form

$$\frac{\partial f_j^{bM}}{\partial \Psi} = -\frac{f_j^{bM}}{T_j} \left[-\frac{T_j}{n_j} \frac{dn_j}{d\Psi} - \left(\varepsilon\chi^2 \frac{T_j}{T_{j\parallel}^2} - \frac{1}{2} \frac{T_j}{T_{j\parallel}} \right) \frac{dT_{j\parallel}}{d\Psi} - \left(\varepsilon\left(1-\chi^2\right) \frac{T_j}{T_{j\perp}^2} - \frac{T_j}{T_{j\perp}} \right) \frac{dT_{j\perp}}{d\Psi} \right], \tag{21}$$

Defining $\omega_{*T_\parallel}^j = -(1/Z_j e)(dT_{j\parallel}/d\Psi)$, and $\omega_{*T_\perp}^j = -(1/Z_j e)\,(dT_{j\perp}/d\Psi)$, then from Eqs. (17) and (18) we find

$$\delta W_K^{bM} = \sum_j \sum_{l=-\infty}^{\infty} \sqrt{\pi} \iiint n_j \left(\frac{1}{T_{j\perp} T_{j\|}^{\frac{1}{2}}}\right) \frac{\hat{\tau}}{B} |\chi| \varepsilon^{\frac{5}{2}} e^{-\varepsilon\chi^2/T_{j\|}} e^{-\varepsilon(1-\chi)^2/T_{j\perp}} d\varepsilon d\chi d\Psi$$
$$\times \left[|\langle HT_j/\varepsilon\rangle|^2 \frac{n\left(\frac{1}{Z_j e n_j}\frac{dn_j}{d\Psi} + \left(\varepsilon\chi^2 \frac{1}{T_{j\|}} - \frac{1}{2}\right)\left(\frac{1}{T_{j\|}}\right)\omega^j_{*T_\|} + \left(\varepsilon\left(1-\chi^2\right)\frac{1}{T_{j\perp}} - 1\right)\left(\frac{1}{T_{j\perp}}\right)\omega^j_{*T_\perp} + \left(\frac{1}{T_{j\|}}\right)\omega_E\right) - \left(\frac{1}{T_{j\|}}\right)\omega}{n\langle\omega^j_D\rangle + l\omega^j_b - i\nu^j_{\rm eff} + n\omega_E - \omega}\right]. \tag{22}$$

One can see that when $T_{j\|} = T_{j\perp} = T_j$, this equation reduces to the usual form for Maxwellian particles, since the exponential terms together become $e^{-\varepsilon/T_j}$, and the ω_{*T} terms become $(\varepsilon/T_j - \frac{3}{2})\omega_{*T}$.

Additionally one could, if desired, derive an expression for the kinetic term under the CGL perturbed pressure assumptions by taking the limit of $\omega \to \infty$, although this is no longer applicable to the RWM. One can show that under this high frequency limit, the CGL perturbed pressures are recovered (see Appendix A).

VI. ANISOTROPIC MODIFICATION OF THE PRESSURE-DRIVEN BALLOONING DESTABILIZATION TERM

Let us now return specifically to the final term in Eq. (15) which is like an anisotropic modification to the pressure-driven ballooning destabilization term. Though this is a fluid term, and is strictly real, it can be evaluated in a similar way to the above method for δW_K

$$\delta W_{A2} = \iint \left(m_j v_\|^2 + \frac{1}{2} m_j v_\perp^2\right) \frac{\mu}{2}\frac{\partial f_j}{\partial \mu} d^3\mathbf{v} |\nabla\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp|^2 d\mathbf{V} \tag{23}$$

$$= \frac{\sqrt{2}\pi^2}{m_j^{\frac{3}{2}}} \iiint \left(1-\chi^4\right)\left(-\frac{\partial f_j}{\partial \chi}\right) |\nabla\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp|^2 \frac{\hat{\tau}}{B} \varepsilon^{\frac{3}{2}} d\varepsilon d\chi d\Psi. \tag{24}$$

For isotropic particles δW_{A2} is zero because $\partial f_j/\partial\chi = 0$.

In the bi-Maxwellian case $\partial f_j/\partial\chi \neq 0$, but rather,

$$\frac{\partial f_j^{bM}}{\partial \chi} = -f_j^{bM}(2\varepsilon|\chi|)\left(\frac{1}{T_{j\|}} - \frac{1}{T_{j\perp}}\right), \tag{25}$$

so

$$\delta W_{A2}^{bM} = -\sqrt{\pi} \iiint n_j \frac{1}{T_{j\perp} T_{j\|}^{\frac{1}{2}}} \left(\frac{1}{T_{j_\|}} - \frac{1}{T_{j\perp}}\right) |\nabla\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp|^2$$
$$\times \left(\chi^4 - 1\right)|\chi| e^{-\frac{\varepsilon\chi^2}{T_{j\|}}} e^{-\frac{\varepsilon\left(1-\chi^2\right)}{T_{j\perp}}} \varepsilon^{\frac{5}{2}} d\varepsilon d\chi d\Psi. \tag{26}$$

The δW_{A2} term does not involve a frequency resonance fraction with various energy dependent terms in the same way that δW_K does. Therefore, we can simply perform the energy integration, using:[52] $\int_0^\infty \varepsilon^{\frac{5}{2}} e^{-a\varepsilon} d\varepsilon = (15/8)\sqrt{\pi} a^{-\frac{7}{2}}$, for $a > 0$. Then we have

$$\delta W_{A2}^{bM} = \frac{15\pi}{8} \iint n_j \frac{1 - \frac{T_{j\|}}{T_{j\perp}}}{T_{j\perp} T_{j\|}^{\frac{3}{2}}} |\nabla\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp|^2 \left(1-\chi^4\right)|\chi|$$
$$\times \left[\frac{\chi^2}{T_{j\|}} + \frac{1-\chi^2}{T_{j\perp}}\right]^{-\frac{7}{2}} d\chi d\Psi. \tag{27}$$

Another consequence of the lack of a frequency resonance fraction is that, unlike the kinetic term, the anisotropy term makes no distinction between ions and electrons if $n_i \approx n_e$ and $T_i \approx T_e$.

VII. CALCULATIONS USING THE `MISK` CODE

The fluid, anisotropy, and kinetic δW terms will be calculated numerically in four steps. First, δW_V and δW_F are calculated by the `PEST` code (as long as we assume that $p_{\rm avg} = p$) in the standard way.[47] Here, we first find the marginally stable RWM eigenfunction and then use it to find δW_F from Eq. (14) and δW_V^∞ by setting the wall position to infinity and δW_V^b by specifying a physical wall position. Second, the $\sigma - 1$ terms of δW_A can be calculated through a modification of `MISK` which separates out the various stabilizing and destabilizing terms, and multiplies the three relevant terms by $\mu_0(p_\perp - p_\|)/B^2$ inside the volume integral of Eq. (15).

Finally, in steps three and four, the `MISK` code is used to calculate δW_K and δW_{A2} according to the methods outlined in Secs. V and VI. `MISK` has been used previously for various machines to calculate kinetic effects on stability of Maxwellian thermal particles[9–11,14,51,53–56] as well as for isotropic or simple anisotropic distributions of trapped energetic particles.[11,53–55] Here, it is expanded to include the anisotropic bi-Maxwellian distribution for thermal particles, the $\sigma - 1$ fluid corrections, and the δW_{A2} correction to the pressure-driven ballooning destabilization term.

A. Solov'ev analytical equilibrium

For the present study, we wish to determine the effect of changing $T_\|/T_\perp$ of thermal particles on RWM stability generally. In principle, this could be a Ψ dependent quantity, but for simplicity we will use constant ratios across the entire radial profile. This is an artificial situation, not based on experimental reality, but it will give insight into the general effect of thermal particle anisotropy. To that end, we will use an analytical Solov'ev equilibrium solution to the Grad-Shafranov equation[57,58] and scan $T_\|/T_\perp$ while keeping the density and the average pressure constant (i.e., $T_\|$ and $T_\perp$ are changed such that $T = \frac{1}{2}T_{\parallel} + \frac{1}{2}T_{\perp}$ stays constant). We will use an equilibrium that was also used in Refs. 59 and 60. This equilibrium is meant to represent a generic, but realistic, example of a tokamak plasma. The equilibrium is shaped, has a conformal wall at 1.1 times the minor radius (see Fig. 2), contains the $q = 2$ and 3 rational surfaces within the plasma, and has a $q_{\rm edge} = 3.263$ (see Fig. 3(a)). It is specified by the parameters elongation: $\kappa = 1.6$, $q_0 = 1.9$, inverse aspect ratio: $\epsilon_a = a/R_0 = 0.33$, $R_0 = 1$ m, and $B_0 = 1$ T in Eqs. (3)–(5) of Ref. 59. Additionally, collisions are neglected, and the profiles of density, $n = n_0(1 - 07\Psi_n)$, temperature, $T = T_0(1 - \Psi_n)/(1 - 0.7\Psi_n)$, and $E \times B$ frequency, $\omega_E = \omega_{E0}(1 - \Psi_n)$, are the same as in Ref. 59. The normalized profiles of density (which doesn't change), temperature, and pressure are shown in Fig. 3 for the isotropic case as well as an example anisotropic case with $T_{\parallel}/T_{\perp} = 1.5$.

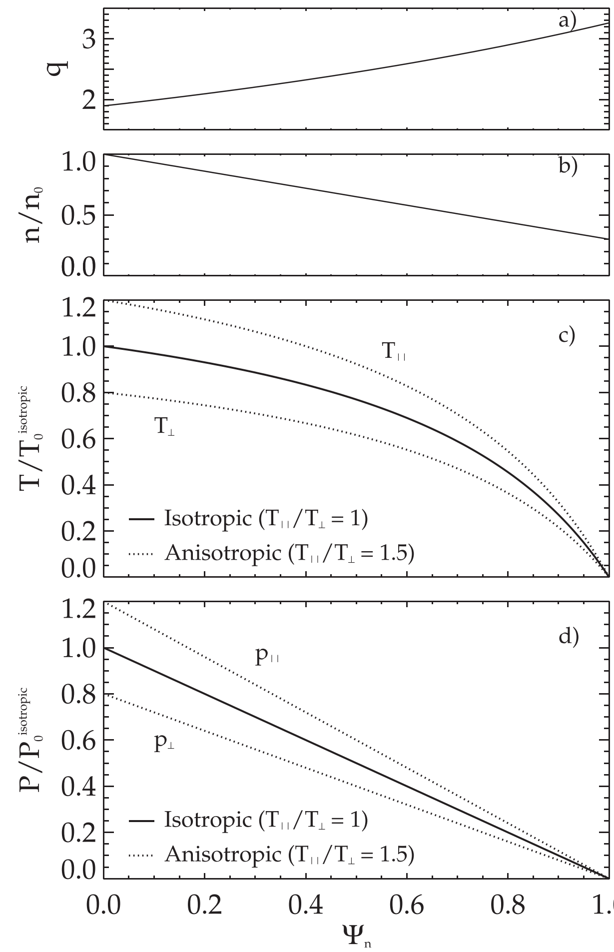

FIG. 3. Profiles of (a) safety factor, (b) density, (c) temperature, and (d) pressure for the isotropic Solov'ev equilibrium as well as an example anisotropic case with $T_{\parallel}/T_{\perp} = 1.5$.

B. Trends of distribution function derivatives with anisotropy

Before delving into code calculations, one can gain insight on the dependencies of the fluid and kinetic effects on anisotropy by examining the trends of the derivatives of the bi-Maxwellian distribution function on $T_{\parallel}/T_{\perp}$. From Eq. (24), it is clear that δW_{A2} depends upon $-\partial f^{\rm bM}/\partial\chi$, while from Eq. (18) δW_K depends upon $-\partial f^{\rm bM}/\partial\varepsilon$ (multiplied by ω_E) and $-\partial f^{\rm bM}/\partial\Psi$. These three derivatives can be easily analytically calculated at a given ε, χ, and Ψ, especially for the Solov'ev equilibrium with prescribed density and temperature profiles. In Fig. 4 we show these three derivatives, normalized by the Maxwellian distribution function $f^{\mathbf{M}}$, for $\varepsilon = T$ (the isotropic temperature), $\chi = 1/\sqrt{2}$ (for convenience, so that $\chi^2 = 1 - \chi^2$ in Eqs. (19) and (21)), and $\Psi = 0.65$ (again for convenience, as it is where the bracketed term in the Maxwellian equivalent of Eq. (21) is equal to $T/2$). The isotropic values of $(-1/f)(\partial f/\partial\chi)$, $(-1/f)(\partial f/\partial\varepsilon)$, and $(-1/f)(\partial f/\partial\Psi)$ are 0, 0.5, and 1.0, respectively, at this point in parameter space.

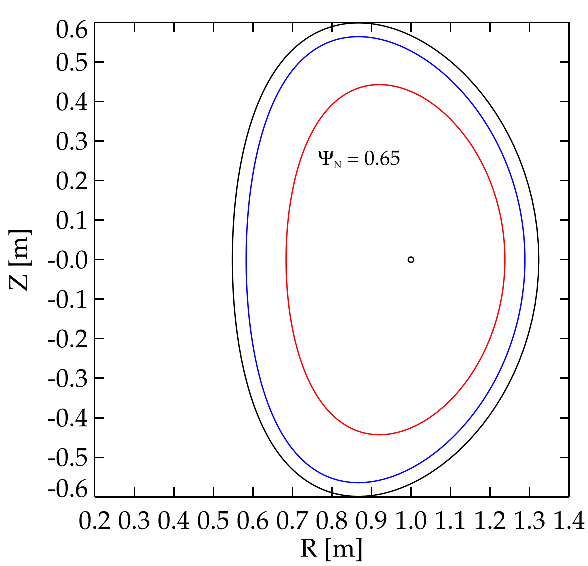

FIG. 2. The Solov'ev analytical equilibrium, showing flux surfaces at the edge (blue) and $\Psi_n = 0.65$ (red), and a conformal wall having $r_w/a = 1.10$ (black).

First, the kinetic effects depend upon the ε and Ψ derivatives. Here, $-\partial f/\partial\varepsilon$ is larger than the isotropic case for $T_{j\parallel}/T_{j\perp} < 1$ and smaller for $T_{j\parallel}/T_{j\perp} > 1$, while the relationship of $-\partial f/\partial\Psi$ is less clear, at least at this point in parameter space for this equilibrium. The kinetic terms therefore

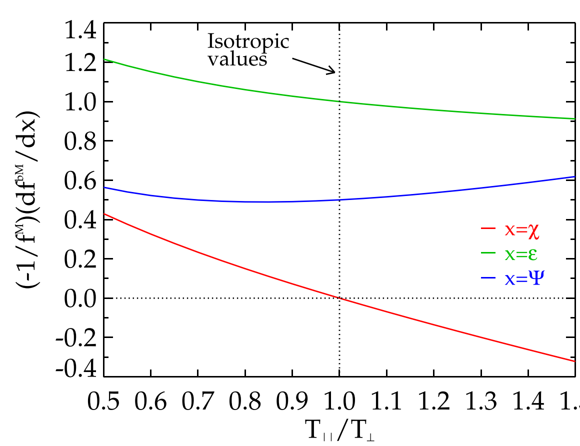

FIG. 4. Derivatives of the bi-Maxwellian distribution function vs. χ, ε, and Ψ, normalized by the negative of the Maxwellian distribution, for the Solov'ev case at $\varepsilon = T$, $\chi = 1\sqrt{2}$, and $\Psi = 0.65$.

will have complex dependencies on anisotropy, depending on ω_E and other factors, but here simplistically we might expect larger kinetic effects at lower $T_{j\|}/T_{j\perp}$.

Next, let us examine the fluid ballooning correction, which depends on $-\partial f/\partial\chi$. In Fig. 4, this quantity is positive for $T_\|/T_\perp < 1$ and negative for $T_{j\|}/T_{j\perp} > 1$. Then one can see from Eq. (24) (and (27)) that since $|\chi| \leq 1$, δW_{A2}^{bM} is positive (and therefore stabilizing), when $T_\|/T_\perp < 1$ and negative (destabilizing) when $T_{j\|}/T_{j\perp} > 1$. It is perhaps somewhat unintuitive, but here the fluid ballooning instability drive is enhanced by increasing parallel pressure, not perpendicular pressure. This result has been previously found under certain assumptions of the poloidal variation of $p_\perp$ and therefore of pressure weighting in the unfavorable curvature regions,[43,44,61,62] although not in others[63] (see Ref. 61 for a discussion of this topic). In our case, everything is considered a flux function, and p_{avg} is held constant. Therefore the fluid effect of anisotropy on the pressure-driven term is not through changes to the last term of Eq. (14), but rather through the last term of Eq. (15). In this way, we are isolating the effect of anisotropy to explore its dependencies. One can see that the negative, destabilizing ballooning drive is enhanced by positive $\partial p_{\text{avg}}/\partial B$ coupled with the RWM displacement quantity, and reduced by negative $\partial p_{\text{avg}}/\partial B$. Enhanced parallel pressure leads to positive $\partial p_{\text{avg}}/\partial B$ (one can show that $\partial p_\|/\partial B = (p_\| - p_\perp)/B$,[20,24,26] and that $\partial p_\perp/\partial B$) takes its sign from $(1 - p_\perp/p_\|)$.

C. Fluid terms

We note that the corrections to the shear Alfvén, fast magneto-acoustic, and kink terms cannot be larger than $\delta W_A/\delta W_F = \sigma_0 - 1$, where σ_0 is the value on axis. This is because we have considered $T_\|/T_\perp$ to be a constant, and pulling the factor $1 - \sigma_0$ out of the integral leaves only a factor of B_0^2/B^2 inside. This factor serves to weight the contribution from the low field regions more heavily, but when integrated over the volume it doesn't have a very large effect. Since the on-axis β ($\beta_0 = 2\mu_0 p_0/B_0^2$) is ≈10.7% for this Solov'ev case, it can be shown that the corrections to the shear Alfvén, fast magneto-acoustic, and kink terms are $<\pm 3\%$ for the range $0.5 < T_\|/T_\perp < 1.5$. The fact that σ_0 is close to 1 ($\sim\pm 3\%$) justifies the use of the isotropic equilibrium as discussed in Sec. III (with the caveat discussed in Appendix C).

Figure 5 shows $(\delta W_F + \delta W_A^{bM})/\delta W_F$ for the various components of the fluid term, vs. $T_\|/T_\perp$. Each component is normalized by its own δW_F, not the total. Not surprisingly, the correction to the pressure-driven ballooning destabilization term is considerably larger than $\sigma_0 - 1$. Here, higher perpendicular pressure (lower $T_\|/T_\perp$) leads to an enhancement of the bending and kinking effects, but a smaller ballooning effect (as expected from Sec. VII B). Higher parallel pressure (higher $T_\|/T_\perp$) leads to an increased ballooning effect, but the increase is not as strong as the decrease at low $T_\|/T_\perp$. The δW_{A2} term is half from ions and half from electrons, and additionally we have found it to be dominated by trapped particles over circulating particles.

Note that Fig. 5 shows normalized quantities. In absolute terms (normalized by $-\delta W_\infty = -\delta W_V^\infty - \delta W_F$), the

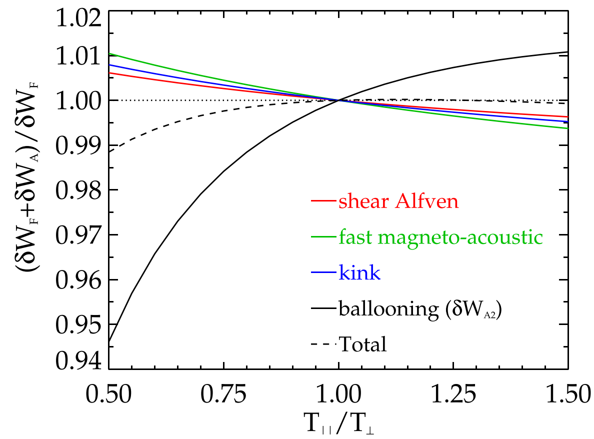

FIG. 5. $(\delta W_F + \delta W_A)/\delta W_F$ vs. scaled $T_\|/T_\perp$ for the Solov'ev equilibrium. Each component is normalized by its own isotropic contribution to δW_F, not the total.

isotropic contributions to $\delta W_F/(-\delta W_\infty)$ from shear Alfvén, fast magneto-acoustic, kink, and ballooning were 2.79, 0.007, −4.10, and −0.717, respectively. Therefore, while the changes to the shear Alfvén, fast magneto-acoustic, and kink terms are essentially the same in a relative sense, the fast magneto-acoustic term is quite small in this case, and therefore unimportant. Also the shear Alfvén term is stabilizing (positive), while the kink term is destabilizing (negative). Therefore the corrections to these two terms partially offset each other. Finally, even though the correction to the ballooning term was considerably larger in a relative sense, in this case the ballooning term is actually only about 25% as large as the shear Alfvén term and 17% as large as the kink term. The total percentage change in $\delta W_F + \delta W_A$ from the isotropic case is also shown in Fig. 5, which reflects all the relative weightings of the terms. The changes essentially cancel in this case for $T_\|/T_\perp > 1$, while leading to a small decrease for $T_\|/T_\perp < 1$.

D. Kinetic effects

The kinetic effects for thermal particles are also calculated, with Eq. (16), resulting in both real and imaginary parts. Figure 6 shows these contributions, normalized by their isotropic cases, plotted vs. $T_\|/T_\perp$. Three particle types, trapped ions and electrons and circulating ions, are shown separately (the contribution from circulating electrons is usually very small[9]). Generally, increased positive $Re(\delta W_K)$ is stabilizing, while increased $|Im(\delta W_K)|$ is always stabilizing (see Fig. 1).

When energy is shifted from parallel to perpendicular motion ($T_\|/T_\perp < 1$), we can see that the stabilizing restoring force provided by the trapped ions and represented by the real part of δW_K increases slightly, but the stabilizing resonance interaction between the mode and the trapped ions ($Im(\delta W_K)$) decreases, and vice versa when the parallel energy is increased ($T_\|/T_\perp > 1$). For electrons, the *destabilizing* force (negative $Re(\delta W_K)$) behaves similarly to the trapped ion term, while the change in the resonance is more complex. It should be noted, however, that the electron term is already smaller than the trapped ion term in magnitude and would be considerably more so if collisions had been

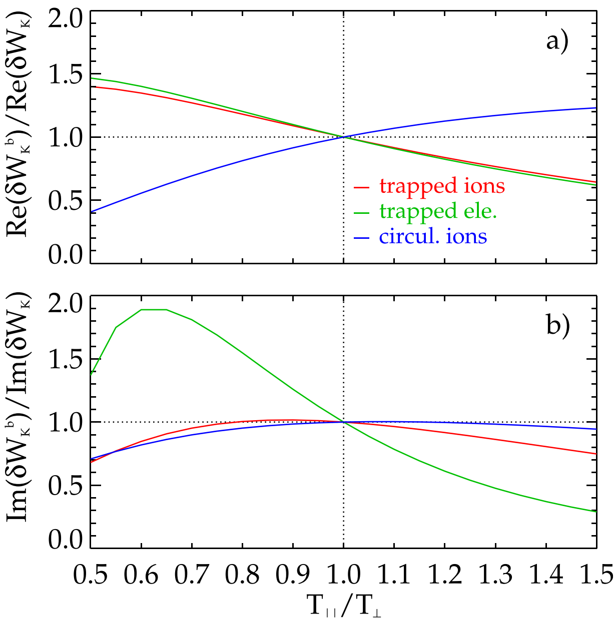

FIG. 6. a) Real and b) imaginary components of δW_K normalized by the corresponding isotropic δW_K components vs. scaled $T_{\|}/T_{\perp}$ for the Solov'ev equilibrium. The effect of anisotropy of three particles types, trapped ions and electrons and circulating ions, are shown.

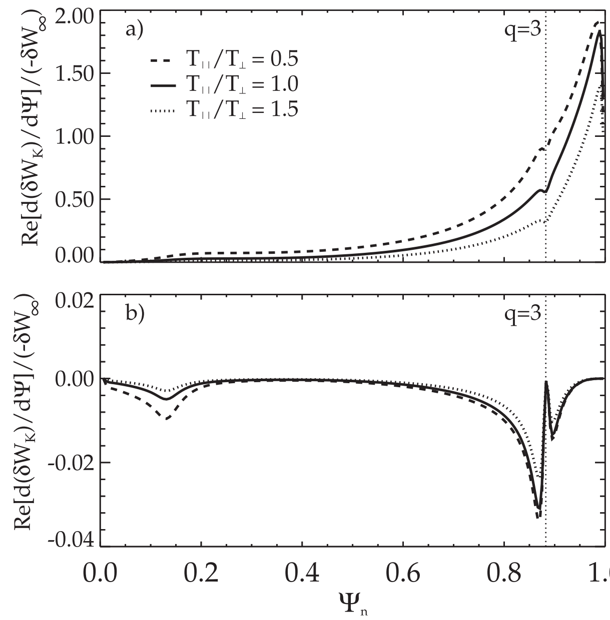

FIG. 7. $Re(d(\delta W_K/(-\delta W_\infty))/d\Psi)$ vs. Ψ for a) $l=0$ trapped thermal ions and b) $l \neq 0$ trapped thermal ions for $T_{\|}/T_{\perp} = 0.5$, 1.0, and 1.5.

considered.[53] For circulating ions (which also have smaller magnitude than thermal ions), smaller parallel energy leads to a decrease in their destabilizing effects (real and imaginary parts), while larger parallel energy increases their effect. Finally, it should be recalled that the assumption of constant p_{avg} has caused the total pressure to change, which affects the kinetic calculation shown here. In this case, the total pressure has increased by ≈11% at the low end of the range of $T_{\|}/T_{\perp}$ and decreased by ≈7% at the high end.

Finally, Fig. 7 shows the breakdown of the trapped ion term from Fig. 6(a), normalized by $-\delta W_\infty$, into the $l=0$ bounce harmonic (precession resonance) and the $l \neq 0$ harmonics (bounce resonances) and their radial dependences, for $T_{\|}/T_{\perp} = 0.5$, 1.0, and 1.5. In this particular case with its analytic profiles there are no singularities at the rational surfaces, so they are integrated across in this work. In cases with singularities, special treatment is necessary.[59]

E. Growth rate

Using Eq. (1), we can finally bring these various effects together by calculating the predicted growth rate of the RWM and plotting it vs. $T_{\|}/T_{\perp}$ in Fig. 8(a). First, without considering kinetic effects, we can see the effect of the fluid anisotropy corrections on the fluid growth rate (shown in red). Once again, we see that lower $T_{\|}/T_{\perp}$ is stabilizing and higher $T_{\|}/T_{\perp}$ is destabilizing, which follows from Fig. 5. When kinetic effects are included, but not the anisotropic fluid effects, the plasma becomes more stable ($\gamma\tau_w \approx 0.43 \rightarrow 0.375$ in the isotropic case). Then, as indicated by the green curve, lower $T_{\|}/T_{\perp}$ is stabilizing, while higher $T_{\|}/T_{\perp}$ is destabilizing. This follows from Fig. 6 with the change to the real part of the trapped thermal ion term being dominant. When the kinetic and fluid corrections are both applied (blue), the effect is even more enhanced at low

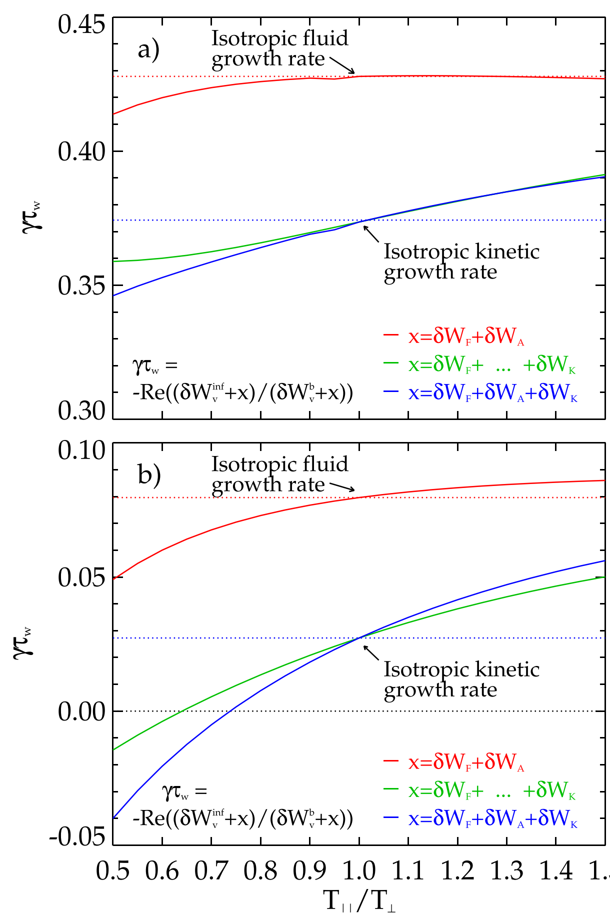

FIG. 8. The effect of anisotropy of thermal particles on the RWM growth rate for a) the nominal Solov'ev equilibrium, and b) the modified Solov'ev equilibrium with $\varepsilon_a = 0.45$. The growth rate normalized to the wall current decay time is given by $\gamma\tau_w = -Re((\delta W_v^\infty + x)/(\delta W_v^b + x))$. Shown is the isotropic fluid growth rate ($x = \delta W_F$, red dashed) and the fluid anisotropic modification to it ($x = \delta W_F + \delta W_A$, red solid). Also shown is the isotropic kinetic growth rate ($x = \delta W_F + \delta W_K$, blue dashed) and two modifications to it: including only the anisotropic kinetic term ($x = \delta W_F + \delta W_K$, green solid) or including both the fluid and kinetic corrections ($x = \delta W_F + \delta W_A + \delta W_K$, blue solid).

$T_\parallel/T_\perp$. This indicates that the fluid δW_A correction, though small (Fig. 5), can have an impact on the predicted growth rate. Finally, in Fig. 8(b), the resulting growth rates for a modified Solov'ev equilibrium are also presented. In this case, ε_a has been increased to 0.45, which increases the aspect ratio, $q_{\rm edge}$, and pressure (here we have correspondingly let the temperature increase while maintaining the same density). One can see that in this case the trends are all the same, though enhanced by the higher pressure. Note that the range of $\gamma\tau_w$ plotted is the same in both frames, but the modified case is closer to marginal stability and the anisotropy effects are enough to change the growth rate from slightly unstable to slightly stable at low $T_\parallel/T_\perp$.

Overall the biggest effect on RWM stability from anisotropy of the thermal particles will be in plasmas with high beta (where σ is larger), a large pressure-driven ballooning instability drive, and with $T_\perp$ larger than $T_\parallel$. In this case a reduction of the ballooning destabilization term can be expected, as well as an enhancement of the stabilizing kinetic effects of the trapped thermal ions.

VIII. CONCLUSIONS AND PHYSICAL IMPLICATIONS

We have derived the effect of anisotropy of the plasma pressure on the resistive wall mode stability energy principle. The fluid anisotropy has been treated as a small perturbation on the plasma equilibrium, which allows a relatively simple treatment of the problem. Fluid treatment with CGL pressures is akin to consideration of the high frequency mode rotation limit. More complete treatment leads to kinetic terms in addition to anisotropy corrections to the fluid terms. Specifically, the shear Alfvén, fast magneto-acoustic, and kink fluid terms are relatively simply modified by a factor of σ. Because of the equilibrium considerations in the perturbative approach, these corrections are necessarily small in our treatment. The kinetic effects depend upon $\partial f/\partial\varepsilon$ and $\partial f/\partial\Psi$, while the anisotropy correction to the fluid pressure-driven ballooning term depends upon $\partial f/\partial\chi$. We have derived expressions for these terms for a perturbed bi-Maxwellian distribution function for thermal particles.

For thermal particles with larger perpendicular energy than parallel the ballooning destabilization term is reduced, while for $T_\parallel > T_\perp$ it is enhanced. This leads to a reduction of the fluid growth rate of the RWM for $T_\parallel/T_\perp < 1$ and an increase for $T_\parallel/T_\perp > 1$. The stabilizing kinetic effects of the trapped thermal ions can also be enhanced for $T_\parallel/T_\perp < 1$, leading to a further increase in the RWM kinetic stability.

Finally, in the analysis presented here (for an analytical Solov'ev equilibrium) $T_\parallel/T_\perp$ was assumed to be uniformly changed over the whole plasma volume for both collisionless ions and electrons while maintaining $T_\parallel + T_\perp =$ constant, and had to be relatively significantly different from unity to see a large effect. Such a scenario would be difficult to achieve in experimental reality. In this light, the effect of realistic thermal particle anisotropy on RWM stability is likely to be modest. Extension of this relevant physics to energetic ions heated by ICRH and to less global modes, such as the internal kink mode, is possible, however.

ACKNOWLEDGMENTS

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APPENDIX A: DERIVATION OF CGL PERTURBED FLUID PRESSURES USING THE HIGH FREQUENCY LIMIT OF THE PERTURBED BI-MAXWELLIAN DISTRIBUTION FUNCTION

The perturbed fluid pressures, $\tilde{p}_\perp$ and $\tilde{p}_\parallel$ can be derived by using double-polytropic laws[62,64,65] as replacements for the adiabatic equation: $d(p_\parallel B^{\gamma_\parallel - 1}\rho^{-\gamma_\parallel})/dt = 0$, and $d(p_\perp B^{1-\gamma_\perp}\rho^{-1})/dt = 0$. The CGL double-adiabatic equations,[31,35,36,66,67] which are derived from the first and second adiabatic invariants[68] under the assumption of negligible heat flux[69,70] have $\gamma_\parallel = 3$ and $\gamma_\perp = 2$. Here, we will demonstrate that in fact the CGL $\tilde{p}_\perp$ and $\tilde{p}_\parallel$ expressions can also be recovered from Eq. (11) (neglecting the electrostatic contribution) using a bi-Maxwellian equilibrium distribution function and our form of $\tilde{f}_j$ from Eq. (12), with the assumption of fast mode rotation.

Let us now examine $\tilde{f}_j$ in the limit of large ω (which in reality pertains to high frequency modes, *not the RWM*). Then we can take a gyro-average ($\langle\cdot\rangle$) of f_j and in Eq. (12), $\omega(\partial f_j/\partial\varepsilon) \gg n(\partial f_j/\partial P_\phi)$, $\langle \mathbf{v}_\perp \cdot \boldsymbol{\xi}_\perp\rangle = 0$, $\langle \mathbf{v}_\perp \cdot \tilde{\mathbf{B}}\rangle = 0$ and

$$\langle \tilde{s}_j\rangle = \left\langle \int_{-\infty}^{t}\left(\mathbf{v}\cdot\frac{d\boldsymbol{\xi}_\perp}{dt'} - \frac{\tilde{\mathcal{Z}}}{m_j}\right)dt'\right\rangle \approx \frac{\langle HT_j\rangle}{im_j\omega}, \tag{A1}$$

so that

$$\tilde{f}_j^{\omega\to\infty} = -\boldsymbol{\xi}_\perp\cdot\nabla f_j + \frac{\partial f_j}{\partial\varepsilon}\big(\langle HT_j\rangle - Z_j e\big(\tilde{\Phi} + \boldsymbol{\xi}_\perp\cdot\nabla\Phi_0\big)\big) - \mu\frac{\tilde{\mathbf{B}}_\parallel}{B}\frac{\partial f_j}{\partial\mu}. \tag{A2}$$

Now,

$$\tilde{f}_j^{\omega\to\infty} + \frac{\partial f_j}{\partial B}\boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B} = -\boldsymbol{\xi}_\perp\cdot\nabla f_j + \frac{\partial f_j}{\partial\varepsilon}\left(\langle HT_j\rangle - \tilde{\mathcal{Z}}\right) - \frac{\mu}{B}\frac{\partial f_j}{\partial\mu}\left(\tilde{\mathbf{B}}_\parallel + \boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B}\right) \tag{A3}$$

$$= -\boldsymbol{\xi}_\perp\cdot\nabla f_j + m_j\frac{\partial f_j}{\partial\varepsilon}\left(\frac{1}{2}v_\perp^2\nabla\cdot\boldsymbol{\xi}_\perp + \frac{1}{2}v_\perp^2\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp - v_\parallel^2\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right) + \mu\frac{\partial f_j}{\partial\mu}(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp + \nabla\cdot\boldsymbol{\xi}_\perp), \tag{A4}$$

where in the last line we have used Eq. (6) of Ref. 14 for $\langle H\rangle$ and $\tilde{\mathbf{B}}_\parallel = -B(\nabla\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp) - \boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B}$.

From Eq. (19) for the bi-Maxwellian distribution $\partial f_j/\partial\varepsilon = -f_j/T_{j\parallel}$, and $\partial f_j/\partial\mu = -f_j B\left(\frac{1}{T_{j\perp}} - \frac{1}{T_{j\parallel}}\right)$. Now making these substitutions we find that

$$\tilde{f}_j^{\omega\to\infty} + \frac{\partial f_j}{\partial B}\boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B}$$
$$= -\boldsymbol{\xi}_\perp\cdot\nabla f_j - f_j m_j\left[\frac{1}{T_{j\parallel}}\left(-v_\parallel^2\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right) + \frac{1}{T_{j\perp}}\left(\frac{1}{2}v_\perp^2\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp + \frac{1}{2}v_\perp^2\nabla\cdot\boldsymbol{\xi}_\perp\right)\right]. \quad \text{(A5)}$$

We then define the quantities R_1, R_2, and R_3 as in Ref. 71

$$R_1 = \sum_j m_j \int v_\parallel^4 f_j d^3\mathbf{v} = \frac{3p_\parallel^2}{\rho}, \quad \text{(A6)}$$

$$R_2 = \frac{1}{2}\sum_j m_j \int v_\parallel^2 v_\perp^2 f_j d^3\mathbf{v} = \frac{p_\parallel p_\perp}{\rho}, \quad \text{(A7)}$$

$$R_3 = \frac{1}{2}\sum_j m_j \int v_\perp^4 f_j d^3\mathbf{v} = \frac{4p_\perp^2}{\rho}. \quad \text{(A8)}$$

Now from Eq. (11)

$$\tilde{p}_\parallel = \sum_j m_j \int v_\parallel^2\left(\tilde{f}_j^{\omega\to\infty} + \frac{\partial f_j}{\partial B}\boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B}\right)d^3\mathbf{v} \quad \text{(A9)}$$

$$= -\boldsymbol{\xi}_\perp\cdot\nabla p_\parallel + \frac{m_j}{T_{j\parallel}}R_1\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp - \frac{m_j}{T_{j\perp}}R_2(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp + \nabla\cdot\boldsymbol{\xi}_\perp) \quad \text{(A10)}$$

$$= -\boldsymbol{\xi}_\perp\cdot\nabla p_\parallel - p_\parallel\nabla\cdot\boldsymbol{\xi}_\perp + 2p_\parallel\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp, \quad \text{(A11)}$$

and

$$\tilde{p}_\perp = \sum_j \frac{1}{2}m_j \int v_\perp^2\left(\tilde{f}_j^{\omega\to\infty} + \frac{\partial f_j}{\partial B}\boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B}\right)d^3\mathbf{v} \quad \text{(A12)}$$

$$= -\boldsymbol{\xi}_\perp\cdot\nabla p_\perp + \frac{m_j}{T_{j\parallel}}R_2\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp - \frac{m_j}{2T_{j\perp}}R_3(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp + \nabla\cdot\boldsymbol{\xi}_\perp) \quad \text{(A13)}$$

$$= -\boldsymbol{\xi}_\perp\cdot\nabla p_\perp - 2p_\perp\nabla\cdot\boldsymbol{\xi}_\perp - p_\perp\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp. \quad \text{(A14)}$$

Note that these $\tilde{p}$ could then be used in Eq. (2) to find a δW_{CGL},[34,35] but this would only be applicable in the high frequency limit.

APPENDIX B: SELF-ADJOINTNESS OF THE ANISOTROPIC FLUID δW

In order to show that $\delta W_F + \delta W_A$ is self-adjoint, we must look at both the ballooning term from Eq. (14), and the kink term from Eqs. (14) and (15) together. Beginning with the kink term we write

$$\delta W_{F+A}^{\text{kink}} = \frac{1}{2}\int\frac{\sigma j_\parallel}{2B}\left((\boldsymbol{\xi}_\perp^*\times\mathbf{B})\cdot\tilde{\mathbf{B}}_\perp + (\boldsymbol{\xi}_\perp^*\times\mathbf{B})\cdot\tilde{\mathbf{B}}_\perp\right)d\mathbf{V}, \quad \text{(B1)}$$

$$= \frac{1}{2}\int\frac{\sigma j_\parallel}{2B}\left((\boldsymbol{\xi}_\perp^*\times\mathbf{B})\cdot\nabla\times(\boldsymbol{\xi}_\perp\times\mathbf{B}) + (\boldsymbol{\xi}_\perp^*\times\mathbf{B})\cdot\tilde{\mathbf{B}}_\perp\right)d\mathbf{V}, \quad \text{(B2)}$$

$$= \frac{1}{2}\int\frac{\sigma j_\parallel}{2B}\big(\nabla\cdot\big((\boldsymbol{\xi}_\perp\times\boldsymbol{\xi}_\perp^*\cdot\mathbf{B})\mathbf{B}\big) + (\boldsymbol{\xi}_\perp\times\mathbf{B})\cdot\nabla\times(\boldsymbol{\xi}_\perp^*\times\mathbf{B}) + (\boldsymbol{\xi}_\perp^*\times\mathbf{B})\cdot\tilde{\mathbf{B}}_\perp\big)d\mathbf{V}, \quad \text{(B3)}$$

$$= \frac{1}{2}\int\left(\nabla\cdot\left(\frac{\sigma j_\parallel}{2B}(\boldsymbol{\xi}_\perp\times\boldsymbol{\xi}_\perp^*\cdot\mathbf{B})\mathbf{B}\right) - (\boldsymbol{\xi}_\perp\times\boldsymbol{\xi}_\perp^*\cdot\mathbf{B})\mathbf{B}\cdot\nabla\left(\frac{\sigma j_\parallel}{2B}\right) + \frac{\sigma j_\parallel}{2B}\left((\boldsymbol{\xi}_\perp\times\mathbf{B})\cdot\tilde{\mathbf{B}}_\perp^* + (\boldsymbol{\xi}_\perp^*\times\mathbf{B})\cdot\tilde{\mathbf{B}}_\perp\right)\right)d\mathbf{V}. \quad \text{(B4)}$$

The first term integrates to zero over the volume. The last terms together are self-adjoint. Defining $\boldsymbol{\xi}_\perp = \xi_\Psi\hat{\mathbf{e}}_\Psi + \xi_\chi\hat{\mathbf{e}}_\chi$, with $\hat{\mathbf{e}}_\Psi = \nabla\Psi/|\nabla\Psi|$ and $\hat{\mathbf{e}}_\chi = \hat{\mathbf{b}}\times\nabla\Psi/|\nabla\Psi|$, and rewriting, we have

$$\delta W_{F+A}^{\text{kink}} = \frac{1}{2}\int\left(\frac{\sigma j_\parallel}{2B}\left((\boldsymbol{\xi}_\perp\times\mathbf{B})\cdot\tilde{\mathbf{B}}_\perp^* + (\boldsymbol{\xi}_\perp^*\times\mathbf{B})\cdot\tilde{\mathbf{B}}_\perp\right) - \left(\xi_\Psi\xi_\chi^* - \xi_\chi\xi_\Psi^*\right)\mathbf{B}\nabla\cdot\left(\frac{\sigma j_\parallel}{2}\hat{\mathbf{b}}\right)\right)d\mathbf{V}. \quad \text{(B5)}$$

Now let us return to the ballooning term and write

$$\delta W_{F+A}^{\text{ballooning}} = \int\left(\boldsymbol{\kappa}_\Psi\xi_\Psi^* + \boldsymbol{\kappa}_\chi\xi_\chi^*\right)\times\left(\boldsymbol{\xi}_\perp\cdot\nabla\Psi\frac{\partial p_{\text{avg}}}{\partial\Psi} + \boldsymbol{\xi}_\perp\cdot\nabla B\frac{\partial p_{\text{avg}}}{\partial B}\right)d\mathbf{V} \quad \text{(B6)}$$

$$= \int\left(\left(\xi_\Psi\xi_\Psi^*\boldsymbol{\kappa}_\Psi + \xi_\Psi\xi_\chi^*\boldsymbol{\kappa}_\chi\right)\left(|\nabla\Psi|\frac{\partial p_{\text{avg}}}{\partial\Psi} + \hat{\mathbf{e}}_\Psi\cdot\nabla B\frac{\partial p_{\text{avg}}}{\partial B}\right) + \left(\xi_\chi\xi_\chi^*\boldsymbol{\kappa}_\chi + \xi_\chi\xi_\Psi^*\boldsymbol{\kappa}_\Psi\right)\left(\hat{\mathbf{e}}_\chi\cdot\nabla B\frac{\partial p_{\text{avg}}}{\partial B}\right)\right)d\mathbf{V}. \quad \text{(B7)}$$

Then let us momentarily consider

$$\nabla\times\hat{\mathbf{b}}\cdot\nabla p_{\text{avg}} = \nabla\times\hat{\mathbf{b}}\cdot\nabla\Psi\frac{\partial p_{\text{avg}}}{\partial\Psi} + \nabla\times\hat{\mathbf{b}}\cdot\nabla B\Psi\frac{\partial p_{\text{avg}}}{\partial B}, \quad \text{(B8)}$$

$$= \nabla\times\hat{\mathbf{b}}\cdot\nabla\Psi\frac{\partial p_{\text{avg}}}{\partial\Psi} + \nabla\times\hat{\mathbf{b}}\cdot\nabla\Psi\frac{\partial B}{\partial\Psi}\frac{\partial p_{\text{avg}}}{\partial B} + (\nabla\times\hat{\mathbf{b}}\cdot\hat{\mathbf{b}})(\hat{\mathbf{b}}\cdot\nabla B)\frac{\partial p_{\text{avg}}}{\partial B} + \left(\nabla\times\hat{\mathbf{b}}\cdot\hat{\mathbf{e}}_\chi\right)(\hat{\mathbf{e}}_\chi\cdot\nabla B)\frac{\partial p_{\text{avg}}}{\partial B}, \quad \text{(B9)}$$

$$= -\boldsymbol{\kappa}_\chi|\nabla\Psi|\frac{\partial p_{\text{avg}}}{\partial\Psi} - \boldsymbol{\kappa}_\chi(\hat{\mathbf{e}}_\Psi\cdot\nabla B)\frac{\partial p_{\text{avg}}}{\partial B} + \frac{j_\parallel}{B}(\hat{\mathbf{b}}\cdot\nabla B)\frac{\partial p_{\text{avg}}}{\partial B} + \boldsymbol{\kappa}_\Psi(\hat{\mathbf{e}}_\chi\cdot\nabla B)\frac{\partial p_{\text{avg}}}{\partial B}, \quad \text{(B10)}$$

so that now

$$\delta W_{F+A}^{\text{ballooning}} = \int\left(|\xi_\Psi|^2\boldsymbol{\kappa}_\Psi\left(|\nabla\Psi|\frac{\partial p_{\text{avg}}}{\partial\Psi} + \hat{\mathbf{e}}_\Psi\cdot\nabla B\frac{\partial p_{\text{avg}}}{\partial B}\right) + |\xi_\chi|^2\boldsymbol{\kappa}_\chi\left(\hat{\mathbf{e}}_\chi\cdot\nabla B\frac{\partial p_{\text{avg}}}{\partial B}\right) + \xi_\Psi\xi_\chi^*\left(\frac{j_\parallel}{B}(\hat{\mathbf{b}}\cdot\nabla B)\frac{\partial p_{\text{avg}}}{\partial B} - \nabla\times\hat{\mathbf{b}}\cdot\nabla p_{\text{avg}}\right)\right)d\mathbf{V}. \quad \text{(B11)}$$

From the equilibrium considered in Sec. III, one can show that the last term can be replaced so that

$$\delta W_{F+A}^{\mathrm{ballooning}} = \int\left(|\xi_\Psi|^2\boldsymbol{\kappa}_\Psi\left(|\nabla\Psi|\frac{\partial p_{\mathrm{avg}}}{\partial\Psi} + \hat{\mathbf{e}}_\Psi\cdot\nabla B\frac{\partial p_{\mathrm{avg}}}{\partial B}\right) + |\xi_\chi|^2\boldsymbol{\kappa}_\chi\left(\hat{\mathbf{e}}_\chi\cdot\nabla B\frac{\partial p_{\mathrm{avg}}}{\partial B}\right) + \xi_\Psi\xi_\chi^*\mathbf{B}\nabla\cdot\left(\frac{\sigma j_\parallel}{2}\hat{\mathbf{b}}\right)\right)d\mathbf{V}. \tag{B12}$$

Then one can see that the following is self-adjoint

$$\begin{aligned}\delta W_{F+A}^{\mathrm{kink}} + \delta W_{F+A}^{\mathrm{ballooning}} = \frac{1}{2}\int&\left(\frac{\sigma j_\parallel}{2B}\left((\boldsymbol{\xi}_\perp\times\mathbf{B})\cdot\tilde{\mathbf{B}}_\perp^* + (\boldsymbol{\xi}_\perp^*\times\mathbf{B})\cdot\tilde{\mathbf{B}}_\perp\right) + \left(\xi_\Psi\xi_\chi^* + \xi_\chi\xi_\Psi^*\right)\mathbf{B}\nabla\cdot\left(\frac{\sigma j_\parallel}{2}\hat{\mathbf{b}}\right)\right.\\ &\left.+2|\xi_\Psi|^2\boldsymbol{\kappa}_\Psi\left(|\nabla\Psi|\frac{\partial p_{\mathrm{avg}}}{\partial\Psi} + \hat{\mathbf{e}}_\Psi\cdot\nabla B\frac{\partial p_{\mathrm{avg}}}{\partial B}\right) + 2|\xi_\chi|^2\boldsymbol{\kappa}_\chi\left(\hat{\mathbf{e}}_\chi\cdot\nabla B\frac{\partial p_{\mathrm{avg}}}{\partial B}\right)\right)d\mathbf{V}.\end{aligned} \tag{B13}$$

APPENDIX C: ORDERING OF δW IN EXPANSION PARAMETER $\sigma - 1$

It was demonstrated in Appendix B that the full fluid anisotropic δW is self-adjoint. Here we will show that this means only the lowest order eigenfunction is necessary. If we consider an expansion parameter ϵ and write the plasma displacement as an expansion in ϵ, $\boldsymbol{\xi} = \boldsymbol{\xi_0} + \epsilon\boldsymbol{\xi_1} + \epsilon^2\boldsymbol{\xi_2}$. Then since δW is self-adjoint, $\delta W(\boldsymbol{\xi}) = \delta W(\boldsymbol{\xi_0}) + \epsilon^2\delta W(\boldsymbol{\xi_0},\boldsymbol{\xi_1})$.

If we now take the expansion parameter to be $\epsilon = \sigma - 1$, then the full anisotropic fluid δW can be written $\delta W(\boldsymbol{\xi}) = \delta W_F(\boldsymbol{\xi_0}) + \delta W_A(\boldsymbol{\xi_0}) + O(\epsilon^2)$. Here, the δW_A term is really first order in ϵ, as can be seen by the $\sigma - 1$ term in the integral in Eq. (15). Therefore, only the lowest order eigenfunction is required to calculate δW to first order in ϵ.

However, anisotropy also modifies the equilibrium, as was discussed in Sec. III. Consider Eq. (9), rewritten to explicitly show the isotropic equilibrium and its anisotropic correction

$$\begin{aligned}&\nabla_\perp\left(\frac{\mathbf{B}^2}{2\mu_0} + p\right) - \boldsymbol{\kappa}\frac{B^2}{\mu_0}\\ &\quad= \nabla_\perp\left((\sigma-1)\frac{\mathbf{B}^2}{2\mu_0} + (p_{\mathrm{avg}} - p)\right) - (\sigma-1)\boldsymbol{\kappa}\frac{B^2}{\mu_0}.\end{aligned} \tag{C1}$$

If $p_{\mathrm{avg}} - p \sim (\sigma - 1)p$ (or, in our case, $p_{\mathrm{avg}} - p = 0$) then the right hand side is an anisotropy correction to the equilibrium of first order in ϵ. Therefore, any equilibrium quantity Q can be written as $Q = Q_0 + \epsilon Q_1 + \cdots$, where Q_0 is the term calculated by the isotropic equilibrium solver. Since each term in Eq. (2) includes an equilibrium quantity and an eigenfunction quantity, we find that $\delta W(\boldsymbol{\xi}) = \delta W_F(Q_0,\boldsymbol{\xi_0}) + \epsilon\delta W_F(Q_1,\boldsymbol{\xi_0}) + \delta W_A(Q_0,\boldsymbol{\xi_0}) + O(\epsilon^2)$. The first term is the usual isotropic fluid δW, and the third term is the first order anisotropic correction presented in this paper, but the second term is also a first order anisotropic correction to the fluid δW that arises from the anisotropy correction to the equilibrium. This term is not calculated here, but may be of the same order as the fluid corrections presented here and therefore should be explored by calculating the anisotropic equilibrium.[20–30] It is worth noting, however, that the anisotropic *kinetic* effects were found here to be larger than either of these first order fluid corrections.

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