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Parallel electron force balance and the L-H transition

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In one popular paradigm for the L-H transition, energy transfer to the mean flows directly depletes turbulence fluctuation energy, resulting in suppression of the turbulence and a corresponding transport bifurcation. To quantitatively evaluate this mechanism, one must remember that electron parallel force balance couples nonzonal velocity fluctuations with electron pressure fluctuations on rapid timescales, comparable with the electron transit time. For this reason, energy in the nonzonal velocity stays in a fairly fixed ratio to the free energy in electron density fluctuations, at least for frequency scales much slower than electron transit. In order for direct depletion of the energy in turbulent fluctuations to cause the L-H transition, energy transfer via Reynolds stress must therefore drain enough energy to significantly reduce the sum of the free energy in nonzonal velocities and electron pressure fluctuations. At low $k_\perp$, the electron thermal free energy is much larger than the energy in nonzonal velocities, posing a stark challenge for this model of the L-H transition. *Published by AIP Publishing.*
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Although the edge transport barrier regime known as H-mode was discovered experimentally over 30 years ago[1] and is necessary in order for ITER to achieve its performance goals, we still lack a clear, universally accepted explanation of the physics underlying this enhanced confinement regime.[2,3] However, most models in current discussion are based around the effect of radially sheared $\boldsymbol{E}\times\boldsymbol{B}$ velocities.[4] In one popular variant of this picture, turbulent fluctuations directly lose energy via transfer to the shear flows.[5] Experimental attempts to validate this variant have used energy balance between the zonal and nonzonal portions of the $\boldsymbol{E}\times\boldsymbol{B}$ velocity to estimate the effect of energy transfer via Reynolds stress.[6,7] In the following, I will briefly demonstrate that such analysis must be corrected to include the nonzonal portion of the free energy in electron density fluctuations. Since this energy may often be much larger than the energy in the $\boldsymbol{E}\times\boldsymbol{B}$ velocity, its inclusion may significantly affect the viability of the direct energy-transfer paradigm in explaining the L-H transition.

We will treat the problem in a very simple two-fluid flux-tube model, with isothermal electrons, a single species of singly ionized cold ions, purely resistive parallel dynamics, frequencies fast relative to ion transit ($\omega \gg c_s/qR$), and a shearless, simple-circular, large-aspect-ratio magnetic geometry. [Although this model must be generalized for quantitative treatments of edge turbulence in experiment, these generalizations do not affect the basic structure underlying our conclusions. Some effects of omitted generalizations will be mentioned throughout the text.] The resulting equations appear in SI units[8] as

$$(\partial_t + \boldsymbol{v}_E\cdot\nabla)(n_e+n_0) = \frac{1}{e}\nabla_\parallel j_\parallel + \frac{1}{e}\mathcal{K}(n_0 e\phi - n_e T_{e0}), \quad (1)$$

$$\frac{n_0 m_i}{B^2}(\partial_t + \boldsymbol{v}_E\cdot\nabla)\nabla_\perp^2\phi = \nabla_\parallel j_\parallel - \mathcal{K}(n_e T_{e0}), \quad (2)$$

$$\eta j_\parallel = \frac{T_{e0}}{n_0 e}\nabla_\parallel n_e - \nabla_\parallel\phi. \quad (3)$$

Equation (1) shows the evolution of fluctuating electron density n_e under advection by the $\boldsymbol{E}\times\boldsymbol{B}$ drift $\boldsymbol{v}_E = B^{-1}\hat{\boldsymbol{b}}\times\nabla\phi$, divergence of the parallel current $\nabla_\parallel j_\parallel$ (equivalent to $-e$ times parallel electron density flux, due to our orderings), and the toroidal effects due to $\mathcal{K} \doteq -(2/B^2)\hat{\boldsymbol{b}}\times\nabla B\cdot\nabla$, both the curvature and ∇B drifts $\propto e^{-1}\mathcal{K}(n_e T_{e0})$ and the divergence of the $\boldsymbol{E}\times\boldsymbol{B}$ drift $-\mathcal{K}(\phi)$. $\boldsymbol{E}\times\boldsymbol{B}$ advection down the gradient of the mean density n_0 is the sole free energy source in this model. Eq. (2) essentially states that the current must be divergence-free: the LHS is (minus) the divergence of the ion polarization current (both linear and nonlinear), while the RHS consists of the divergence of the electron curvature current $-\mathcal{K}(n_e T_{e0})$ and the parallel current $\nabla_\parallel j_\parallel$. Eq. (3) determines the parallel current by a balance between resistive drag $n_0 e\eta j_\parallel$ and the parallel forces on electrons, due to electron pressure gradient $-T_{e0}\nabla_\parallel n_e$ and electric force $n_0 e\nabla_\parallel\phi$. After evaluation of $\mathcal{K}$ and the substitution $\boldsymbol{v}_E\cdot\nabla n_0 \to -(n_0/L_n)v_E^x$ [for $v_E^x \doteq \boldsymbol{v}_E\cdot\nabla x$, radial coordinate x, and $1/L_n \doteq -n_0^{-1}(dn_0/dx)$], we may take n_0, T_{e0}, and B to be constants. Eqs. (1)–(3) may be seen as a generalization of the Hasegawa-Wakatani equations[9] to include some toroidal effects, or as a simplification of isothermal Braginskii or gyrofluid equations.[10]

In many cases, the parallel resistivity η is small enough that parallel electron diffusion is rapid relative to other physical processes. In this case, if the terms on the RHS of Eq. (3) do not approximately cancel, then a large parallel current $j_\parallel$ is needed to make $\eta j_\parallel$ large enough to satisfy Eq. (3), so Eqs. (1) and (2) will simplify to[11]

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$$\partial_t n_e \approx \frac{1}{e}\nabla_\| j_\| = \frac{1}{\eta e}\left(\frac{T_{e0}}{n_0 e}\nabla_\|^2 n_e - \nabla_\|^2\phi\right), \tag{4}$$

$$\frac{n_0 m_i}{B^2}\partial_t \nabla_\perp^2\phi \approx \nabla_\| j_\| = \frac{1}{\eta}\left(\frac{T_{e0}}{n_0 e}\nabla_\|^2 n_e - \nabla_\|^2\phi\right). \tag{5}$$

Considering a single Fourier mode in the perpendicular direction so $\nabla_\perp^2 \to -k_\perp^2$, we may combine Eqs. (4) and (5) to evolve the dimensionless combination $h_e \doteq n_e/n_0 - e\phi/T_{e0}$

$$\partial_t h_e = \frac{T_{e0}}{\eta n_0 e^2}\left(1 + \frac{1}{k_\perp^2\rho_s^2}\right)\nabla_\|^2 h_e, \tag{6}$$

showing dissipation of h_e due to parallel conduction. The basic rate can easily be seen to follow from (collisional) parallel electron diffusion: $k_\|^2 T_{e0}/\eta n_0 e^2 = k_\|^2 v_{te}^2/\nu_{ei}$ for $v_{te} \doteq (T_{e0}/m_e)^{1/2}$ and ν_{ei} the appropriate electron-ion collision rate. The factor of $1/k_\perp^2\rho_s^2$, with the "sound radius" ρ_s just the ion gyroradius evaluated at the electron temperature $\rho_s \doteq c_s m_i/eB$ for $c_s \doteq (T_{e0}/m_i)^{1/2}$,[12] can substantially increase the damping rate for h_e, in the common case of fluctuations at $k_\perp\rho_s \ll 1$, that is, for cross-field spatial scales larger than the sound radius. The physical reason is more straightforward than it may appear: at these larger spatial scales, the ion polarization response becomes weak, so that even a small parallel current can cause a large change in ϕ. This means that for whatever parallel current divergence we get from Ohm's Law [Eq. (3)], the change in the potential must be larger by $1/k_\perp^2\rho_s^2$ to bring in enough ions (across the field) to maintain quasineutrality. From this analysis, we may conclude that at timescales longer than the very short timescale $\nu_{ei}k_\perp^2\rho_s^2/k_\|^2 v_{te}^2$ corresponding to parallel electron diffusion, the RHS of Eq. (3) must be in approximate balance, so that $\nabla_\|(n_e/n_0) \approx \nabla_\|(e\phi/T_{e0})$. If we were to include electron inertia or electromagnetic fluctuations [neglected in Eqs. (1)–(3)], we would slow the short timescale to the longest of three parallel electron timescales: resistive (given here), collisionless (going with electron transit time), or Alfvénic (going with Alfvén damping time), but the ultimate relaxation to electron adiabatic response would be unchanged, as long as our other frequencies were still much slower than the slowest electron parallel rate.

Since the magnetic field is always tangential to the flux surface, the parallel gradient vanishes for anything that is constant over the flux surface. For this reason, it is advantageous to decompose the potential ϕ into its flux-surface average $\langle\phi\rangle$ (the "zonal" potential), which is constant over a flux surface so that $\nabla_\|\langle\phi\rangle = 0$, and the remaining "nonzonal" potential $\tilde\phi \doteq \phi - \langle\phi\rangle$, which trivially satisfies $\langle\tilde\phi\rangle = 0$. In our simple geometry, we may define a radial coordinate x and binormal coordinate y such that $v_E^x \doteq \boldsymbol{v}_E\cdot\nabla x = -B^{-1}\partial_y\phi$, $v_E^y \doteq \boldsymbol{v}_E\cdot\nabla y = B^{-1}\partial_x\phi$, and $\nabla_\perp^2 \approx \partial_x^2 + \partial_y^2$, in which ∂_x and ∂_y are partial derivatives with respect to x and y. We may then decompose $v_E^y = \langle v_E^y\rangle + \tilde v_E^y$ with $\langle v_E^y\rangle = B^{-1}\partial_x\langle\phi\rangle$ and $\tilde v_E^y = B^{-1}\partial_x\tilde\phi$. Since the flux-surface average involves integrating over the angle-like y, the corresponding decomposition for v_E^x is simply $\langle v_E^x\rangle = 0$ (by periodicity of ϕ in y) so $v_E^x = \tilde v_E^x = -B^{-1}\partial_y\phi = -B^{-1}\partial_y\tilde\phi$.

Turbulent evolution is complicated, due primarily to the advective nonlinearities, in our case the $(\boldsymbol{v}_E\cdot\nabla)n_e$ and $(\boldsymbol{v}_E\cdot\nabla)\nabla_\perp^2\phi$ terms in Eqs. (1) and (2). Some insight may therefore be gained by considering quantities that are invariant under the nonlinearities, that is, quantities whose evolution equations do not contain the nonlinear terms. Much productive analysis has resulted from a focus on fluctuation free energy, a nonlinear invariant that is proportional to the square of the turbulent amplitudes, roughly measuring the strength of the fluctuations. Such equations may be derived for Eqs. (1)–(3) as follows: Multiply Eq. (1) by $T_{e0}n_e/n_0$ and Eq. (2) separately by $-\tilde\phi$ and by $-\langle\phi\rangle$, then integrate each resulting equation over some volume $\int d\mathcal{V}$ and do some integrations by parts, neglecting the fluxes through the boundaries. Note that $\nabla_\|\langle f\rangle = 0$ and $\int d\mathcal{V}\langle f\rangle\tilde g = 0$ for arbitrary functions f and g. The resulting equations are[13]

$$\partial_t E_n = T_{e0}\int d\mathcal{V}\left[\frac{1}{L_n}n_e v_E^x - \phi\mathcal{K}(n_e) - \frac{1}{n_0 e}j_\|\nabla_\| n_e\right], \tag{7}$$

$$\partial_t E_\sim = \int d\mathcal{V}[T_{e0}\tilde\phi\mathcal{K}(n_e) + j_\|\nabla_\|\tilde\phi - n_0 m_i(\tilde v_E^x\tilde v_E^y)\partial_x\langle v_E^y\rangle], \tag{8}$$

$$\partial_t E_z = \int d\mathcal{V}[T_{e0}\langle\phi\rangle\mathcal{K}(n_e) + n_0 m_i(\tilde v_E^x\tilde v_E^y)\partial_x\langle v_E^y\rangle], \tag{9}$$

for density free energy $E_n \doteq \frac{T_{e0}}{2n_0}\int d\mathcal{V}\,n_e^2$, nonzonal $\boldsymbol{E}\times\boldsymbol{B}$ energy $E_\sim \doteq \frac{1}{2}n_0 m_i\int d\mathcal{V}[(v_E^x)^2 + (\tilde v_E^y)^2]$, and zonal $\boldsymbol{E}\times\boldsymbol{B}$ energy $E_z \doteq \frac{1}{2}n_0 m_i\int d\mathcal{V}\langle v_E^y\rangle^2$. [For emphasis, E_n, $E_\sim$, and E_z refer to portions of the free energy, *not* to electric field components.] The only free energy source is the $n_e v_E^x/L_n$ term in Eq. (7), due to density transport down ∇n_0. The curvature term $\phi\mathcal{K}(n_e)$ conservatively transfers energy between E_n and both $E_\sim$ and E_z, with the $\tilde\phi\mathcal{K}(n_e)$ portion often referred to as "curvature drive" and the $\langle\phi\rangle\mathcal{K}(n_e)$ portion important for geodesic acoustic modes. (The curvature-mediated energy transfer will not play a central role in the following analysis.) The Reynolds work term $n_0 m_i(\tilde v_E^x\tilde v_E^y)\partial_x\langle v_E^y\rangle$ conservatively transfers energy between $E_\sim$ and E_z, capturing the energy transfer that plays a key role in many models of the L-H transition. The parallel current plays a dual role: In the evolution of the total energy $(E_n + E_\sim + E_z)$, the summed parallel current terms contribute positive-definite dissipation, $-(T_{e0}/e)\int d\mathcal{V}j_\|\nabla_\| h_e = -\eta\int d\mathcal{V}j_\|^2 = -(T_{e0}^2/e^2\eta)\int d\mathcal{V}(\nabla_\| h_e)^2 < 0$. If $\tilde h_e$ becomes small, so $\tilde n_e/n_0 \approx e\tilde\phi/T_{e0}$, the individual parallel current terms $-(T_{e0}/n_0 e)j_\|\nabla_\| n_e$ and $j_\|\nabla_\|\tilde\phi = j_\|\nabla_\|\phi$ become nearly equal and opposite, representing free energy transfer between E_n and $E_\sim$. If we suppose now that η is small enough that $\tilde n_e/n_0 \approx e\tilde\phi/T_{e0}$, we may immediately estimate the ratio of free energy in nonzonal $\boldsymbol{E}\times\boldsymbol{B}$ velocity as compared with that in the nonzonal fluctuating density, $E_{\tilde n} \doteq \frac{T_{e0}}{2n_0}\int d\mathcal{V}\,\tilde n_e^2$

$$\frac{E_\sim}{E_{\tilde{n}}} = \frac{\int d\mathcal{V}\left[(v_E^x)^2 + (\tilde{v}_E^y)^2\right]/c_s^2}{\int d\mathcal{V}\,\tilde{n}_e^2/n_0^2} = \frac{m_i}{T_{e0}B^2}\frac{\int d\mathcal{V}|\nabla_\perp\tilde{\phi}|^2}{\int d\mathcal{V}\,\tilde{n}_e^2/n_0^2} \sim k_\perp^2\rho_s^2, \tag{10}$$

where $k_\perp$ is a representative wavenumber averaged over the turbulent fluctuation spectrum. In the typical edge turbulence case of fluctuations at scales much larger than the sound radius ($k_\perp\rho_s \ll 1$), we see that the energy in the nonzonal velocities is much smaller than the free energy in the electron density fluctuations.

What does this imply for quantitative evaluation of the predator-prey model[5] of the L-H transition? In the predator-prey model, direct energy transfer to the mean flows depletes the energy content of the turbulence, resulting in suppression of the turbulence and the transition to H-mode. In Eqs. (7)–(9), this energy transfer appears explicitly as the Reynolds work term, $n_0 m_i\langle\tilde{v}_E^x\tilde{v}_E^y\rangle\partial_x\langle v_E^y\rangle$. Even if the correlations are optimal for energy transfer to mean flows, this energy transfer term cannot deplete $E_\sim$ faster than $[\int d\mathcal{V}\,n_0 m_i\langle\tilde{v}_E^x\tilde{v}_E^y\rangle\partial_x\langle v_E^y\rangle]/E_\sim \le \max|\partial_x\langle v_E^y\rangle|$.[14] However, our Eq. (6) and subsequent discussion shows that the parallel current acts at a rapid rate $\sim k_\parallel^2 v_{te}^2/(\nu_{ei}k_\perp^2\rho_s^2)$ to enforce $\tilde{n}_e/n_0 \approx e\tilde{\phi}/T_{e0}$, thus also $E_\sim/E_{\tilde{n}} \sim k_\perp^2\rho_s^2$. So, assuming the parallel electron rate is fast relative to $\max|\partial_x\langle v_E^y\rangle|$, parallel electron physics will effectively cause the sum $(E_\sim + E_{\tilde{n}})$ to move as a unit, with $E_\sim$ and $E_{\tilde{n}}$ staying in a roughly fixed ratio to one another. So, in order for the Reynolds work term to deplete the free energy from turbulent fluctuations, it must transfer an amount of energy comparable with $(E_\sim + E_{\tilde{n}})$, rather than with $E_\sim$ alone. Since $E_{\tilde{n}}/E_\sim \sim 1/(k_\perp\rho_s)^2$ is often much larger than one in edge turbulence, this makes it significantly more difficult for the Reynolds work term to directly deplete the turbulent fluctuations' free energy.

Alternatively, consider now a transition that is slower than a typical instability growth rate γ, so that the $\partial_t E_n$ and $\partial_t E_\sim$ terms in Eqs. (7) and (8) become small relative to at least some of the energy source and transfer terms on the RHS. In this slow-transition limit, we must balance the free energy source $\gamma E_{\tilde{n}} \doteq T_{e0}\int d\mathcal{V}\,n_e v_E^x/L_n$ against energy transfer via the Reynolds stress $\int d\mathcal{V}\,n_0 m_i\langle\tilde{v}_E^x\tilde{v}_E^y\rangle\partial_x\langle v_E^y\rangle \le E_\sim\max|\partial_x\langle v_E^y\rangle|$, leading to a criterion for turbulence suppression $\max|\partial_x\langle v_E^y\rangle| \ge \gamma(E_{\tilde{n}}/E_\sim) \sim \gamma/(k_\perp\rho_s)^2$, resembling a "Waltz rule"[15] that has been modified by the factor $1/(k_\perp\rho_s)^2$. Since $1/(k_\perp\rho_s)^2$ is typically large in edge turbulence, this formula requires a much larger flow shear than the usual criterion ($|\partial_x\langle v_E^y\rangle| \gtrsim c_1\gamma$, for constant c_1 of order unity), suggesting that other mechanisms must typically dominate in order to get turbulence suppression at the more easily accessible shearing rate $\max|\partial_x\langle v_E^y\rangle| \sim \gamma$.

It is important to recall that depletion of the turbulence via energy transfer to E_z is only one of several possible mechanisms through which $\boldsymbol{E}\times\boldsymbol{B}$ flow shear could suppress turbulence. For example, sheared $\boldsymbol{E}\times\boldsymbol{B}$ flows could distort the turbulent fluctuations to reduce their effective growth rate γ, or could increase their mean perpendicular wave number $k_\perp$ and thereby enhance cross-field dissipation.[4] In that latter case, energy would also be redistributed between $E_{\tilde{n}}$ and $E_\sim$ as the mean $k_\perp$ changed. When η is small enough, this transfer is approximately conservative, so it has little effect on the suppression criterion for the rapid L-H transition case, since the Reynolds stress must still transfer the L-mode level of $(E_{\tilde{n}} + E_\sim)$ to E_z in order to suppress the turbulence. However, it could become significant for the slow-transition case if $k_\perp\rho_s$ became a significant fraction of unity during the transition. Indirect effects, such as those due to modification of the effective γ or $k_\perp$ by eddy shearing, are not directly addressed by the energy-balance arguments presented in this article, and could act to suppress turbulence even if the direct energy depletion by Reynolds work is negligible.

Note also that the nonadiabatic response h_e can become order unity for some edge parameter values. However, the parallel current still acts strongly enough that $\tilde{n}_e/n_0$ and $e\tilde{\phi}/T_{e0}$ must remain comparable in magnitude at frequencies lower than the parallel electron rate.[16] This is enough to cause the general ordering given in Eq. (10) and to support the subsequent analysis.

In summary, parallel electron conduction causes a relaxation of turbulent fluctuations towards adiabatic electron response ($\tilde{n}_e/n_0 \approx e\tilde{\phi}/T_{e0}$) on rapid parallel electron transit timescales. At longer timescales, the energy in nonzonal $\boldsymbol{E}\times\boldsymbol{B}$ flows $E_\sim$ is held in an approximately fixed relationship to the free energy in nonzonal density fluctuations $E_{\tilde{n}}$, with $E_\sim/E_{\tilde{n}} \sim (k_\perp\rho_s)^2$. For the typical case that the electron transit time is fast relative to the background $\boldsymbol{E}\times\boldsymbol{B}$ shearing rate, electron density fluctuations rapidly restore nonzonal $\boldsymbol{E}\times\boldsymbol{B}$ energy lost by Reynolds-stress transfer, implying that the Reynolds work term must deplete the energy not only from nonzonal $\boldsymbol{E}\times\boldsymbol{B}$ flows but also from $E_{\tilde{n}}$ in order to suppress the turbulence. Because $E_{\tilde{n}} \gg E_\sim$ for the typical edge turbulence case of fluctuations at scales rather larger than ρ_s, $(k_\perp\rho_s)^2 \ll 1$, this makes it much more difficult to suppress the turbulence via direct energy transfer to sheared flows.

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[11]For example, assuming no approximate cancellation so $\tilde{h}_e \sim e\tilde{\phi}/T_{e0}$, we may compare $\boldsymbol{v}_E \cdot \nabla n_0$ with $e^{-1}\nabla_\| j_\|$ in Eq. (1), using Eq. (3) and substituting for resistivity $\eta = m_e \nu_{ei}/n_0 e^2$

$$\frac{\boldsymbol{v}_E \cdot \nabla n_0}{e^{-1}\nabla_\| j_\|} \sim \frac{(k_\perp \tilde{\phi}/B)(n_0/L_n)}{\frac{1}{\eta e} k_\|^2 \tilde{\phi}} \sim \frac{\eta k_\perp e n_0}{B L_n k_\|^2} \sim \frac{m_e \nu_{ei} k_\perp T_{e0}}{B e L_n k_\|^2 T_{e0}} \sim \frac{k_\perp \rho_s c_s / L_n}{k_\|^2 v_{te}^2/\nu_{ei}}.$$

Since $k_\| \sim 1/qR$ is roughly fixed by the geometry, the middle form $\eta k_\perp e n_0 / B L_n k_\|^2$ shows that the ratio becomes small for small η. The rightmost form clarifies that this occurs when η is small enough that the parallel electron diffusion rate $\sim k_\|^2 v_{te}^2/\nu_{ei}$ is fast relative to the linear drift wave frequency $k_\perp \rho_s c_s / L_n$. For small $k_\perp \rho_s$, the criterion may be relaxed to $k_\perp \rho_s c_s / L_n \ll k_\|^2 v_{te}^2 / \nu_{ei} k_\perp^2 \rho_s^2$, as shown in Ref. 16.

[12]Note that ρ_s does not represent FLR effects. In Eqs. (1)–(3), the electron gyroradius is taken negligibly small due to smallness of m_e, and ion gyroradius is taken negligibly small since ions are cold. The hybrid quantity ρ_s appears due to the competition [in Eq. (2)] between electron parallel force balance (bringing in T_{e0}) and ion polarization across the field (bringing in m_i).

[13]To derive Eqs. (7)–(9), use the following, in which f and g each indicate an arbitrary scalar function. First some straightforward relations: $f\mathcal{K}(f) = \frac{1}{2}\mathcal{K}(f^2)$, $\mathcal{K}(fg) = f\mathcal{K}(g) + g\mathcal{K}(f)$, $f\boldsymbol{v}_E \cdot \nabla f = \boldsymbol{v}_E \cdot \nabla f^2/2$, and $\boldsymbol{v}_E \cdot \nabla \phi = 0$. We also need some consequences of the simplified fluxtube geometry, discussed for example in Ref. 10, along with neglect of boundary fluxes. Specifically, we use $\int d\mathcal{V}\,\mathcal{K}(f) = 0$ and the fact that the advecting $\boldsymbol{v}_E$ (in $\boldsymbol{v}_E \cdot \nabla$) is divergence-free in the simplified fluxtube geometry [although the contribution of $n_0 \nabla \cdot \boldsymbol{v}_E \approx -\mathcal{K}(n_0\phi)$ is explicitly retained in Eq. (1)], so, e.g., $\int d\mathcal{V}\,\boldsymbol{v}_E \cdot \nabla n_e^2 = \int d\mathcal{V}\,\nabla \cdot (\boldsymbol{v}_E n_e^2)$, which is zero upon neglect of boundary fluxes. Due to the large-aspect-ratio approximation that B is spatially constant, we also have $\int d\mathcal{V}\,\nabla_\| f = 0$ and $\int d\mathcal{V} f \nabla_\| g = -\int d\mathcal{V} g \nabla_\| f$. Since $\boldsymbol{B}$ is tangential to a flux surface, $\nabla_\| \langle f \rangle = 0$. By definition of the nonzonal portion, we may infer that $\int d\mathcal{V} \langle f \rangle \tilde{g} = 0$. As an example of the necessary manipulations, we derive the Reynolds work term for Eq. (8)

$$\begin{aligned}
-\int d\mathcal{V}\,\tilde{\phi}(\boldsymbol{v}_E \cdot \nabla)\nabla_\perp^2 \phi &= -\int d\mathcal{V}\,\tilde{\phi}\nabla \cdot \left(\boldsymbol{v}_E \nabla_\perp^2 \phi\right) \\
&= -\int d\mathcal{V}\,\nabla \cdot \left(\tilde{\phi}\boldsymbol{v}_E \nabla_\perp^2 \phi\right) + \int d\mathcal{V}\left(\nabla_\perp^2 \phi\right)\boldsymbol{v}_E \cdot \nabla \tilde{\phi} \\
&= \int d\mathcal{V}\left(\nabla_\perp^2 \phi\right)\boldsymbol{v}_E \cdot \nabla(\phi - \langle \phi \rangle) \\
&= -\int d\mathcal{V}\left(\nabla_\perp^2 \phi\right)\boldsymbol{v}_E \cdot \nabla \langle \phi \rangle \\
&= -\int d\mathcal{V}\,\nabla \cdot [(\nabla_\perp \phi)\boldsymbol{v}_E \cdot \nabla \langle \phi \rangle] \\
&\quad + \int d\mathcal{V}\,(\nabla_\perp \phi) \cdot \nabla_\perp [\boldsymbol{v}_E \cdot \nabla \langle \phi \rangle] \\
&= \int d\mathcal{V}\,(\nabla_\perp \phi) \cdot \nabla_\perp \left[-B^{-1}\left(\partial_y \phi\right)\partial_x \langle \phi \rangle\right] \\
&= -B^{-1}\int d\mathcal{V}\,(\partial_x \langle \phi \rangle)(\nabla_\perp \phi) \cdot \partial_y \nabla_\perp \phi \\
&\quad - B^{-1}\int d\mathcal{V}\left(\partial_y \phi\right)(\nabla_\perp \phi) \cdot \nabla_\perp \partial_x \langle \phi \rangle \\
&= -B^{-1}\int d\mathcal{V}\,(\partial_x \langle \phi \rangle)\partial_y \frac{1}{2}|\nabla_\perp \phi|^2 \\
&\quad - B^{-1}\int d\mathcal{V}\left(\partial_y \tilde{\phi}\right)(\partial_x \phi)\partial_x^2 \langle \phi \rangle \\
&= -B^{-1}\int d\mathcal{V}\left(\partial_y \tilde{\phi}\right)\left(\partial_x \tilde{\phi}\right)\partial_x^2 \langle \phi \rangle \\
&= B^2 \int d\mathcal{V}\left(\tilde{v}_E^x \tilde{v}_E^y\right)\partial_x \langle v_E^y \rangle.
\end{aligned}$$

[14]This bound follows from the fact that $0 \le (|\tilde{v}_E^x| - |\tilde{v}_E^y|)^2$.

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