Ellipticity Criterion for Flowing Two-Fluid High Beta Equilibria

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Outline

-Normal mode and axisymmetric flowing equilibrium of neutral gas

-Axisymmetric flowing equilibrium of one-fluid MHD

-Axisymmetric flowing equilibrium of two-fluid plasma

-Normal mode property in two-fluid plasma

Motivation

The transition from elliptic to hyperbolic types of equation is related to shock formation. As the shock may degrade plasma confinement, it is useful to know this transition criterion.

1. Introduction

Let us consider at first the normal mode and the ellipticity criterion for axisymmetric flowing equilibrium of neutral gas:

Governing equations:

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{u}) = 0 \qquad \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \bullet \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p \qquad p\rho^{-\gamma} = const.$$

Normal mode:

$$\left(\left(\frac{\omega}{kC_S} \right)^2 - 1 \right) (\mathbf{k} \bullet \tilde{\mathbf{u}}) = 0$$

compressible

C_S **sound velocity**

Axisymmetric flowing equilibrium of neutral gas:

$$S_{xx}\frac{\partial^2\psi}{\partial x^2}+S_{xz}\frac{\partial^2\psi}{\partial x\partial Z}+S_{zz}\frac{\partial^2\psi}{\partial Z^2}=G\left(x,\psi,\frac{\partial\psi}{\partial x},\frac{\partial\psi}{\partial Z}\right)$$

ψ **: stream function for the mass flow** $\rho\mathbf{u}$

$$x=R^2/2$$

The type of this equation is determined by

$$S_{xx}S_{zz}-\left(S_{xz}/2\right)^2=\frac{2x}{1-\left(\mathbf{u}_p/C_S\right)^2}$$

results from compressibility

If the poloidal flow speed is less (larger) than the *sound velocity*, it is elliptic (hyperbolic).

It is understood that this transition is related to the shock formation due to the nonlinear excitation of the *sound wave* shown above.

Note that the normal mode is valid for $k\approx 1/d$ **where** d **is the shock width.**

2. Axisymmetric flowing equilibrium of one-fluid MHD:

Generalized G-S equation:

$$S_{xx}\frac{\partial^2\psi_m}{\partial x^2}+S_{xz}\frac{\partial^2\psi_m}{\partial x\partial Z}+S_{zz}\frac{\partial^2\psi_m}{\partial Z^2}=G\left(x,\psi_m,\frac{\partial\psi_m}{\partial x},\frac{\partial\psi_m}{\partial Z}\right)$$

ψ_m**: magnetic flux function**

H.P.Zehrfeld and B.J.Green (1972)
E.Hameiri (1983)
R.Betti and J.P.Fridberg (2000)

Two elliptic regions exist:

$$u_p{}^2\big/C_s{}^2<B_p{}^2\left(B_\phi{}^2+B_p{}^2+\mu_0\gamma p\right)^{-1}\equiv F_1$$

$$G_-<u_p{}^2\big/C_s{}^2<G_+ \qquad G_\mp\equiv\frac{1}{2}\left[(V_A/C_s)^2+1\mp\sqrt{\left((V_A/C_s)^2+1\right)^2-4(V_{Ap}/C_s)^2}\right]$$

$G_\mp$ **is related to the slow and fast magnetosonic waves which are compressible.**

3. Axisymmetric flowing equilibrium of two-fluid plasma:

Governing equations: See the November issue of PoP

$$2x\frac{\partial^2\Psi_e}{\partial x^2}+\frac{\partial^2\Psi_e}{\partial Z^2}=G_e\left(x,\Psi_e,\Psi_i,\frac{\partial\Psi_i}{\partial x},\frac{\partial\Psi_i}{\partial Z}\right)$$

always elliptic

$$S_{xx}\frac{\partial^2\Psi_i}{\partial x^2}+S_{xz}\frac{\partial^2\Psi_i}{\partial x\partial Z}+S_{zz}\frac{\partial^2\Psi_i}{\partial Z^2}=G_i\left(x,\Psi_e,\frac{\partial\Psi_e}{\partial x},\frac{\partial\Psi_e}{\partial Z},\Psi_i,\frac{\partial\Psi_i}{\partial x},\frac{\partial\Psi_i}{\partial Z}\right)$$

where $\Psi_e=\psi_m$ **and** $\Psi_i=\psi_m+\frac{m_i}{q_i}Ru_{i\phi}$

Ion's toroidal flow

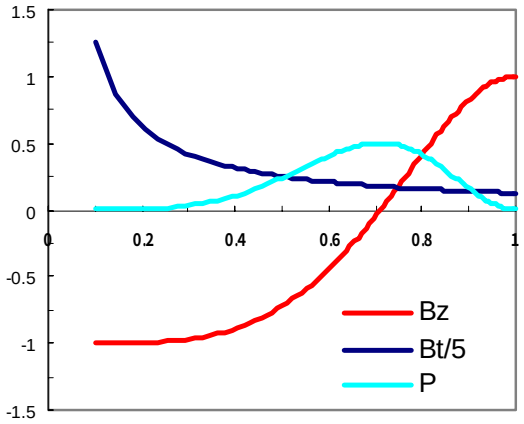
Since $S_{xx}S_{zz}-(S_{xz}/2)^2=\frac{2x}{1-(\mathbf{u}_{ip}/C_S)^2}$

results from compressibility

Elliptic criterion: the ion poloidal flow speed is less than the sound velocity.

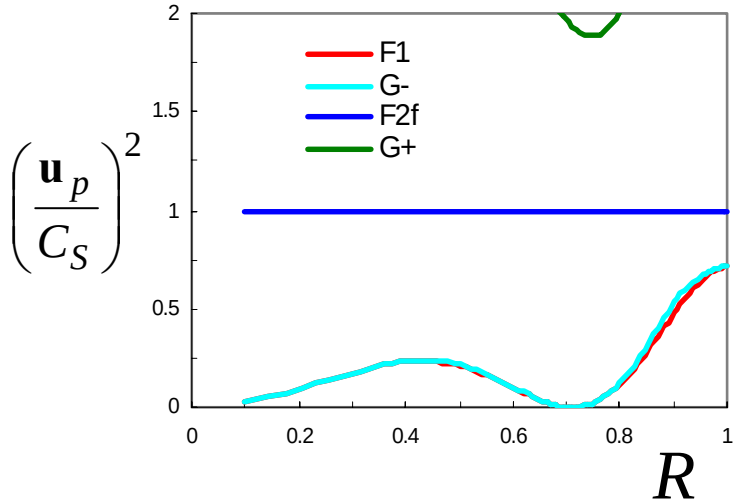
4. Comparison of Type of Equilibrium Equations

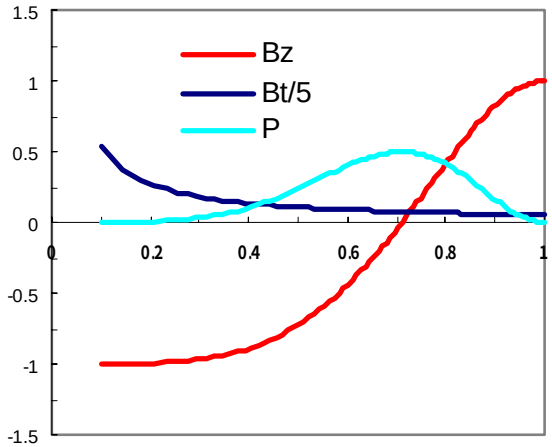
ST equilibria



$$\beta_T = 0.38$$

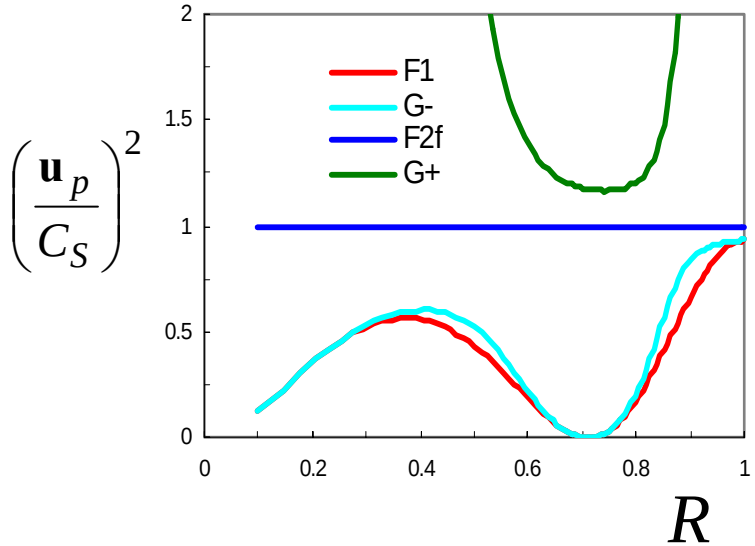
$$\beta = 0.20$$





$$\beta_T = 2.0$$

$$\beta = 0.56$$



5. Normal Mode Property in Two-Fluid Plasma

This part is done in collaboration with C.Z.Cheng and Y-K.Martin Peng

Purpose:

To understand the above mentioned discrepancy between the one- and two-fluid criteria, we have examined the normal mode property in a two-fluid plasma.

Model used here: **massless electron**

uniform and static equilibrium

linear mode analysis

Normal mode equation:

$$\left(\omega^2 - k_{\|}{}^2 V_A{}^2\right)\mathbf{k}\bullet(\xi_i \times \mathbf{b}) - i\omega V_A \ell_i k_{\|}{}^2\left(1 - \frac{k^2 C_S{}^2}{\omega^2}\right)(\mathbf{k}\bullet\xi_i) = 0 \qquad \textbf{(5.1)}$$

$$i\omega V_A \ell_i k^2\,\mathbf{k}\bullet(\xi_i \times \mathbf{b}) + \left(\omega^2 - k^2\left(C_S{}^2 + V_A{}^2\right) + k^2 V_A{}^2\left(\frac{k_{\|}C_S}{\omega}\right)^2\right)(\mathbf{k}\bullet\xi_i) = 0 \qquad \textbf{(5.2)}$$

where $\ell_i = c/\omega_{pi}$ **is the ion skin depth which represents the two-fluid effect.**

In MHD limit: (5.1) gives the *incompressible* Alfven wave.

(5.2) gives the *compressible* magnetosonic waves.

Dispersion relation:

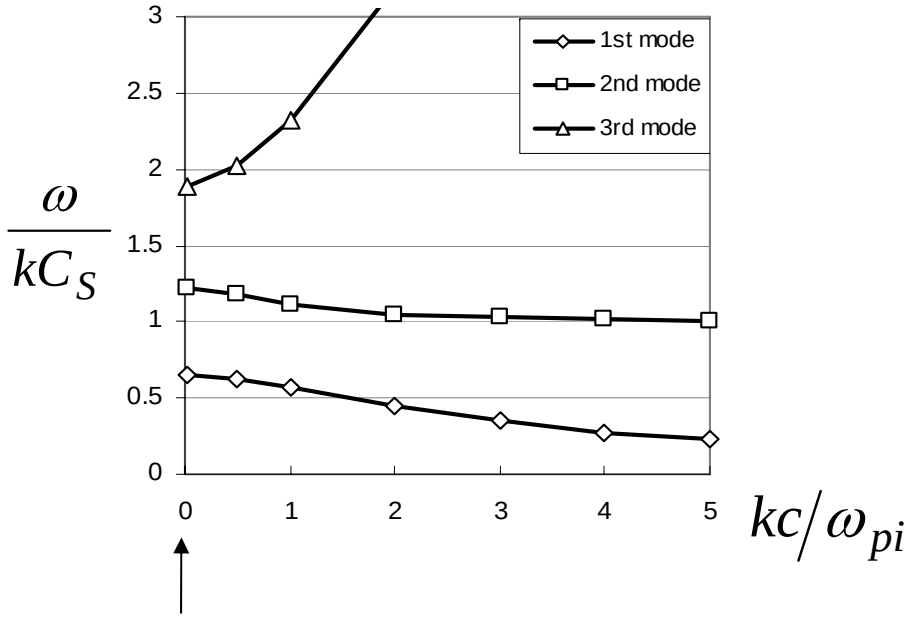
$$\left(\left(\frac{\omega}{\omega_s}\right)^2 - 1\right)\left(\left(\frac{\omega}{k_{\|}V_A}\right)^2 - 1\right)\left(\left(\frac{\omega}{\omega_f}\right)^2 - 1\right) - (k\ell_i)^2\left(\frac{\omega}{k_{\|}V_A}\right)^2\left(\left(\frac{\omega}{kC_S}\right)^2 - 1\right) = 0$$

$$\beta = 0.4$$

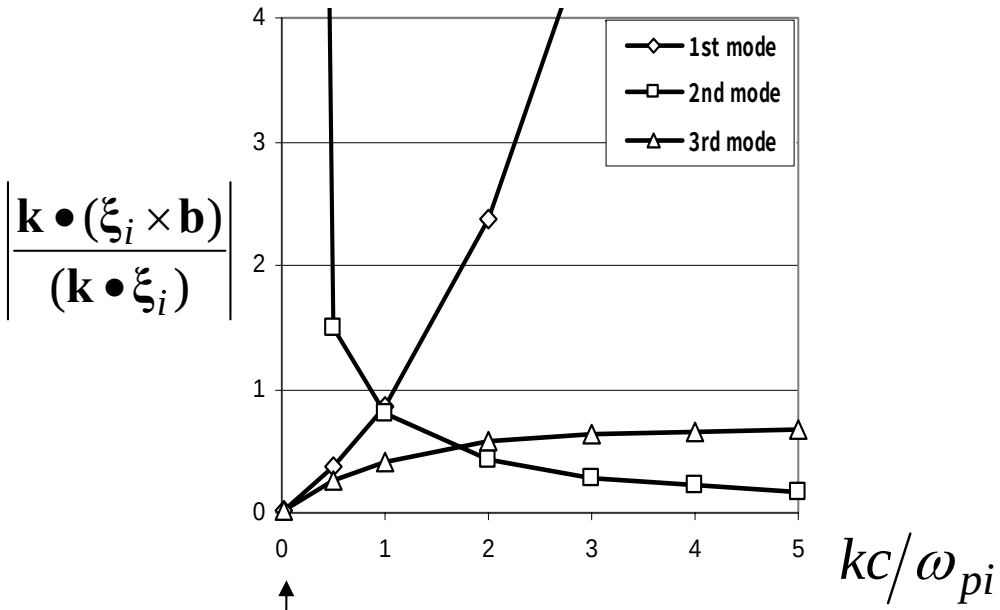
$$k_{||}/k = \cos(\pi/4)$$

$$\beta = 2\mu_0(p_{i0} + p_{e0})/B^2$$

$$(V_A/C_S)^2 = \frac{6}{5\beta}$$



Incompressibility index



MHD limit

Mode frequency and compressibility depend strongly on the two-fluid parameter, $k\ell_i$

As $k\ell_i$ increases,

ω/kC_S of the 2nd mode (the Alfven mode in MHD limit) approaches to unity and the mode becomes *compressible*.

ω/kC_S of the 1st mode (the slow magnetosonic in MHD limit) decreases and the mode becomes *incompressible*.

ω/kC_S of the 3rd mode (the fast magnetosonic in MHD limit) increases and approaches to the whistler frequency. The mode becomes *mixed.*

The above results suggest that if the shock width is of the order of ℓ_i or smaller than that, only the 2nd mode can be a candidate for shock formation. This is consistent with our two-fluid prediction.

6. Conclusion

- **Elliptic criterion for flowing two-fluid axisymmetric equilibrium is studied.**

- **As the axisymmetry is assumed, the criterion for the toroidal flow speed does not appear in the present theory.**

  The present theory, however, implies that:

  If small irregularity of the density exist in the toroidal direction, ellipticity criterion for the toroidal flow speed may be less than the sound velocity.

  Measurement of the ratio of the toroidal flow to the sound velocity as well as the ratio of the poloidal flow to the sound velocity in NSTX are therefore encouraged.