Kinetic Non-Maxwellians, from Theory to Experiments

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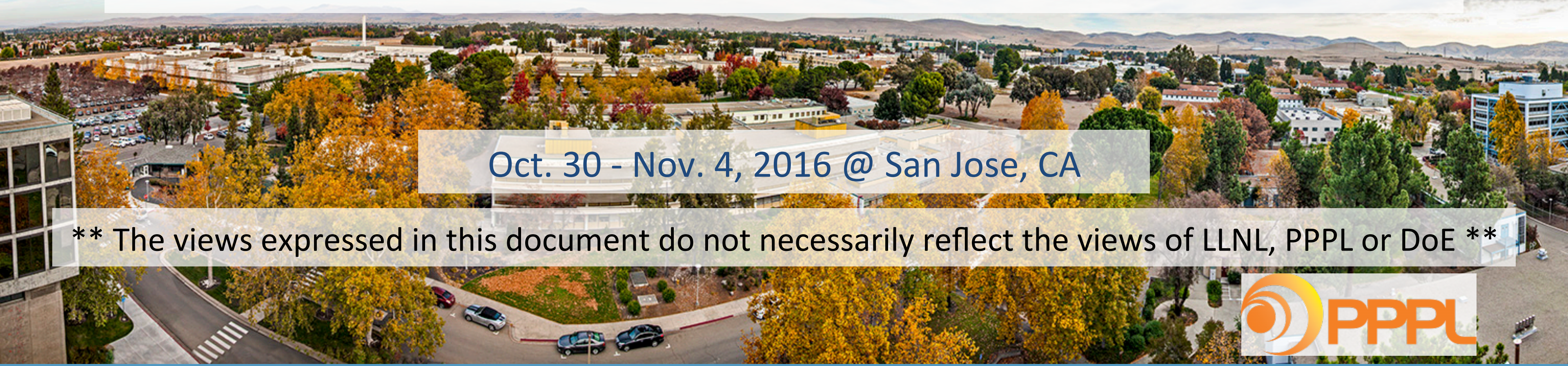

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Common source of transport & turbulence?

Common observation:

1- It is very common to use spatial profiles of transport coefficients in fluid models

2- else, very CPU-consuming (gyro)kinetic code are used
(e.g., turbulence codes to evaluate spatial profiles of transport coefficients)

[**Groth**, *APS-DPP* (2014) CO5.7] (* **Canik**, *PSI* (2014) P1-088)

- In detached conditions, UEDGE (and SOLPS*) consistently under-predicts radiation
- Also observed for other edge fluid codes, i.e., SOLPS* and EDGE2D-EIRENE

However: **Radiation shortfall** associated with detached inner divertor leg, **despite including cross-field drifts**

[**Groth**, et al., *PPCF* **53** (2011) 124017]

- ***"These fundamental issues remain outstanding and require inclusion of other physics process, such as supra-thermal electrons".***

[**Groth**, et al., *NF* **53** (2013) 093016 & private communication July 7 2015]

- Magic ***"morphed profiles of transport coefficients can enhance the radiation"***

GOAL: How to check if the radiation shortfall due to kinetic effects?

- NEED TO DESCRIBE NON-MAXWELLIANS
- CAN WE INCLUDE KINETIC EFFECTS IN FLUID EQUATIONS?
- WHAT IS THE ORIGIN OF DISSIPATIVE COEFFS.?

OUTLINE

Part. 1: Non-Maxwellian Distrib. Funct. (NMDFs)

1. Motivation/measurement of non-Maxwellians
2. Analytic/physical interpreted NMDF (called INMDF)

Part. 2: Kinetic corrections from analytic NMDFs

1. Effects of non-Maxwellians on SEE & Langmuir Probes

Part. 3: Generalized fluid models w/ kinetic effects

1. Generalized fluid models including kinetic effects!
2. Nonlinear Fokker-Planck collision operator

Part. 4: Experimental measurements of INMDFs

1. Diagnostic discrepancies
2. Modified interpretations & new diagnostics

References:
[**O. Izacard**, *Phys. Plasmas* **23** (2016) 082504]
[**O. Izacard**, *J. Plasmas Phys.* (2016) Submitted]

1. Goal: Introduce a New Technique to Develop Next Generation of Fluid Equations Including Some Kinetic Effects

1 - Start from a **fitting** analytic **Non-Maxwellian** Distrib. Funct. (NMDF) using as few hidden parameters as possible (finite number of terms, …)

Kappa f_κ , 2 MDF $f_0 + f_0'$, a new interpreted NM f_I (focus here), or create as many new analytic functions as wanted

(e.g.: runaway, neoclassical, ion orbit loss, tail, …)

2 - **Analytic** computations of velocity phase-space integrals

3 - (1st ?) **Analytic** steady-state solution of the Boltzmann-F-P equation in presence of **sources** (i.e.: it is a statistical description of a non-isolated plasma, it does not break 2nd law of thermodynamics)

4 - **Collisionless** fluid closure (does not appear for a MDF) and **collisional** fluid closure from nonlinear Fokker-Planck collision operator

5 - Foundations of the **next generation of fluid codes** (fluid reduction of velocity-space mesh-free analytic DFs)

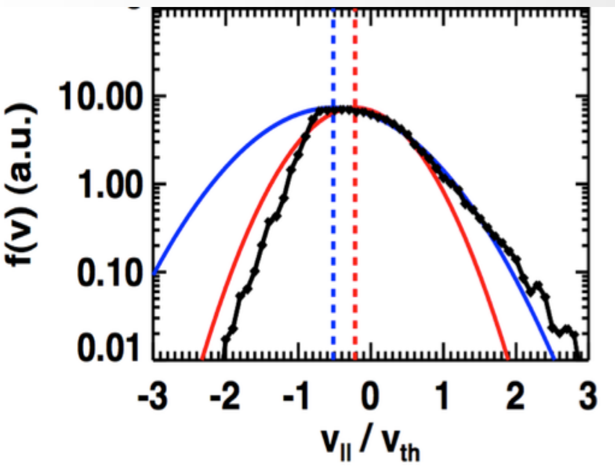
1. Motivation for non-Maxwellian studies

Examples of non-Maxwellians:

Can we reproduce non-Maxwellians with an asymmetry or tail?

Experiment:
Langmuir probes interpretation

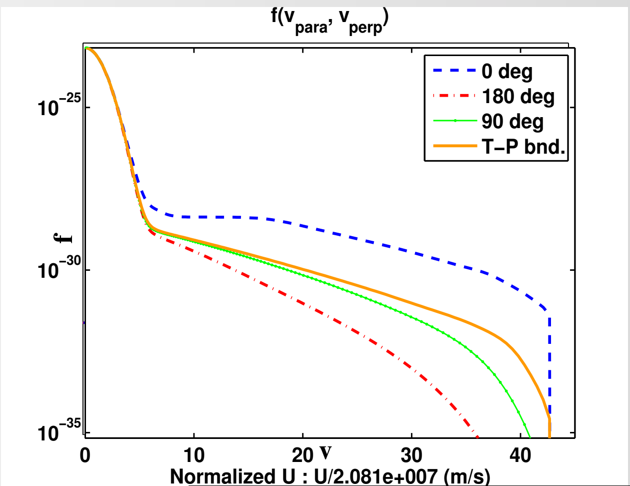
asymmetry



Battaglia APS (2013)

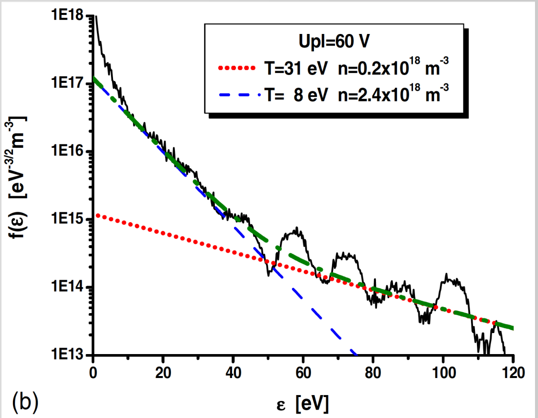
XGC code in presence of ion orbit loss

tail



Meneghini PhD Thesis (2012)

Fokker-Planck code in presence of Lower Hybrid current drive

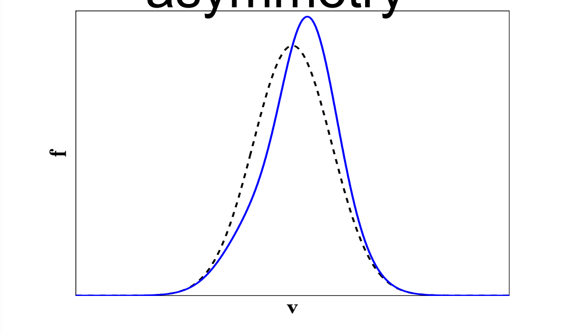


Popov PPCF 51 (2009)

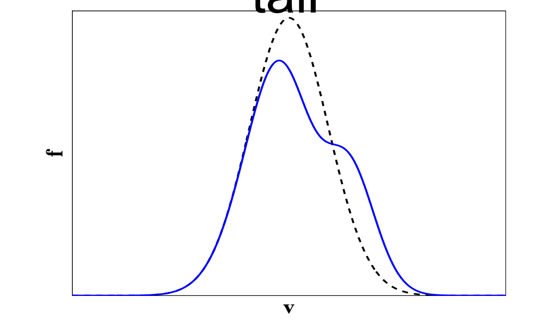
1. “Physical” description of Non-Maxwellians

Simple “physical” understanding of non-Maxwellians:

asymmetry



tail



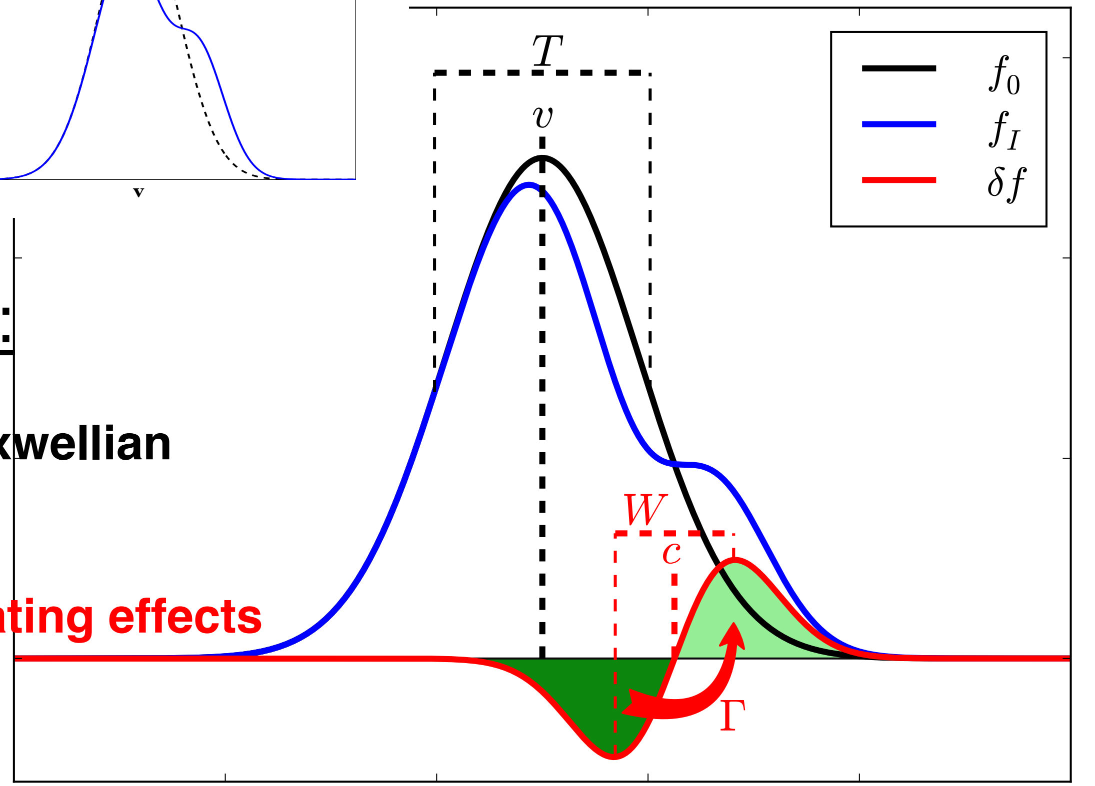
Scheme of heat process:

density n
velocity v
temperature T

Maxwellian

kinetic flux Γ
central flow c
width heat spread W

Heating effects



[**O. Izacard**, *Phys. Plasmas* **23** (2016) 082504]

1. Analytic physical description of Non-Maxwellians

$$f_I(\mathrm{x},\mathrm{v},t) = \frac{\boldsymbol{n}}{(2\pi\,\boldsymbol{T})^{1/2}}\exp\left(-\frac{(\mathrm{v}-\boldsymbol{v})^2}{2\,\boldsymbol{T}}\right) + \frac{\boldsymbol{\Gamma}}{(2\,\boldsymbol{W}^3)^{1/2}}(\mathrm{v}-\boldsymbol{c})\exp\left(-\frac{(\mathrm{v}-\boldsymbol{c})^2}{2\,\boldsymbol{W}}\right)$$

Called the "interpreted NMDF"

[**O. Izacard**, *Phys. Plasmas* **23** (2016) 082504]

Highlights for this analytic Non-Maxwellian:

- (n, v, T, Γ, c, W) are **not** fluid moments
- But they are **HIDDEN VARIABLES** with a physical meaning
- Collisionless **fluid closure** fixed (from hidden variables)
- Some **kinetic effects** are described (tail, asymmetry)
- **Cannot be describe** by a finite number terms using: Hemite, Laguerre, Legendre, Bessel, sum of Maxwellians…
- Generalization to a **completeness** set of basis functions

1. Conceptual differences between INMDF and 2 MDFs

$$f_I(\mathrm{x},\mathrm{v},t) = \frac{\boldsymbol{n}}{(2\pi\,\boldsymbol{T})^{1/2}}\exp\left(-\frac{(\mathrm{v}-\boldsymbol{v})^2}{2\,\boldsymbol{T}}\right) + \frac{\boldsymbol{\Gamma}}{(2\,\boldsymbol{W}^3)^{1/2}}(\mathrm{v}-\boldsymbol{c})\exp\left(-\frac{(\mathrm{v}-\boldsymbol{c})^2}{2\,\boldsymbol{W}}\right)$$

Called the "interpreted NMDF"

[**O. Izacard**, *Phys. Plasmas* **23** (2016) 082504]

- 2 MDFs is the thermodynamics limit at a very specific collisionality: (for each collision *fast-thermalized*, the additional energy is instantaneously redistributed to the full distribution function)

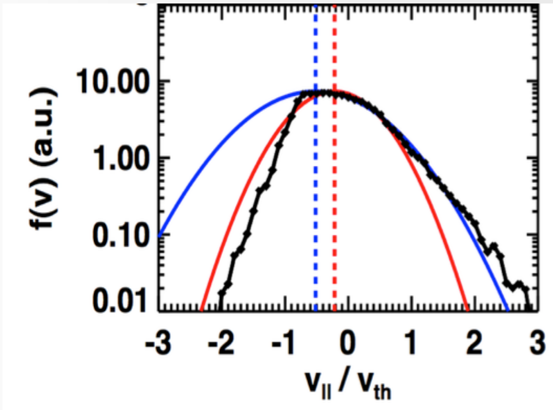
$$\nu_{\mathrm{th-th}} \sim \nu_{\mathrm{f-f}} \gg \nu_{\mathrm{th-f}} \sim \nu_{\mathrm{f-th}}$$

BUT it is inconsistent with numerical & experimental observations of NMDF steady states where thermalization is not "instantaneous"

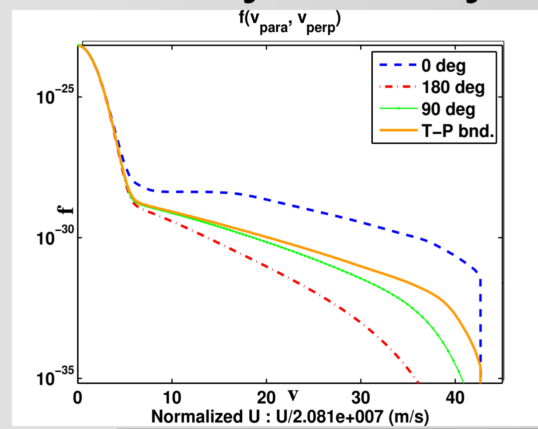
- For other collisionality regimes (particularly in spherical tokamaks) associated with alpha, RFCD, NBI, ion orbit loss, X-points, … there are non-negligible interactions between *th* and *f* populations. Then the *th* and *f* populations cannot be describe by a MDF

1. Qualitative fitting of non-Maxwellians

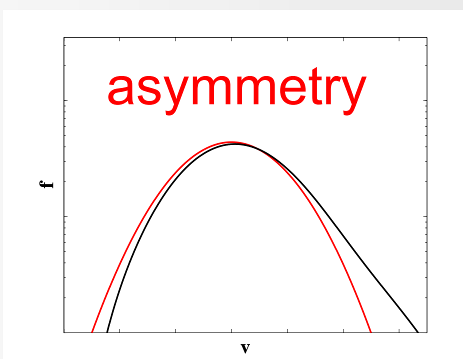
Examples of ***qualitative*** fitting of non-Maxwellians:
Can reproduce non-Maxwellians with an asymmetry or tail

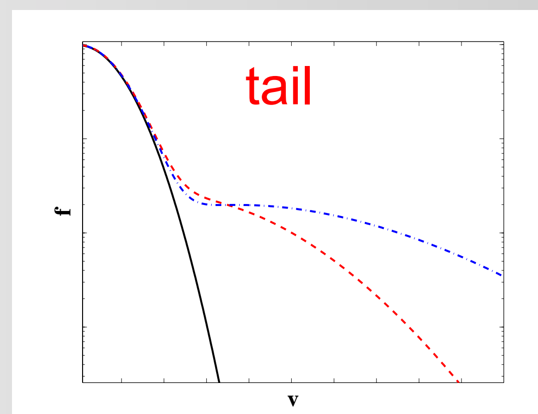


From XGC0 code, NM due to IOL — **Battaglia APS (2013)**



From 3D FP code, NM due to LHCD — **Meneghini PhD Thesis (2012)**





1. Proof of physical reality of INMDFs in tokamaks

Are INMDFs numerically/experimentally observed? **YES, at least in F-P & PIC codes & in JET & TFTR**

1] Numerical observation of NMDFs in:

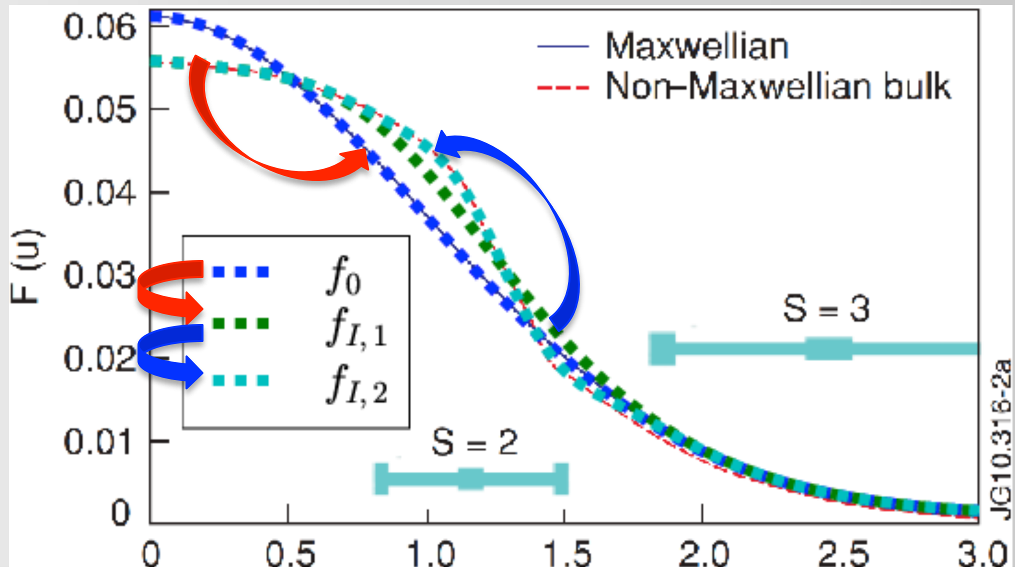
a) PIC code XGC0 due to ion orbit loss (see our fitting of Battaglia APS 2013)

b) 3D Fokker-Planck code due to LHCD (see our fitting of Meneghini Thesis 2012)

2] Experimental discrepancy between ECE and TS interpretations of the electron temperature [**K.V. Beausang** RSI (2011)]:

a) We successfully fit their numerical model NMDF which resolves TS-ECE discrepancy **observed in JET** due to **NBI + ICRF**

b) We detect **heating** and **cooling**



Background figure reprinted from [**K.V. Beausang** RSI (2011)]

[**O. Izacard**, *Phys. Plasmas* **23** (2016) 082504]

Conclusion:

Yes, INMDFs are experimentally/numerically observed

2. Analytic secondary electron emission formula

Secondary electron emission:
when free particles generates the emission of a secondary electron [**Bacharis** PRE (2010)]

$$\boxed{\delta_{see} = \frac{\int_0^\infty \sqrt{\frac{2E}{m_e}} f(E)\delta_s(E)dE}{\int_0^\infty \sqrt{\frac{2E}{m_e}} f(E)dE}}$$

with $\delta_s(E) = \Delta_s \; E \exp\left(-\frac{2}{\sqrt{E_{max}}}\sqrt{E}\right), \quad \Delta_s = (2.72)^2 \frac{\delta_{max}}{E_{max}}$

Empirical formula for a MDF [Bacharis PRE (2010)]:

$$\log_{10}\left[\delta_{see}(T)\right] \approx C_3 x^3 + C_2 x^2 + C_1 x + C_0, \quad x = \log_{10}(T)$$

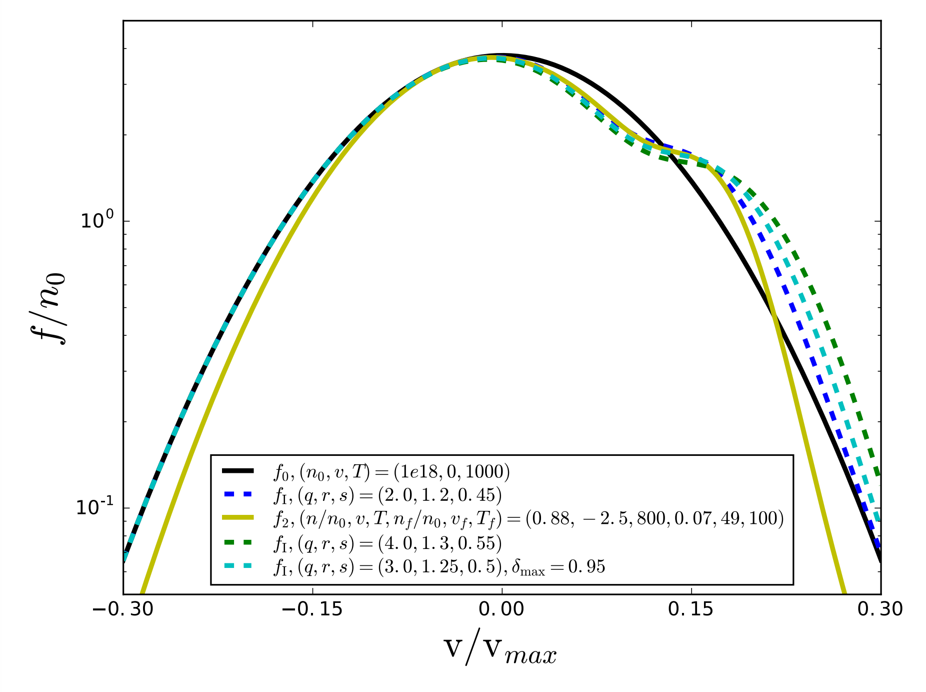
After some analytic computation (w/o limit, truncation, approx.)

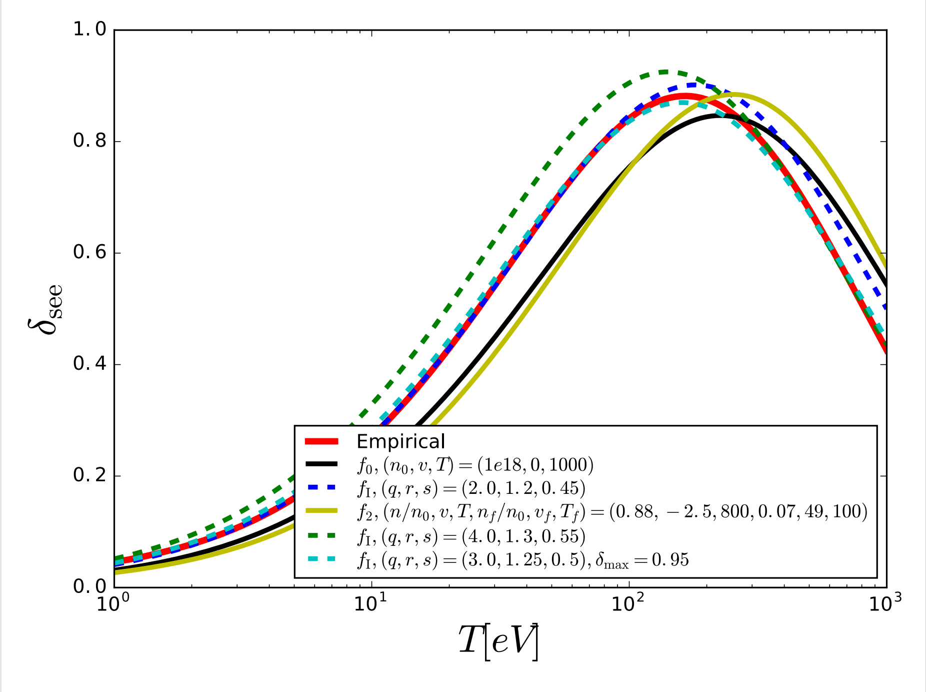
…

Analytic formulas: $\delta_{see,0}$ for a **MDF** and $\delta_{see,I}$ for an **INMDF**

2. Corrections of secondary electron emission

Even 4% of supra-thermal particles can have significant impact on $\delta_{sec}(T)$: supra-thermal particles usually act more **at lower temperature** (i.e., SOL)





[**O. Izacard**, *Phys. Plasmas* **23** (2016) 082504]

RESULTS:

- **Non-Maxwellian correction of** $\delta_{see,I}(n, v, T, \Gamma, c, W)$
- **Detection of tail (or numerical issue) in empirical formula?**

2. Langmuir probes characteristic interpretation

Characteristic curve:

$$\boxed{I_e(U) = -\frac{8\pi e S}{3m^2}\int_{eU}^{\infty}\frac{(E-eU)f(E)}{\gamma(E)\left[1+\frac{E-eU}{E}\psi(E)\right]}dE}$$

in diffusionless regime $\psi(E) \ll 1$ and with $\gamma(E) \approx \frac{4}{3}, \quad \frac{m\mathrm{v}^2}{2} = E$

$$I_e(U) = -\frac{2\pi e S}{m}\int_{u}^{\infty}\left(\frac{m\mathrm{v}^2}{2} - eU\right)\mathrm{v}f(\mathrm{v})d\mathrm{v}, \quad u = \sqrt{\frac{2eU}{m}}$$

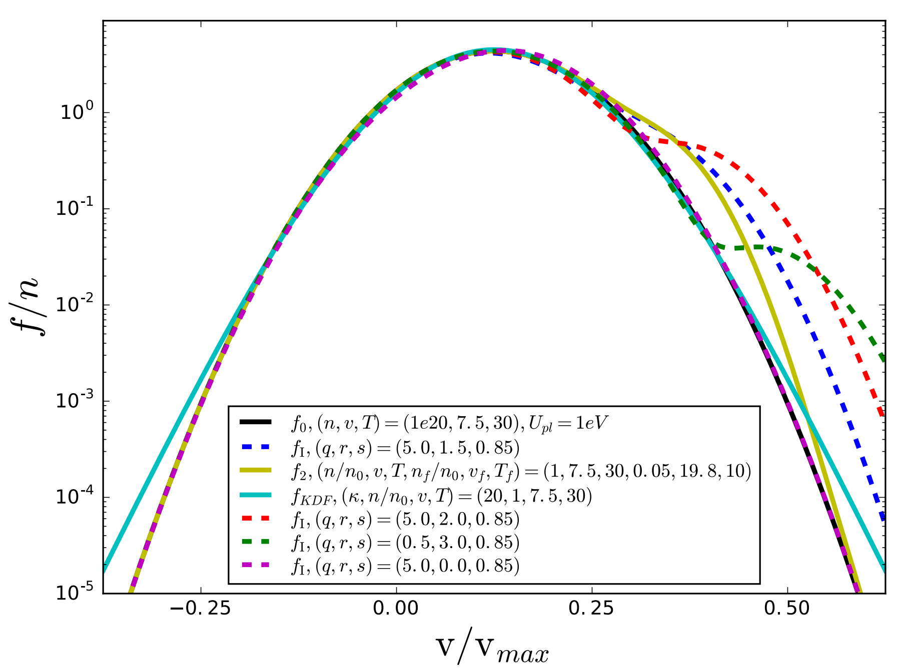
After some analytic computation (w/o limit, truncation, approx.)

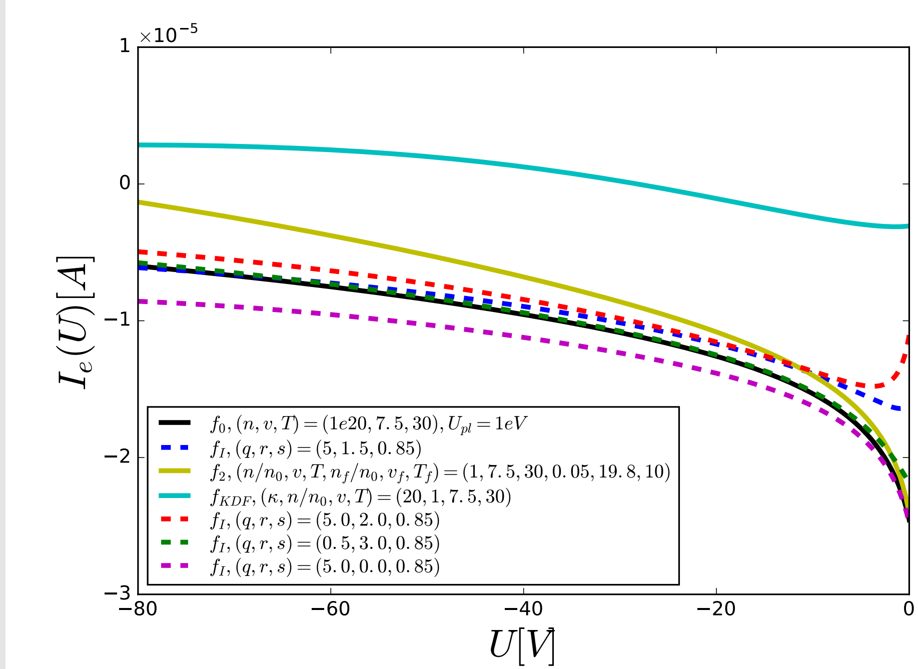
...

Analytic formulas: $I_{e,0}$ for a **MDF** and $I_{e,I}$ for an **INMDF**

2. Corrections of Langmuir probes characteristic

Even 5% of supra-thermal particles can have significant impact on $I_e(U)$:
supra-thermal particles can **replace the diffusion** parameter $\psi(E)$!!!





[**O. Izacard**, *Phys. Plasmas* **23** (2016) 082504]

RESULTS:

- **Non-Maxwellian correction of** $I_{e,I}(U, n, v, T, \Gamma, c, W)$
- **A super-thermal tail can replace ad-hoc diffusion terms!**

3. State of our understanding of the fluid theory

1 ➢ For fluid reduction/computation of collision operator of non-Maxwellians, **everyone** is using one of these "mathematical approximations":

- Hermite polynomials, Laguerre polynomials (and Sonine), Fourier series, Bessel functions, Legendre polynomials

2 ➢ One of the first interesting link between kinetic and fluid models [**Grad**, CPAM (1949)] with Hermite polynomials and the 13-moments model.

in equilibrium, "f" is known exactly as a function of the thermodynamic variables

=> **Commonly accepted facts:**

- Many researchers think (5/2015) "*Fluid codes are useless w.r.t. kinetic codes because they cannot reproduce kinetic effects*"
- fluid closure is needed due to the fluid reduction
- fluid closure assumes a relation of 1 fluid moment with previous ones
- fluid theory is valid only when $\lambda_{mfp}/L \ll 1$
- So many papers show plots f on $\{P_\phi, \mu, E, E_\perp, E_{//}\}$ rather than $\{v_x, v_y, v_z\}$

1) They are bad choices! 2) They are false!

[**O. Izacard**, *J. Plasmas Phys.* (2016) Submitted]

3. SOLUTION: Example of a fluid model including kinetic effects!

Reminder: "Mathematical" moments $M_k(\mathrm{x}, t) = \int f_I(\mathrm{x}, \mathrm{v}, t)\mathrm{v}^k d\mathrm{v}$

All M_k are function of the fitting hidden variables (n, v, T, Γ, c, W)

tri-diagonal matrix !

Dynamical fluid equations relevant to non-Maxwellians: ($\partial_t M_5$ contains $\nabla \cdot M_6$)

$$\partial_t \begin{pmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \end{pmatrix} = \begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \end{pmatrix}$$

See slide 33 for equations

With the **collisionless fluid closure**:

$$M_6 = n\left(15T^3 + 45T^2v^2 + 15Tv^4 + v^6\right) + 6\Gamma c\left(15W^2 + 10Wc^2 + c^4\right)$$

from the INMDF

[**O. Izacard**, *J. Plasmas Phys.* (2016) Submitted]

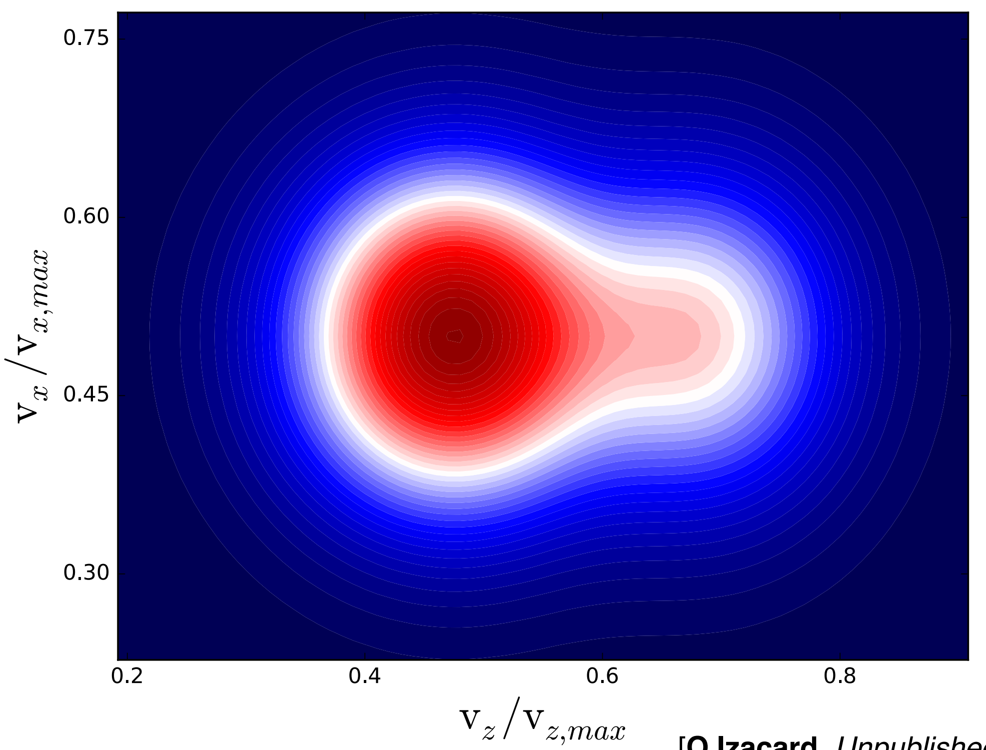
3. Toward the collisional fluid closure of a NMDF

State of the art of collision operator:

- One of the most useful analytic computation in [**Gaffey**, *JPP* (1976)]: computation of the steady state distribution function of an injected neutral beam in response to Fokker-Planck collision operator with a 3D Maxwellian background distribution function of the plasma
- Assumption of the 3D non-Maxwellian by multiplying a 1D tail in v_z and a 2D isotropic Maxwellian in (v_x,v_y)
- Follow analytic computation of Fokker-Planck collision operator of **Gaffey**

=> Collisional fluid closure

dissipative coefficients as fct of hidden variables



[**O.Izacard**, *Unpublished*]

It is more difficult to match plots of f on $\{P_\phi,\mu,E,E_\perp,E_{//},\dots\}$ rather than on $\{v_x,v_y,v_z\}$

3. Summary: Physical interpretation of a non-isolated macro-fluid-particle

Hidden variables of a non-isolated macro-fluid-particle:

- Does not violate the 2nd law of thermodynamics (because not conserved momentum and energy)
- Include kinetic effects in fluid models
- Possible candidate to explain/understand origins: turbulence, diffusion, …

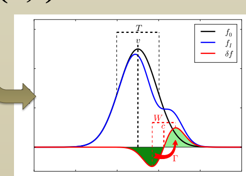
This description can correspond to a local negentropy

Source of energy

Macro-fluid particle

at the position (x,t)

$\boldsymbol{n}$**(x,t)** density
$\boldsymbol{v}$**(x,t)** fluid velocity
$\boldsymbol{T}$**(x,t)** temperature
$\boldsymbol{\Gamma}$**(x,t)** kinetic flux
$\boldsymbol{c}$**(x,t)** central flow
$\boldsymbol{W}$**(x,t)** width of heat spread



Local INMDF

Sink of energy

[**O. Izacard**, *J. Plasmas Phys.* (2016) Submitted]

3. Perspectives for fluid models including kinetic effects

Theoretical/Numerical perspectives:

- Development of the **collisional fluid closures** from Fokker-Plank operator $C(n,v,T,\Gamma,c,W)$: unperturbed limit using our non-Maxwellian & common perturbative methods (Chapman-Enskog, Braginskii, …)
- Modify an **existing fluid code** with non-Maxwellian set of fluid equations
- **Validation**: Use existing diagnostics to fit the hidden variables (Γ,c,W) of our non-Maxwellian
- **Resolve** the long time scale dynamo (**Schekochihin**) by developing an **expanded fluid-MHD** model

Experimental perspectives:

- Could we create **new diagnostics or new diagnostics interpretations** to measure (Γ,c,W) based on:
  - our predictive modifications of SEE, LP, Radiation…?
  - our new description of non-Maxwellian collisional fluid models?

4. Experimental measurement of NMDFs

<u>Current activity to detect NMDFs:</u>

- Diagnostic discrepancies:
  - Langmuir Probe / Thomson Scattering:
    discrepancies of electron temperature
    [Jaworski et al. (2012)]
  - Thomson Scattering / Electron Cyclotron Emission:
    discrepancies in TFTR and JET for electron temperature
    [De La Luna et al. (2003)]

- Other diagnostics:
  - Fast Ion DA
  - Soft X-Rays

4. Diagnostics pros and cons

Pros (+) and cons (-) for different solutions:

Thomson-Scattering:

+ multiple polychromators can be use to extract an approximated shape of the distribution function, analytic predictions with INMDFs

- polychromators not daily modifiable

Langmuir probes:

+ Analytic prediction of characteristic curves already done [Izacard, PoP (2016) 082504]

- Low temperature only

FIDA:

+ Directly linked to the NMDF

- Analytic model for charge-exchange cross section?

ECE:

+ Spectrum related to NMDFs

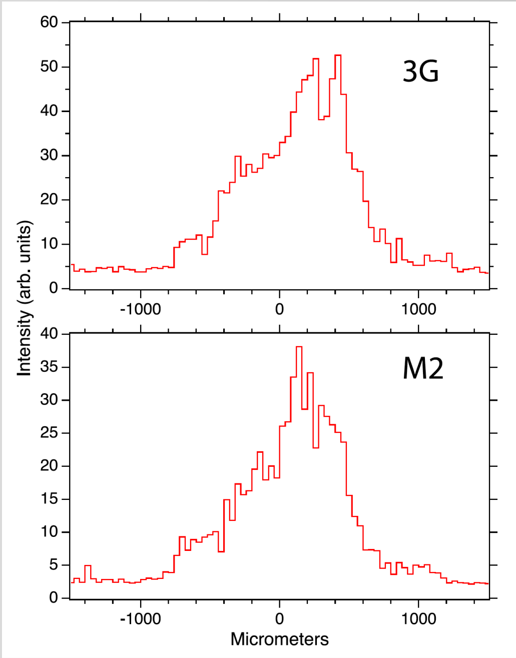
- Not available in NSTX-U (density limit)

4. Universal measurement of NMDFs

Best(?) universal direct measurement of NMDFs

High-resolution spectrometer with bended crystal
+ spectrometer available (P. Bereidober, LLNL)
+ arbitrary range of temperature
 (radial profiles of NMDFs)
+ "spatial profile" on detector directly
 linked to "spectral profile" (Bragg's law)
+ "spectral profile" proportional
 to NMDF (Doppler effect)

Brown, RSI (2016)



- can become very complex with other sources of broadening
 can we evaluate the dominant source of broadening?

Can we propose a DIII-D/NSTX-U experiment in Dec. 2016

Additional slides for questions

Acknowledgement for our discussions:

(on the physical reality of INMDFs) B. Cohen (LLNL)
(on runaway electrons) D.P. Brennan (PPPL)
(on Langmuir probes measurements) M. Jaworski (PPPL)
(on Thomson scattering) A. Diallo (PPPL)
(on gyrokinetic, Hermite, Neoclassical theory) J. Candy (GA)
(on Fokker-Planck code with LHCD) O. Meneghini (GA)
and many others…

Analytic Representation of Non-Maxwellians

Examples of known analytic non-Maxwellians:

- Kappa distribution function:

$$f_\kappa = \Delta_\kappa \left(1 + \frac{\mathrm{v}^2}{W_\kappa}\right)^{-(\kappa+1)}$$

with:

$$\Delta_\kappa = \frac{n}{\sqrt{\pi W_\kappa}} \left(\frac{\Gamma(\kappa+1)}{\Gamma\left(\kappa - \frac{1}{2}\right)}\right)^{1/3}$$

$$W_\kappa = (2\kappa - 3)T$$

- Bi-modal (sum of 2 Maxwellians):

$$f_{2M} = \Delta \exp\left(-\frac{1}{2T}\mathrm{v}^2 + \frac{v}{T}\mathrm{v}\right) + \Delta_f \exp\left(-\frac{1}{2T_f}\mathrm{v}^2 + \frac{v_f}{T_f}\mathrm{v}\right)$$

with: $\Delta = \dfrac{n}{\sqrt{2\pi T}} \exp\left(-\dfrac{v^2}{2T}\right), \quad \Delta_f = \dfrac{n_f}{\sqrt{2\pi T_f}} \exp\left(-\dfrac{v_f^2}{2T_f}\right)$

Analytic physical description of Non-Maxwellians

$$\boxed{f_I(\mathrm{x},\mathrm{v},t) = \frac{\boldsymbol{n}}{(2\pi \boldsymbol{T})^{1/2}} \exp\left(-\frac{(\mathrm{v}-\boldsymbol{v})^2}{2\boldsymbol{T}}\right) + \frac{\boldsymbol{\Gamma}}{(2\boldsymbol{W}^3)^{1/2}}(\mathrm{v}-\boldsymbol{c})\exp\left(-\frac{(\mathrm{v}-\boldsymbol{c})^2}{2\boldsymbol{W}}\right)}$$

"Mathematical" moments: $M_k(\mathrm{x},t) = \int f_I(\mathrm{x},\mathrm{v},t)\mathrm{v}^k d\mathrm{v}$

"Physical" moments: $P_k(\mathrm{x},t) = \frac{1}{M_0}\int f_I(\mathrm{x},\mathrm{v},t)\left(\mathrm{v}-\frac{M_1}{M_0}\right)^k d\mathrm{v}$

Example of computation:

$$M_0 = n$$
$$M_1 = nv + \Gamma$$
$$M_2 = n\left(T+v^2\right) + 2\Gamma c$$
$$P_1 = 0$$

$$P_2 = T - 2(v-c)\frac{\Gamma}{n} - \left(\frac{\Gamma}{n}\right)^2$$

$$P_3 = 3\left(W - T + (v-c)^2\right)\frac{\Gamma}{n} + 6(v-c)\left(\frac{\Gamma}{n}\right)^2 + 2\left(\frac{\Gamma}{n}\right)^3$$

Goal for Langmuir probes interpretations:

Analytic computation of *I(V) and *I'(V)*…** [following **Popov** PPCF '09]

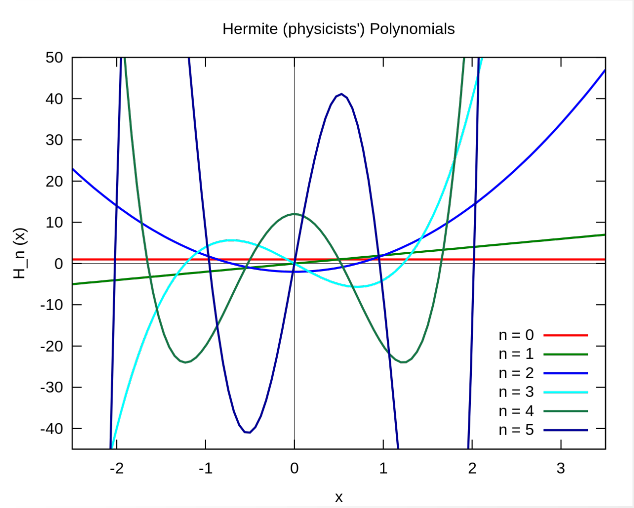
Also for: **floating potential, secondary electron emission (analytic formula*), radiation*, ion drag, neutron-neutron collision rate, …**

*** see next slides**

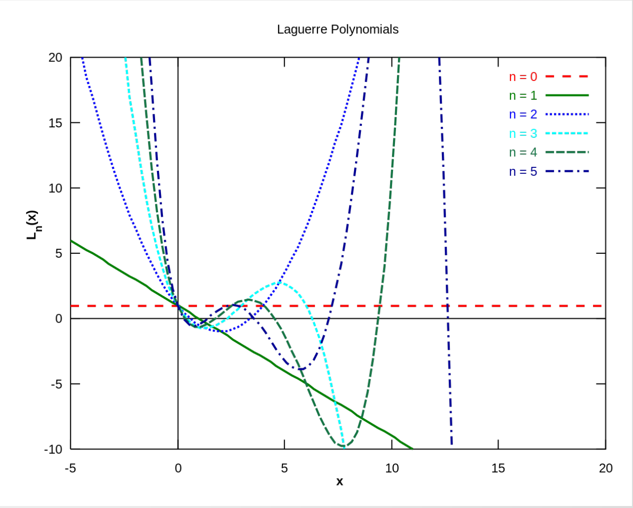
How fluid reduction is historically misunderstood?

<u>Quiz</u>: What is the similarity between these "mathematical" approximations?

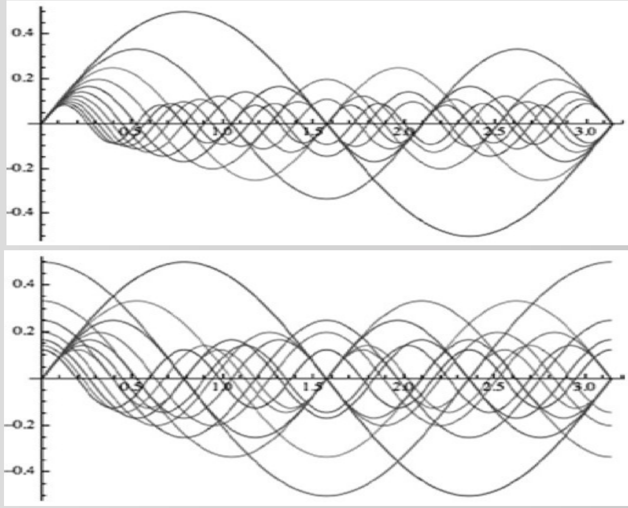
Hermite



Laguerre



Fourier



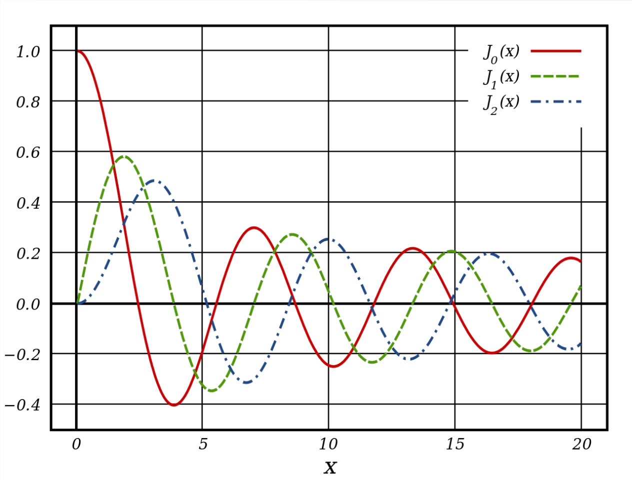
<u>Answer:</u> **Theirs limits are not 0 at infinity**

NOT adapted for localized perturbations

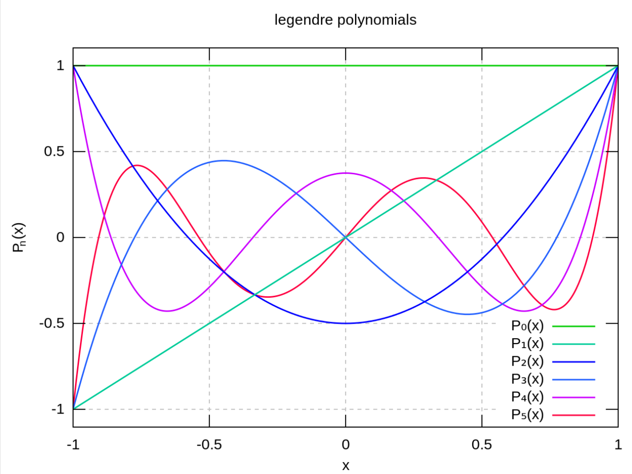
How fluid reduction is historically misunderstood?

Special cases:

Bessel



Legendre



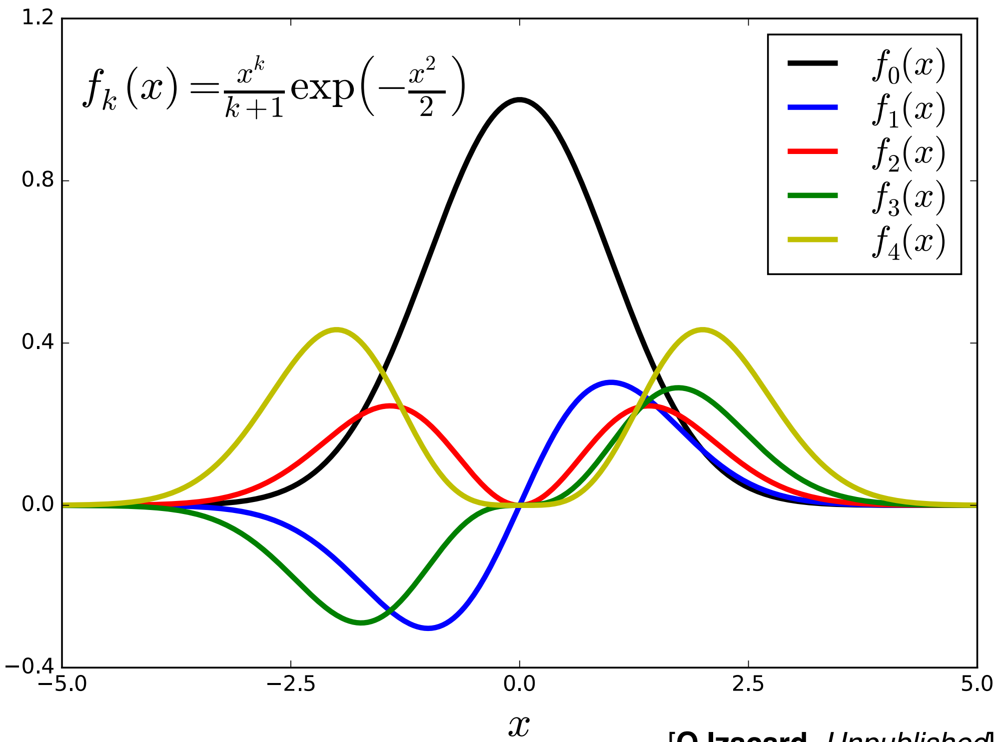
Still not good choices:

Bessel: quasi-periodic

Legendre: useless due to huge discretization,
& non-continuous distribution function

How fluid reduction is historically misunderstood?

PHYSICAL SOLUTION:



[**O.Izacard**, *Unpublished*]

Good choice:

- goes to 0 at ∞ (**localized**)
- **easy** analytically
- physical **interpretation** of hidden variables
- not an orthogonal basis but inherited **completeness**

Remarks: I do not care about orthogonal/normalized basis, can be reduced to sum of MDFs or Hermite polynomials

Is non-local heat transport a solution?

What is the non-local heat transport method?

$$\partial_t \begin{pmatrix} n \\ v \\ T \end{pmatrix} = \begin{pmatrix} \cdots \\ \cdots \\ \cdots \end{pmatrix} + {\color{red}D} \begin{pmatrix} \cdots \\ \cdots \\ \cdots \end{pmatrix} \quad \& \ q = -{\color{red}\kappa} \nabla T$$

$${\color{red}\kappa} = \int F\Big[n(x,t), v(x,t), T(x,t)\Big] d^3x$$

Non-local method is the **1st efficient existing way** to introduce time dependences in ad-hoc dissipative transport coefficients

However, it seems impossible to track back & understand the links between non-locality and non-Maxwellian distribution functions

CONCLUSION:

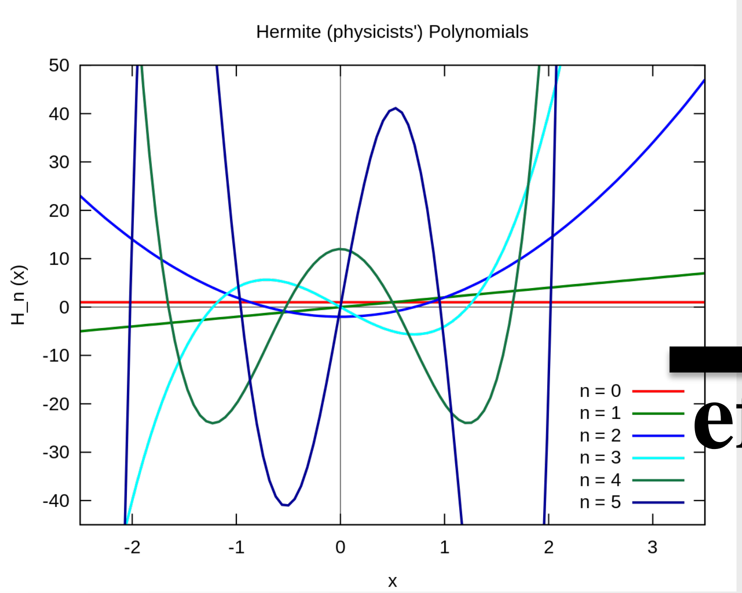
Is there a link with a tail or specific shape of the distribution function?

I personally think that non-locality is an incomplete solution

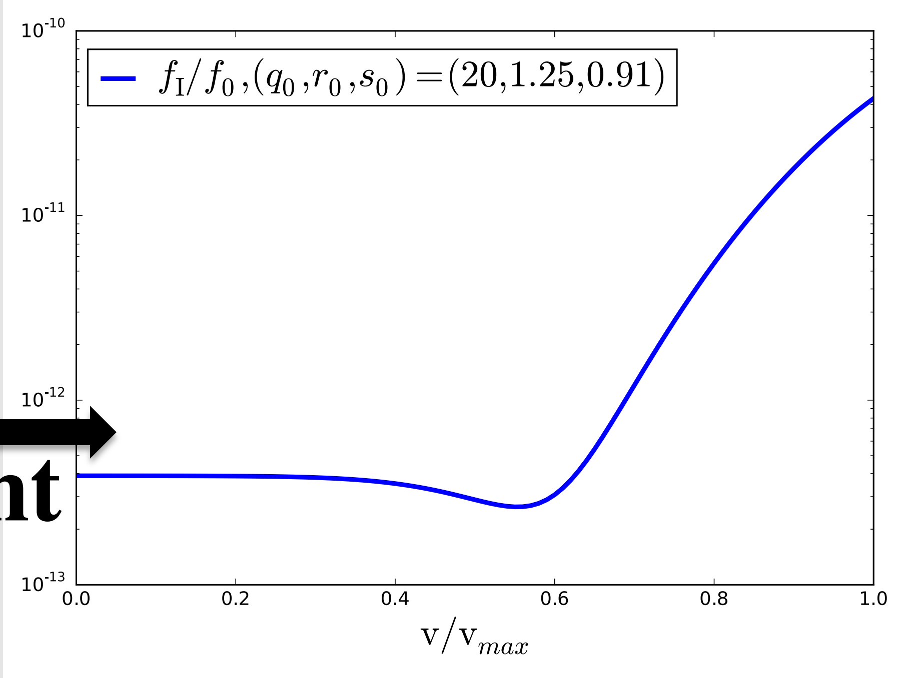
How fluid reduction is historically misunderstood?

<u>Quiz</u>: Why "Mathematical approximation" representations failed to represent a simple tail?

Hermite



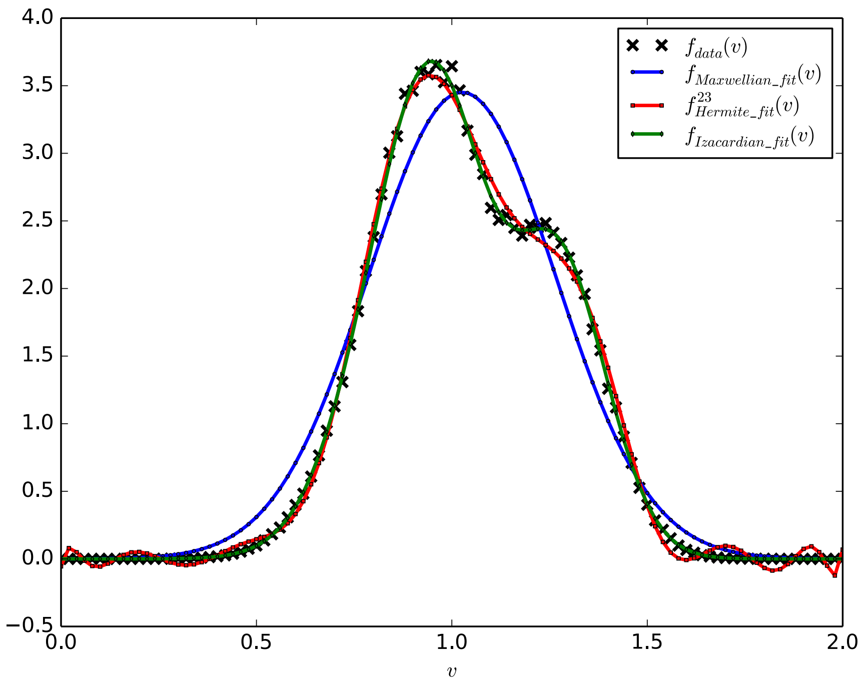
not efficient

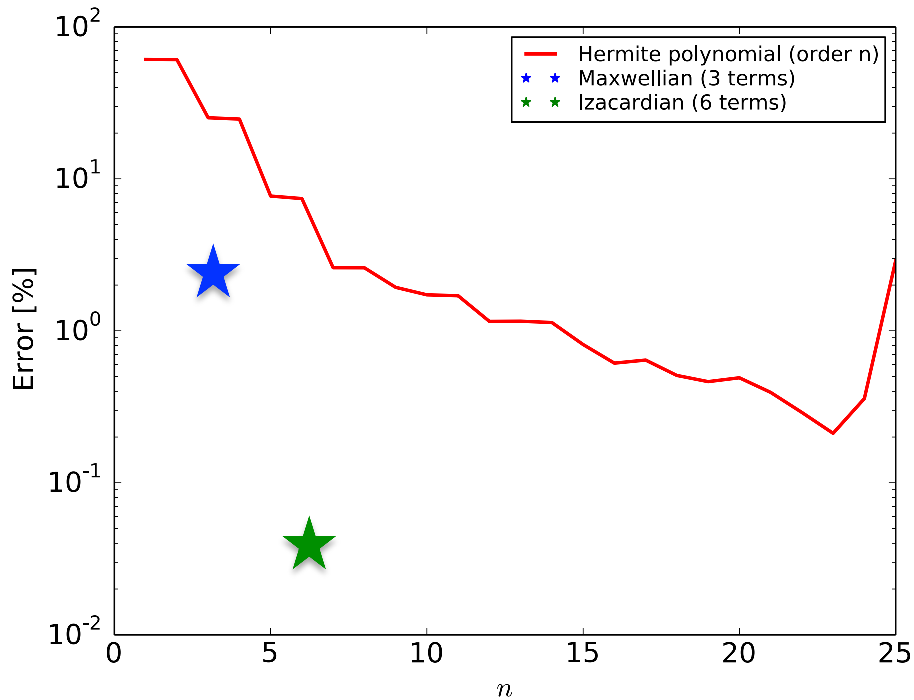


<u>Answer:</u> The curve f_I/f_0 cannot be approximate without a huge number of terms

Comparison of efficiency to describe a tail between Hermite polynomials and our non-Maxwellian

Hermite polynomials is not a good choice





Even in (gyro)-kinetic codes, we should use our "generalized Hermite-Maxwellian basis"

Comparison of efficiency to describe a tail between Hermite polynomials and our non-Maxwellian

- From the explicit expressions for the first few polynomials,

[**Grad H.**, CPAM (1949)]

$$
(4.10) \quad \begin{cases}
\mathcal{H}^{(0)} = 1 \\
\mathcal{H}_i^{(1)} = v_i \\
\mathcal{H}_{ij}^{(2)} = v_i v_j - \delta_{ij} \\
\mathcal{H}_{ijk}^{(3)} = v_i v_j v_k - (v_i \delta_{jk} + v_j \delta_{ik} + v_k \delta_{ij}) \\
\mathcal{H}_{ijkl}^{(4)} = v_i v_j v_k v_l - (v_i v_j \delta_{kl} + v_i v_k \delta_{jl} + v_i v_l \delta_{jk} + v_j v_k \delta_{il} \\
\qquad + v_j v_l \delta_{ik} + v_k v_l \delta_{ij}) + (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),
\end{cases}
$$

Some interesting relations

All M_k are function of the fitting hidden variables $(a_0, a_1, \cdots, a_N)$

$$M_k^{(i_1,\cdots,i_k)}(\mathbf{x},t) = \int f(\mathbf{x},\mathbf{v},t) \prod_{j=1}^{k} \mathrm{v}^{(i_j)} d^3\mathrm{v}$$

$$P_k^{(i_1,\cdots,i_k)}(\mathbf{x},t) = \frac{1}{n(\mathbf{x},t)} \int f(\mathbf{x},\mathbf{v},t) \prod_{j=1}^{k} \left(\mathrm{v}^{(i_j)} - v^{(i_j)}(\mathbf{x},t)\right) d^3\mathrm{v}$$

$$\partial_t M_0 = -\sum_{\alpha=1}^{3} \nabla_\alpha M_1^{(\alpha)}$$

$$\partial_t M_k^{(i_1,\cdots,i_k)} = -\sum_{\alpha=1}^{3} \nabla_\alpha M_{k+1}^{(\alpha,i_1,\cdots,i_k)}$$
$$+\frac{e}{m}\left(E^{(i_1)} M_{k-1}^{(i_2,\cdots,i_k)} + \sum_{\alpha,\beta=1}^{3} \epsilon_{i_1\alpha\beta} M_k^{(\alpha,i_2,\cdots,i_k)} B^{(\beta)} + \underbrace{\circlearrowright}_{(i_k)} \right)$$

Example of a non-Maxwellian fluid model with constraints

Reminder: "Mathematical" moments $M_k(\mathrm{x}, t) = \int f_I(\mathrm{x}, \mathrm{v}, t)\mathrm{v}^k d\mathrm{v}$

All M_k are function of the fitting hidden variables (n, v, T, Γ)

Constraints: $c(n, v, T, \Gamma)$, $W(n, v, T, \Gamma)$

Dynamical fluid equations relevant to non-Maxwellians: ($\partial_t M_3$ contains $\nabla \cdot M_4$)

$$\partial_t \begin{pmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \end{pmatrix} = \begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \end{pmatrix}$$

See slide 33 for equations

With the **collisionless fluid closure**: from the INMDF

$$M_4 = n\left(3T^2 + 6Tv^2 + v^4\right) + 4\Gamma c\left(3W + c^2\right)$$

[**O.Izacard**, *Unpublished*]

Toward the collisional fluid closure of a NMDF

State of the art of collision operator:

- I already reproduced the analytic computation of [**Gaffey**, *JPP* (1976)]:

$$F(x_{ab}) = \frac{1}{n_b v_b}\int_{-\infty}^{\infty} f_b(\mathbf{v}_b) g_{ab} d^3\mathbf{v}_b, \qquad x_{ab} = \mathbf{v}_a/v_b, \qquad g_{ab} = |\mathbf{v}_a - \mathbf{v}_b|$$

$$\mathcal{C}(f_a, f_b) = \frac{2\pi e_a^2 e_b^2 n_b \ln\Lambda}{m_a^2}\frac{\partial}{\partial \mathbf{v}_a}\cdot\left[\frac{\partial f_a(\mathbf{v}_a)}{\partial \mathbf{v}_a}\cdot\left(\frac{\partial^2 v_a}{\partial \mathbf{v}_a \partial \mathbf{v}_a}F'(x_{ab}) + \frac{\mathbf{v}_a\mathbf{v}_a}{v_a^3}x_{ab}F''(x_{ab})\right)\right.$$
$$\left. + 2\frac{m_a}{m_b} f_a(\mathbf{v}_a)\frac{\mathbf{v}_a}{v_a^3}\left(F'(x_{ab}) - x_{ab}F''(x_{ab}) - \frac{x_{ab}^2}{2}F'''(x_{ab})\right)\right]$$

$$f_b = f_0 \;\Rightarrow\; F(x) = F_0(x) = \left(x + \frac{1}{2x}\right)\mathrm{Erf}(x) + \frac{1}{\sqrt{\pi}}\exp(-x^2)$$

Analytic computation of *transport coefficients* with a tail:

- f_a and f_b are the same INMDF
- shift in v_z as function of v_a, with

$$f_I(\mathrm{x}, \mathrm{v}, t) = \frac{\boldsymbol{n}}{(2\pi \boldsymbol{T})^{1/2}}\exp\left(-\frac{(\mathrm{v} - \boldsymbol{v})^2}{2\boldsymbol{T}}\right) + \frac{\boldsymbol{\Gamma}}{(2\boldsymbol{W}^3)^{1/2}}(\mathrm{v}_z - \boldsymbol{c})\exp\left(-\frac{(\mathrm{v}_z - \boldsymbol{c})^2}{2\boldsymbol{W}}\right)$$

1st self-consistent collisional non-Maxwellian fluid model

Nonlinear Fokker-Planck collision operator:

$$\mathcal{C}(f_a,f_b)=\frac{2\pi e_a^2 e_b^2 n_b \ln\Lambda}{m_a^2}\frac{\partial}{\partial \mathbf{v}_a}\cdot\left[\frac{\partial f_a(\mathbf{v}_a)}{\partial \mathbf{v}_a}\cdot\left(\frac{\partial^2 v_a}{\partial \mathbf{v}_a \partial \mathbf{v}_a}F'(x_{ab})+\frac{\mathbf{v}_a\mathbf{v}_a}{v_a^3}x_{ab}F''(x_{ab})\right)\right.$$
$$\left.+2\frac{m_a}{m_b}f_a(\mathbf{v}_a)\frac{\mathbf{v}_a}{v_a^3}\left(F'(x_{ab})-x_{ab}F''(x_{ab})-\frac{x_{ab}^2}{2}F'''(x_{ab})\right)\right]$$

$$f_a=f_b=f_I \;\Rightarrow\; F(x)=F_I(x)=F_0(x)+\Delta_I \mathrm{v}_\perp^2\left[\mathrm{v}_\perp I_1(A,B)+(\mathrm{v}_z-c)I_0(A,B)\right]$$

with $A=\frac{\mathrm{v}_\perp^2}{2W}, B=\frac{\mathrm{v}_\perp(c-\mathrm{v}_z)}{2W}$

Self-consistent **collisional fluid closure** with sources f_s with $s=\{\alpha, NBI, LH, \ldots\}$

$$\partial_t\begin{pmatrix}M_0\\M_1\\M_2\\M_3\end{pmatrix}=\begin{pmatrix}\cdots\\\cdots\\\cdots\\\cdots\end{pmatrix}+\begin{pmatrix}\int\mathcal{C}(f_I,f_I)d\mathrm{v}\\\int\mathcal{C}(f_I,f_I)\mathrm{v}d\mathrm{v}\\\int\mathcal{C}(f_I,f_I)\mathrm{v}^2d\mathrm{v}\\\int\mathcal{C}(f_I,f_I)\mathrm{v}^3d\mathrm{v}\end{pmatrix}+\begin{pmatrix}\int\mathcal{C}(f_I,f_s)d\mathrm{v}\\\int\mathcal{C}(f_I,f_s)\mathrm{v}d\mathrm{v}\\\int\mathcal{C}(f_I,f_s)\mathrm{v}^2d\mathrm{v}\\\int\mathcal{C}(f_I,f_s)\mathrm{v}^3d\mathrm{v}\end{pmatrix}$$

See slide 39 for equations

With the **collisionless fluid closure**: from the INMDF

$$M_4=n\left(3T^2+6Tv^2+v^4\right)+4\Gamma c\left(3W+c^2\right)$$

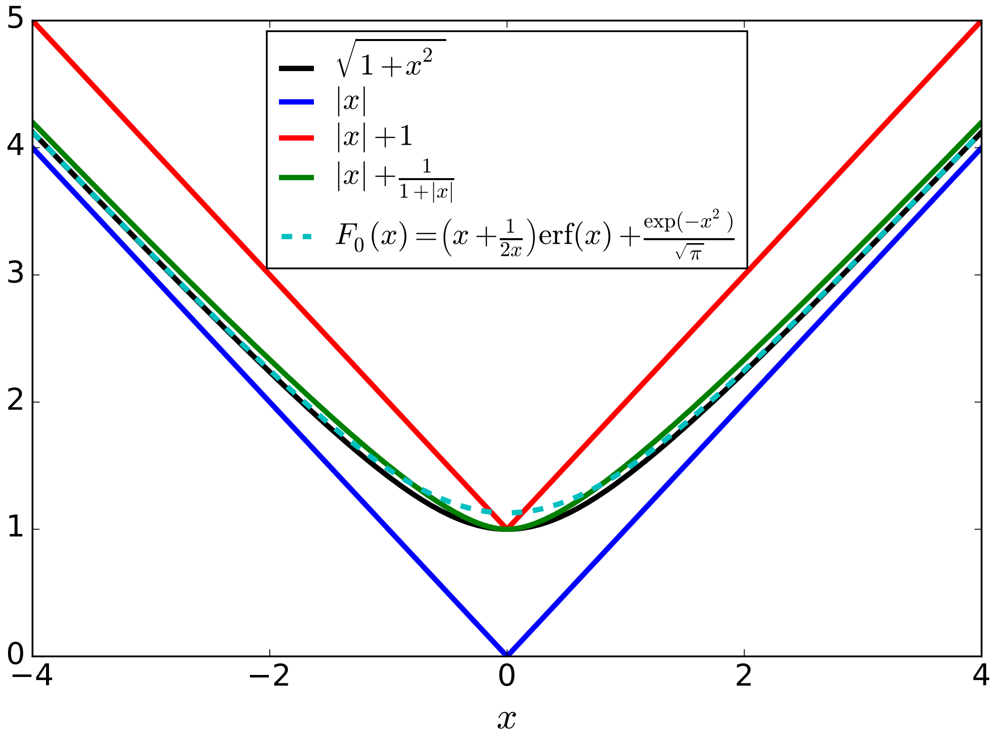
[**O.Izacard**, *Unpublished*]

Toward the collisional fluid closure of a NMDF

Analytic computation of *transport coefficients* with a tail:

For $F_l(x)$ for f_b=f_l we need to compute this "***plasma collisional function***":

$$I_k(a,b) = \int_{-\infty}^{\infty} x^k \sqrt{1+x^2} \exp\left(-ax^2 + bx\right) dx$$



We have min/max boundaries:

$$\boxed{|x| < \sqrt{1+x^2} < |x|+1}$$

$$\boxed{\sqrt{1+x^2} < |x| + \frac{1}{1+|x|}}$$

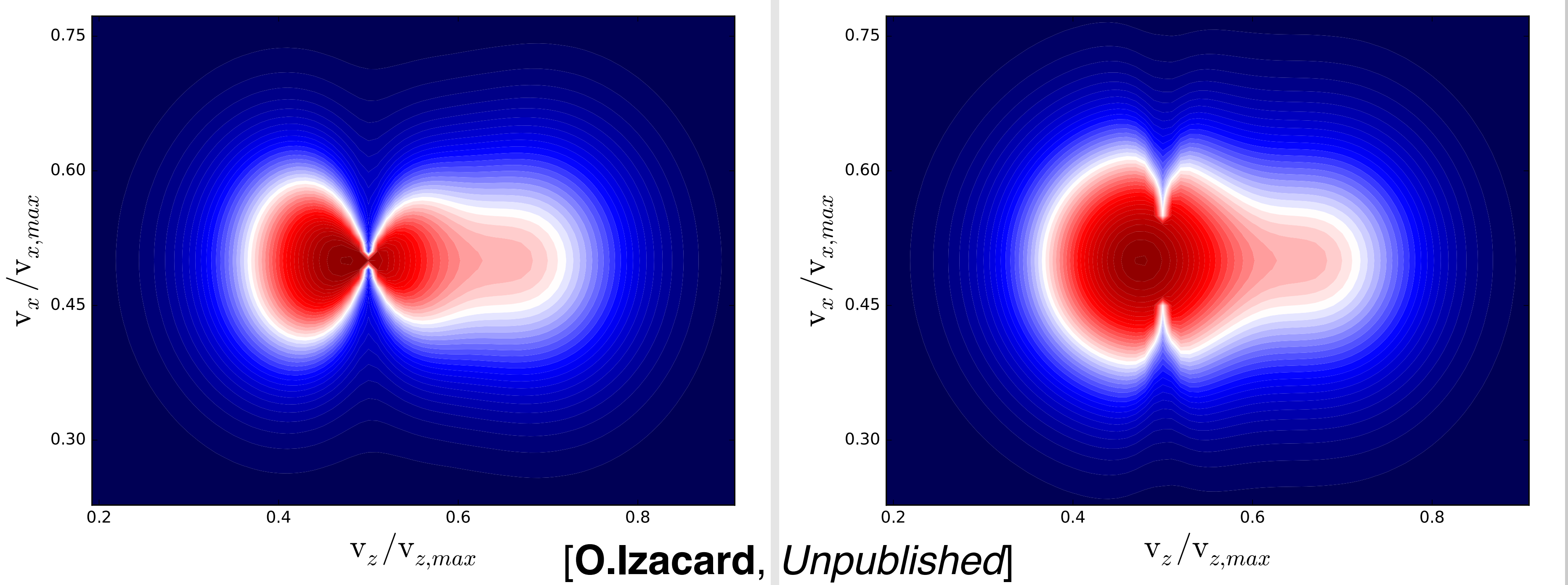
$$\boxed{\sqrt{1+x^2} < F_0(x)}$$

[**O.Izacard**, *Unpublished*]

How to include **neoclassical** particles in fluid equations?

We need to use tractable functions (cosines, sines, Bessel, polynomials) with as less hidden variables as possible

With only a cosines and 3 parameters, we can approximate NC as:



[**O.Izacard**, *Unpublished*]

The fluid reduction of neoclassical particles becomes obviously tractable