Computation of Resistive Instabilities in Tokamaks with Full Toroidal Geometry and Coupling using DCON

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Abstract:

Precise determination of resistive instabilities is an outstanding issue in tokamaks, remaining unsatisfactory for a long time despite its importance for advanced plasma control. This paper presents the first successful computation of such resistive instabilities, including full mode coupling and multiple singular surfaces, by upgrading the DCON ideal stability code with a resonant-Galerkin method using advanced basis functions. Incorporating the resistive layer model of Glasser, Greene, and Johnson (GGJ) and matching the inner-layer solutions to the full outer-layer solutions in MATCH code, a complete picture of resistive instabilities in tokamaks is obtained and studied. Excellent quantitative agreement with the MARS-F code, for both growth rate and outer-layer solutions, has been achieved. Convergence is also a key property of resistive DCON, as tested on challenging NSTX equilibria with strong shaping, high β, and multiple rational surfaces, up to 10. Another advantage of the new DCON is the separation of the inner-layer from the outer-layer regions, which allows us to extend inner-layer model efficiently to more advanced fluid equations as well as drift kinetic effects and to perform more precise calculations of non-ideal stability and 3D perturbed equilibria in the future.

1 Introduction

Tokamak performance is limited by various plasma instabilities. Global ideal MHD instabilities, such as kink modes, typically set the upper bound of operational space, leading to disruption. Other ideal but local MHD instabilities, such as ballooning or peeling modes, are also not acceptable, especially in next step devices such as ITER, since these edge localized modes (ELMs) can severely damage the plasma facing components. Fine control of these ideal modes is still an important issue, but the predictability of ideal modes is at a matured level at present, so that the focus is rather on profile changes to avoid or mitigate the instabilities by tuning transport or micro-instabilities.

Resistive instabilities, however, are not well predicted in tokamaks, although their control is also critical to achieving high-performance plasmas. The most important consequence of finite resistivity is the reconnection of magnetic field lines and tearing modes (TMs) [1, 2] characterized by magnetic islands in the neighborhood of mode rational surfaces where $q = m/n$. These magnetic islands can grow nonlinearly after an initial linear phase, and in the nonlinear phase, the mode activities can be changed along with the evoluton of profiles, leading to neoclassical tearing modes (NTMs) [3]. TMs or NTMs tend to be more unstable at higher β; they can have lower stability threshold than ideal modes and can be β-limiting [4]. They can also interact with the external magnetic perturbations, such as error fields [5] or applied resonant magnetic perturbations (RMPs) [6], inducing either a disruptive or locally stabilizing transport process. These are all currently important topics, but their physics studies are mostly built upon a poor approximation in the linear phase, a cylindrical Δ' [1].

The scalar quantity Δ' is well known as a tearing mode index, and defined by the jump in the derivative of the normal magnetic field $[\partial(\vec{b} \cdot \nabla\psi)/\partial\psi]$ across the singular layer. It assumes only a single rational surface, ignoring all others, and also the constant-ψ across the rational surface, assuming continuity in the normal magnetic field. However, the theoretical work of Dewar *et al.* [7, 8] has shown that a single Δ' cannot determine or characterize resistive instabilities in toroidal geometry, but instead a full matrix is required that couples all the rational surfaces. In addition, the left and right sides of the rational surfaces are disconnected in the outer region and so the coupling matrix becomes $\Delta'_{ip;jq}$ with $1 \leq i, j \leq N$ where N is the number of the rational surfaces and p and q are either L or R representing the left or the right side of the rational surfaces. The $2N \times 2N$ coupling matrix $\Delta'_{ip;jq}$ can be determined completely by the solutions in the outer ideal MHD region, then coupled to a corresponding solution in the inner regions in the neighborhood of the mode rational surfaces.

The asymptotic matching method between the inner region and the outer region, using the matching data $\Delta'_{ip;jq}$, is the most precise description of resistive instabilities, revealing the nature of the resisitve instabilities. By comparison, for instance, the MARS-F code [9], one of the most successful codes for resisitive instabilities, cannot provide information about stable resisitive modes that can be used to study resistive plasma response to external perturbations. The asymptotic matching theory for resisitive instabilities in toroidal geometry has been comprehensively developed, and its numerical implementation has also been achieved by PEST3 code [7, 8]. The PEST3 code, however, implements only the lowest-order Frobenious expansion of the resonant solutions, resulting in the limited applicability only to low-β, strongly sheared, and weakly shaped cases.

This paper presents the first successful computation of such resistive instabilities, obtained by upgrading the DCON [10], based on the precise asymptotic matching method, including full mode coupling and multiple singular surfaces. Section 2 describes the separation of the resonant solutions through the coupling matrix and the improved numerical methods with a modified Galerkin method, resulting in robust and stable numerical properties for the outer region solutions. Section 3 explains the asymptotic matching with the inner region solutions and shows a successful benchmark with the MARS-F code both on the growth rates and global eigenfunctions of resistive instabilities, using the layer model of Glasser, Greene, and Johnson (GGJ) [11, 12] is used. Sec 4 discusses the applications and summarizes the results.

2 Outer region solutions and the matching matrix

In the outer, ideal MHD region, we use the singular Galerkin approach introduced by Dewar *et al.* [7, 8], but with essential improvements. There is a Hilbert space of finite ideal MHD energy solutions, $\delta W < \infty$. For an ideally stable equilibrium, there are no finite-energy solutions that satisfy the boundary conditions at the plasma edge. In the neighborhood of each singular surface, there are large and small resonant solutions for $m = nq$, as well as nonresonant solutions for the Fourier components $m \neq nq$. The large resonant solutions are not elements of the Hilbert space, but they drive finite-energy responses. Treating the large solutions as inhomogeneities, we compute the response in the Hilbert space by solving an inhomogeneous linear equation, the Euler-Lagrange equation for minimizing δW, which is also the equation of motion for zero-frequency modes.

The Euler-Lagrange equation has the form

$$\mathbf{L}\Xi \equiv (\mathbf{F}\Xi + \mathbf{K}\Xi)' - (\mathbf{K}^{\dagger}\Xi' + \mathbf{G}\Xi) = 0, \tag{1}$$

where ψ is a flux coordinate; $\Xi(\psi)$ is a complex M-vector of Fourier coefficients of the normal plasma displacement; and $\mathbf{F}$, $\mathbf{K}$, and $\mathbf{G}$ are $M \times M$ complex matrices derived from equilibrium quantities. The inhomogeneous equation is

$$\mathbf{L}\Xi_{\mathrm{Hilbert}} = -\mathbf{L}\Xi_{\mathrm{large}}. \tag{2}$$

We express Eq. (2) in matrix form by expanding Ξ in a set of Galerkin basis functions. The choice of basis functions determines the rate of convergence. In the PEST III code, Dewar *et al.* use linear finite elements on a packed grid in the surface coordinate ψ; and Fourier series in the PEST θ coordinate. We make an improved choice of radial basis functions: C^1 Hermite cubics to resolve the nonresonant solutions; and Frobenius power series to resolve the large and small resonant solutions; on a more flexible packed grid in ψ. We use poloidal Fourier series in a variety of straight-fieldline coordinates, of which the best convergence is obtained with Hamada coordinates. Equilibrium quantities are fit to bicubic splines in ψ and θ, which allows us to compute convergent Frobenius expansions to arbitrarily high order.

The matching matrix $\mathbf{\Delta}'$ is contructed from the coefficients of the small resonant solutions driven by the large resonant solutions. For N singular surfaces, there are $2N$ large solutions, on the left and right of each singular surface, each of which drives $2N$ small solutions, making $\mathbf{\Delta}'$ a $2N \times 2N$ matrix.

After expansion in basis functions, Eq. (2) is solved using LAPACK routines ZGETRF and ZGETRS. For a challenging equilibrium with high β, strong shaping, and many singular surfaces, this can be done in a few seconds on a scalar processor. The robustness, speed, and accuracy of the method is a consequence of the improved choice of basis functions. After computing the $\mathbf{\Delta}'$ matrix once, it can be coupled to a large range of inner region models and parameters, which can be computed much faster.

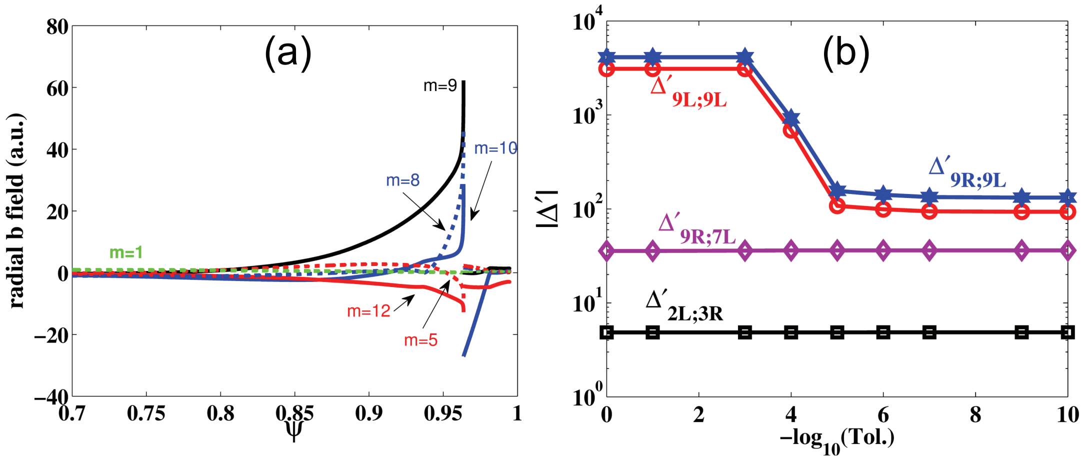

FIG. 1: (a) The solutions driven by the big solution $\Xi^b_{9L}(\psi)$ on the left side of $q = 9$ rational surface and (b) the numerical convergence for the stability index component as a function of numerical tolerance, in a strongly shaped, high-β NSTX case. Note the strong mode coupling in (a).

It is instructive to compare this singular Galerkin method with the method used in DCON to determine ideal MHD stability, a generalization to an axisymmetric torus. of Newcomb's cylindrical method. [13] In that method, Eq. (1) is treated as a coupled system of ordinary differential equations in ψ and adaptively integrated from the magnetic axis to the first singular surface, where boundary conditions excluding large solutions are applied to cross to the other side, then on to the plasma edge. If the generalized Newcomb criterion is violated, there is a fixed-boundary instability. If it is satisfied, then the solution at the boundary is coupled to a vacuum solution to determine free-boundary stability. Many attempts were made to generalize that method to resistive instabilities, until it was recognized that this extension makes the method numerically unstable. DCON retains that method fo ideal stability, but now uses the singular Galerkin method for resistive stability. An alternative method of determining ideal stability would be to find the smallest eigenvalues of the Hermitian $\mathbf{L}$ matrix in the Hilbert space, using LAPACK routine ZHBEVX. This was tried and found to be much slower than the generalized Newcomb method.

The New DCON can now accurately provides the non-resonant and small solutions driven by the big solutions even in an extreme case such as NSTX. Figure 1(a) shows the solutions driven by the big solution $\Xi^b_{9L}(\psi)$ on the left side of $q = 9$ rational surface in an NSTX example with 13 rational surfaces and $\beta_N \sim 3.3$. The convergence properties are also very good, as shown in Figure 1(b), where the coupling coefficients Δ' are tested at the multiple rational surfaces in the NSTX case.

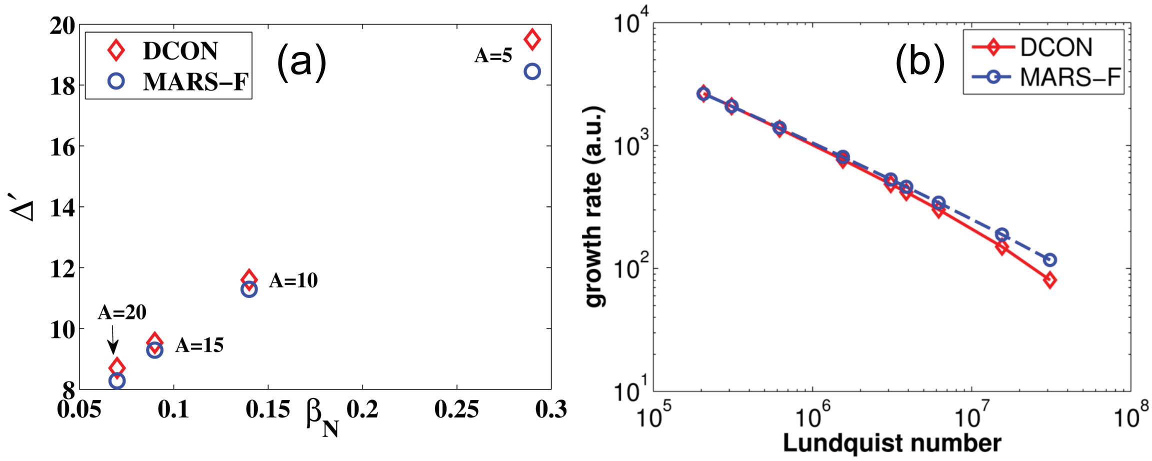

FIG. 2: Comparison of (a) the reduced stability index $\Delta_{ij} \propto [\partial(\vec{b}\cdot\nabla\psi)/\partial\psi]$ as a function of β_N (also the aspect ratio A) in a circulr plasma, and (b) the resistive growth rate as a function of Lundquist number in a D shaped plasma, between DCON and MARS-F.

3 Asymptotic matching and benchmark

The outer region solutions provide the stability index matrix, or equivalently the coupling matrix between the big and the small solutions. The coupling matrix is only the information needed to determine the inner region solutions, and the asymptotic matching yields the complete solution structure as well as the growth rate of resistive instabilities. Presently the inner region is described by the model developed by Glasser, Greene, and Johnson (GGJ) [11, 12], which includes the effects by inertia $\rho\partial^2\vec{\xi}/\partial t^2$ and the resistivity η at the layer. GGJ produces the even and odd inner region solutions as

$$\Xi_{i\pm}(x) = \Xi^b_{i\pm}(x) + \Delta_{i\pm}(Q)\Xi^s_{i\pm}(x), \tag{3}$$

where Q is a resistively scaled complex growth rate and x is the scaled length across the layer. The total inner region solution at each layer is $\Xi_i(x) = d_{i+}\Xi_{i+}(x) + d_{i-}\Xi_{i-}(x)$ with the constants of integration $d_{i\pm}$ and must be asymptotically matched with the outer region solution at $x \to \infty$. Note the outer region fomulation with the left and the right side solutions $p, q = L, R$ can be replaced by even and odd solutions $p, q = +, -$ simply by adding and substracting one to the other. The even and odd matching conditions at each rational surface are given by

$$\lim_{z\to 0}\sum_{ip} c_{ip}\left(\delta_{ij}\delta_{pq}\Xi^b_{ip}(z) + \sum_{jq}\Delta'_{ip;jq}\Xi^s_{jq}(z)\right) = \lim_{x\to\infty} d_{ip}\left(\Xi^b_{ip}(x) + \Delta_{ip}(Q)\Xi^s_{ip}(x)\right). \tag{4}$$

Matching the big solutions simply gives $d_{ip} = c_{ip}$ and matching the small solutions yields

$$\sum_{ip}\left[\Delta'_{ip;jq} - \delta_{ip}\Delta_{jq}(Q)\right] c_{ip} = 0. \tag{5}$$

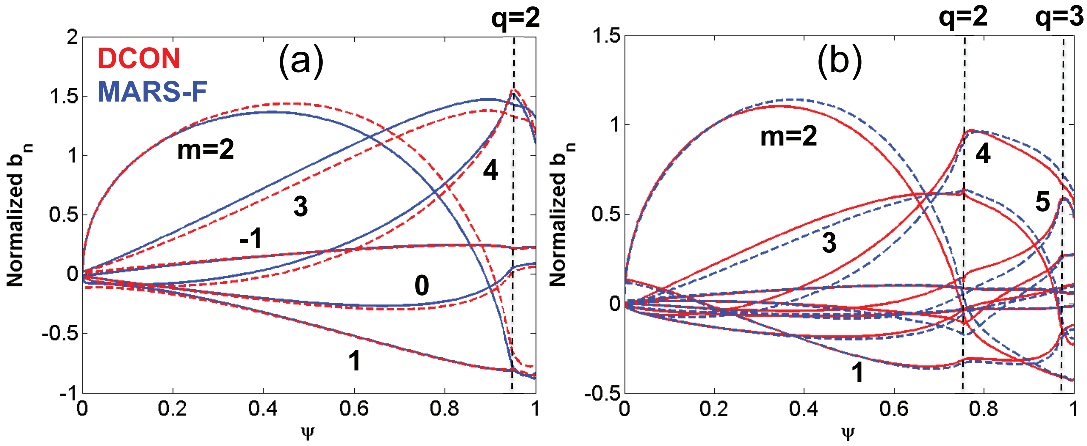

FIG. 3: (a) Comparison of the resistive modes between DCON with GGJ and MARS-F, and (b) Recontruction of the MARS-F resistive modes using DCON with proper coupling coefficients.

A non-trivial solution exists only when

$$\det|\Delta'_{ip;jq} - \delta_{ip}\Delta_{jq}(Q)| = 0. \tag{6}$$

This is the dispersion relation for determining the complex global rate growth Q, completing the stability analysis. There is an instability if the dispersion relation has a root with $\Re Q > 0$.

Resistive stability is determined not only by one layer but through the coupling among all the others. The classical tearing mode index Δ' becomes relevant for stability only in a cylindrical limit, but the jump in the derivative of the normal magnetic field can be estimated by

$$\Delta'_{ij} \equiv \Delta'_{i-;j-} = \frac{1}{2}(\Delta'_{iR;jR} - \Delta'_{iL;jR} - \Delta'_{iR;jL} + \Delta'_{iL;jL}). \tag{7}$$

This reduced index can be useful for benchmarking purpose. Figure 2(a) shows the comparison and the good agreement between Δ'_{ij} in DCON and $[\partial(\vec{b}\cdot\nabla\psi)/\partial\psi]$ in MARS-F, for circular cross-section plasmas with a single rational surface. Note DCON can calculate Δ'_{ij} in any case, but only the resistively unstable cases were selected for benchmark with MARS-F. The resistive growth rates were also agreed very well with the scaling $\gamma \sim S^{-3/5}$ that is expected by tearing modes with the Lundquist number, as shown in Figure 2(b), where a D-shaped plasma is tested. The finally constructed solutions by asymptotic matching in DCON are also agreed in sufficient accuracies as shown in Figure 3(a). The solutions become almost identical in circular plasmas although it is not shown here. Further benchmarking efforts will be made with more extreme targets, and also with PEST3 code, but the disagreement is possibly expected in higher-β and stronger shaping as DCON is formulated with the most precise method without a limitation, at least in the purely resistive limit.

4 Discussion and summary

The newly developed DCON for resistive instabilities will have diverse areas of applications. The calculation of the resistive stability boundary for existing tokamaks and ITER will the first, in the context of the global parameters such as P', q', and the resistivity η. The correlation of the actual growth rate with the scalar tearing mode index Δ' will also be of interest, since it can possibly modify the basis of many theories for TMs or NTMs. For high-β applications on the other hand, it is important to explore the influence of local pressure flattening on resistive instabilities. The Pfirsch-Schlüter currents by the finite pressure gradient can be exteremly stabilizing and thus the stability prediction can be widely different when they are excluded by local pressure flattening, as other literature discussed in a cylindrical case [14]. When the pressure is locally flattened near the rational surfaces, the two resonant solutions become degenrate and a logarthimic correction $\ln z$ is expected as is known in Frobenious expansion, but a toroidal case is more complicated due to the coupling and the formulation should be carefully tested.

The structure of DCON code, separating the inner regions from the outer regions, provides an important advantage. New physics required to describe the inner region has little effect in the outer region, in which the complication comes from strong shaping and complicated external boundary conditions. The ideal and outer region solutions are completely determined by new DCON, leaving only the coupling matrix for the inner region, which can be considered independently. Figure 3(b) shows an example, and that DCON can reproduce MARS-F solutions with proper choices of the coupling coefficients, even without the inner region model such as GGJ. In practice, GGJ may be not a sufficiently complete model to describe the inner layer dynamics. As studied in other literature, viscous, rotational, drfit-MHD [15, 16], and also ion polarization current [17] effects may eventually be required to achieve the predictablity of tearing modes or island opening in perturbed equilibria. The inner region model can be extended to incorporate such advanced physics, and furthermore even to non-linear regime where the degree of perturbation can still be sufficiently small for the outer region.

In summary, DCON has been successfully upgraded to determine precise resistive instability calculations in full toroidal geometry and for high-β, based on the asymptotic matching method. Stable convergence and robust solution behavior are verified even in an extreme case such as NSTX. For resistively unstable cases, the solutions and mode growth rates are successfully compared with MARS-F code. This new developement will enable us to address the resisitve stability boundary in advanced tokamaks such as ITER and also to improve theories and predictions of TMs or NTMs, and island opening dynamics, eliminating a cylindrical approximation with the classical tearing mode index.

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References

[1] FURTH, H. P., KILLEEN, J., and ROSENBLUTH, M. N., Phys. Fluds **4** (1963) 459.

[2] RUTHERFORD, P., Phys. Fluids **11** (1973) 1903.

[3] LA HAYE, R. J., Phys. Plasmas **13** (2006) 05501.

[4] SAUTER, O., LA HAYE, R. J., and ET. AL., C. Z., Phys. Plasmas **4** (1997) 1654.

[5] FITZPATRICK, R. and HENDER, T., Phys. Fluids B **3** (1991) 644.

[6] WAELBROECK, F. L., JOSEPH, I., NARDON, E., et al., Nucl. Fusion **52** (2012) 074004.

[7] PLETZER, A. and DEWAR, R. L., J. Plasma Phys. **45** (1991) 427.

[8] PLETZER, A., BONDESON, A., and DEWAR, R. L., J. Comp. Phys. **115** (1994) 530.

[9] LIU, Y. Q., BONDESON, A., and FRANSSON, C., Phys. Plasmas **7** (2000) 3681.

[10] GLASSER, A. H. and CHANCE, M. S., Bull. Am. Phys. Soc. **42** (1997) 1848.

[11] GLASSER, A. H., GREENE, J. M., and JOHNSON, J. L., Phys. Fluids **18** (1975) 875.

[12] GLASSER, A. H., JARDIN, S. C., and TESAURO, G., Phys. Fluids **27** (1984) 1225.

[13] NEWCOMB, W. A., Ann. Phys. **10** (1960) 232.

[14] BISHOP, C. M., CONNER, J. W., HASTIE, R. J., et al., Plasma Phys. Control. Fusion **33** (1991) 339.

[15] HEGNA, C., Phys. Plasmas **5** (1997) 1767.

[16] COLE, A. and FITZPATRICK, R., Phys. Plasmas **13** (2006) 932503.

[17] FITZPATRICK, R., Plasma Phys. Contol. Fusion **54** (2012) 094002.