NSTX-U

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The Use of DCON for Computation of Outer Region Matching Data for Singular MHD Modes in Axisymmetric Toroidal Plasmas

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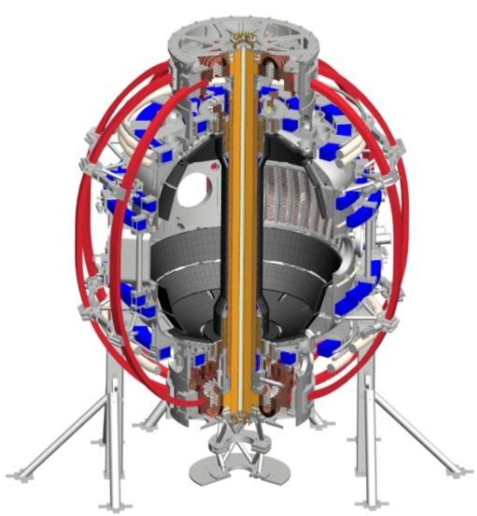

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Motivation

- DCON is widely used for computing the ideal MHD stability of axisymmetric torodial plasmas.

  It is thoroughly verified and validated, robust, reliable, easy to use.

- To calculate the outer region solution of singular MHD modes (e.g. tearing modes), **Resistive DCON** based on the asymptotic matching method allows non-vanishing large solutions.

  Asymptotic matching between inner and outer regions yields a dispersion relation for the complex growth rate and the associated eigenfunction.

- DCON based on the shooting method has been applied to determine outer (ideal) region matching conditions (Δ' – like quantities) for resistive and related singular modes. But despite many efforts, the matching data were noisy and unreliable.

  The cause of this problem has been identified: the shooting method is subject to a numerical instability due to non-vanishing large solutions on left and right of resonant surfaces.

- Resistive DCON replaces the shooting method by extending the finite element method developed by Pletzer & Dewar (1991) to compute the outer region matching data for toroidal plasmas.

  A very robust convergence of matching data is achieved even in a challenging NSTX plasmas by an improved choice of Galerkin basis function.

Outline

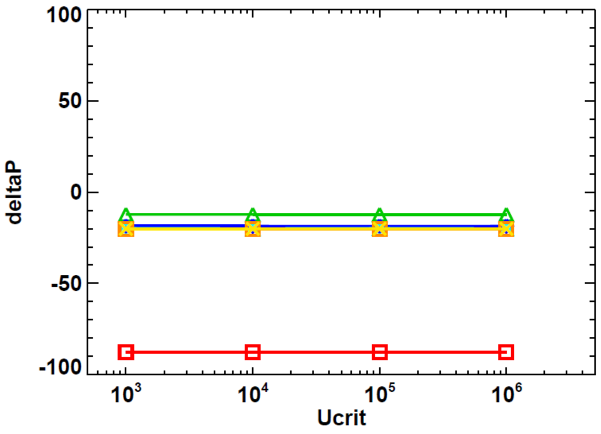
- Improved convergence of DCON in solving outer region solution
- Benchmark of eigenfunction between DCON and MARS-F(PEST 3)

  MARS-F: the single fluid resistive MHD equations
  no separation of outer region and inner layer.
- Resonant Galerkin method in newly developed DCON
- Summary

The Convergence Issue in Shooting Method While Solving Outer Region

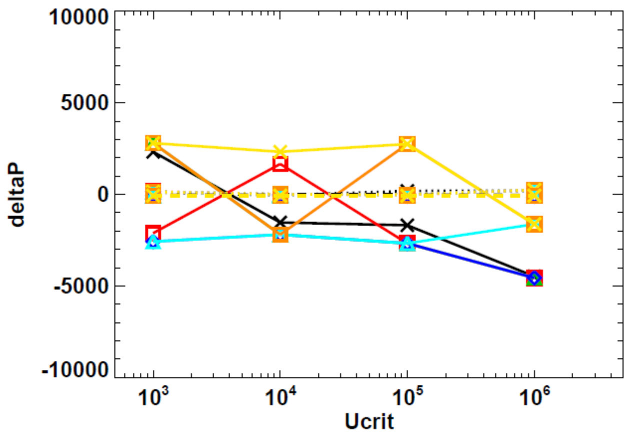
Shooting method is used to solve Δ' of n=1 tearing mode
Three tokamak equilibria are considered.

Circular tokamak case (A=3)



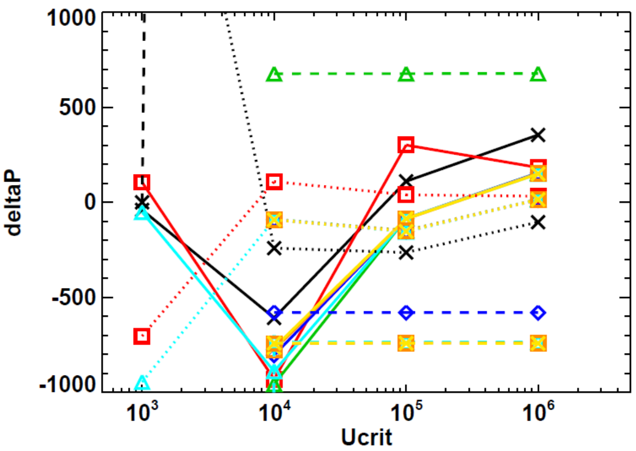
The convergence of Δ' is good.

DIII-D case: g097741.01605



The value of Δ' starts to diverge.

DIII-D case:g126006.03600



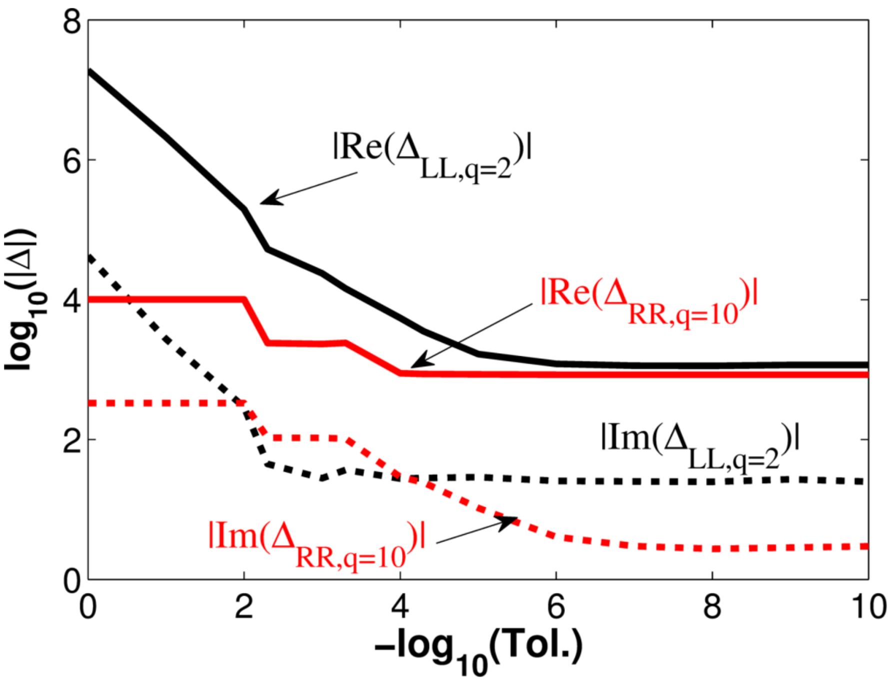
The convergence of Δ' value is bad.

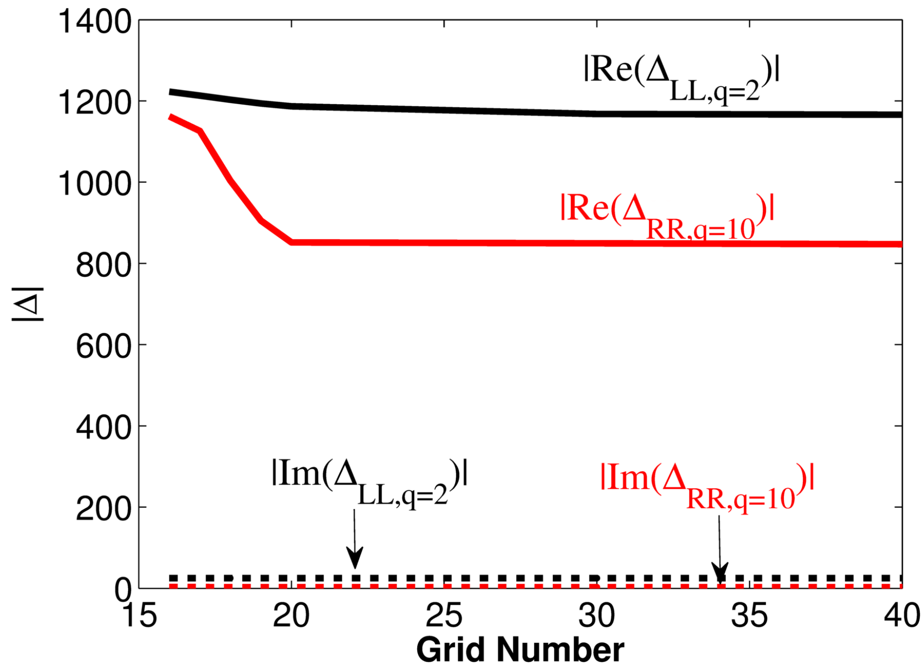
The value of Δ' becomes sensitive to the numerical parameters (fixing up criteria and integral tolerance) and diverge.

The Convergence of Matching Data in newly developed DCON

A very robust convergence of matching data is achieved in a challenging NSTX plasma.

The NSTX case (q_0=2.13, q_a=14.9, β_N=3.29) is studied

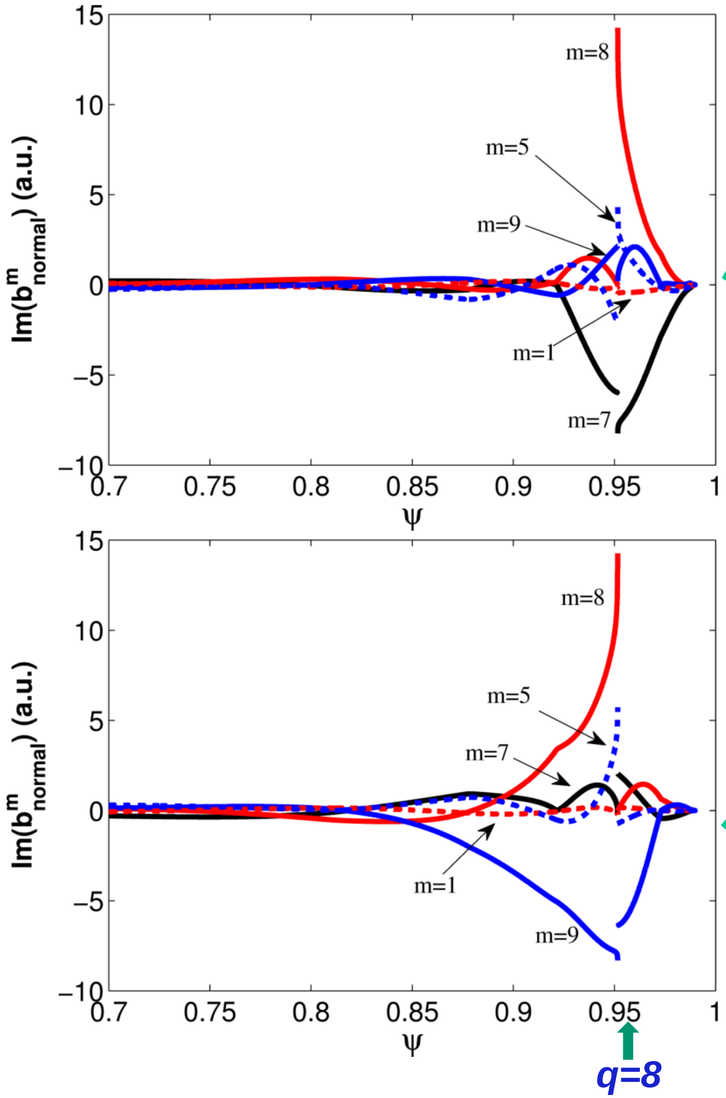




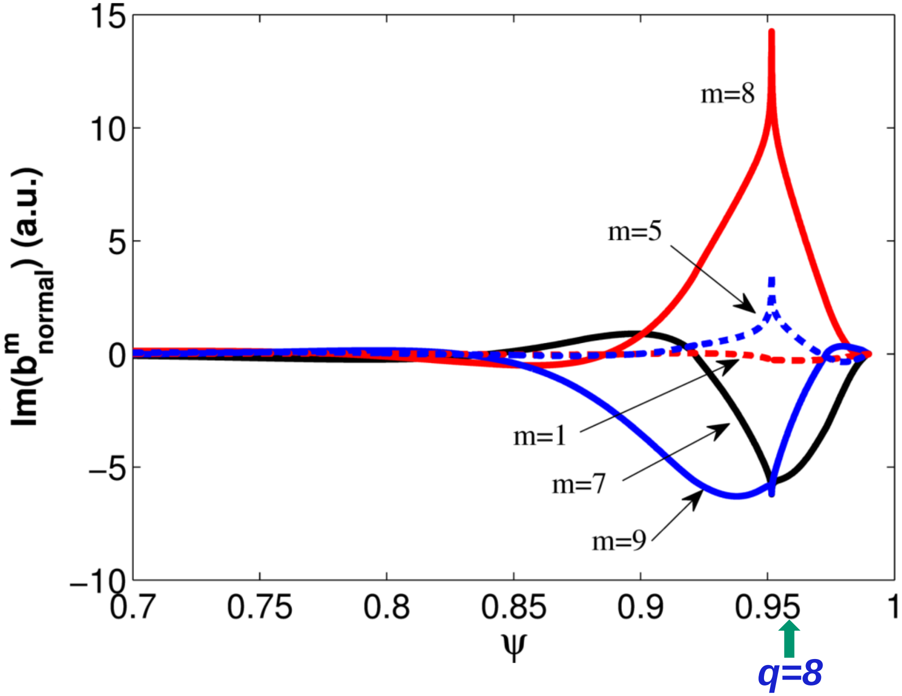
The scan of integral tolerance shows the stable convergence of Δ value.

A very quick convergence is observed due to the inclusion of the resonant element and Hermite cubic basis.

Construction of the Solution Driven by Large Solution at q=8 Resonant Surface in NSTX Plasmas



The strong coupling effect between resonant and non-resonant components can be observed.

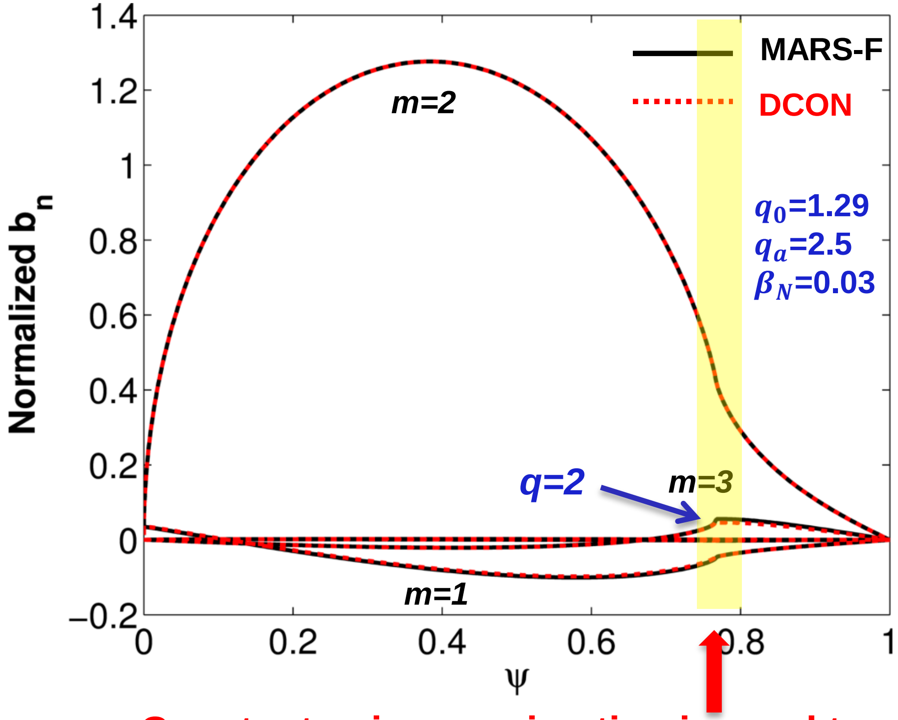


In the construction of solution, the constant ψ approximation is used. The freedom should be determined by matching with an inner layer model.

Benchmark between DCON and MARS-F

In low beta case, the outer region solution of DCON gets a good agreement with MARS-F result.

Low beta case: Circular Tokamak A=10



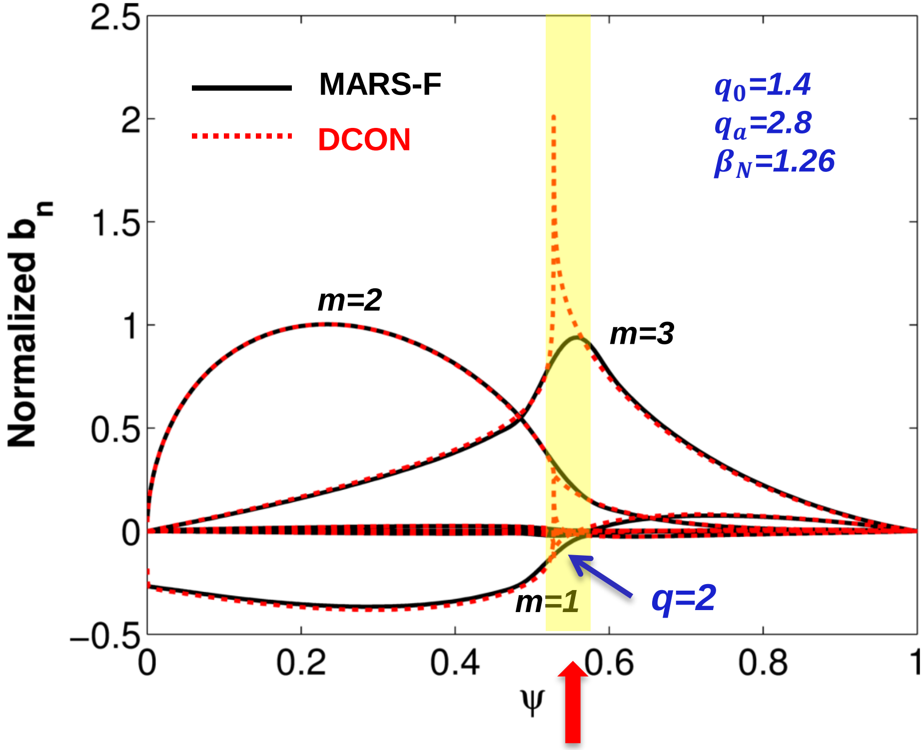
Constant psi approximation is used to construct the outer region solution in DCON.

- In low beta case, The constant approximation is valid in the construction of outer region solution in DCON.
- Mode coupling is weak due to low beta and large aspect ratio.
- The outer region solution of DCON gets a very good agreement with MARS-F result.

Benchmark between DCON and MARS-F

The eigenfunction calculated by DCON is compared with MARS-F result. DCON solves the outer region

High beta case: Circular Tokamak A=10



Solution near resonant surface should be resolved by inner layer model in DCON.

- In higher beta case, the outer region solution of DCON also gets a good agreement with MARS-F result in higher beta case.
- Mode coupling is strong due to high beta.
- Constant psi approximation failed in high beta case.
- The solution of DCON is reconstructed with an adjustable coefficients which should be determined by matching the outer and inner regions.

Formulation of DCON in Outer Region

In the approach of asymptotic matching method, the outer region plasma is modeled with zero-frequency ideal MHD.

$$\vec{j} \times \vec{B} + \vec{J} \times \vec{b} - \nabla p = 0$$

$$p = -\vec{\xi} \cdot \nabla P - \Gamma P \nabla \cdot \vec{\xi}$$

$$\vec{b} = \nabla \times (\vec{\xi} \times \vec{B})$$

$$\vec{j} = \nabla \times \vec{b}$$

Euler-Lagrange Equation

$$\mathbf{Lu} = -\left(\mathbf{Fu}' + \mathbf{Ku}\right)' + \left(\mathbf{K}^{\dagger}\mathbf{u}' + \mathbf{Gu}\right) = 0$$

$$\boldsymbol{u}(\psi) = \{\boldsymbol{\xi}_m(\psi) \mid m \in [m_{min}, m_{max}]\}$$

Fourier Representation of plasma displacement

$$\xi \cdot \nabla \psi(\psi, \theta, \zeta) = \sum_{m=m_{\text{low}}}^{m_{\text{high}}} \xi_m(\psi) e^{i(m\theta - n\zeta)}$$

Ordering: $\mathbf{F} \sim z^2$ $\mathbf{K} \sim \mathbf{K}^{\dagger} \sim z$ $\mathbf{G} \sim \mathbf{K}^{\dagger}\mathbf{F}^{-1}\mathbf{K} \sim 1$

$$z = \begin{cases} \psi - \psi_R & (\psi \geq \psi_R) \\ \psi_R - \psi & (\psi < \psi_R) \end{cases}$$

ψ_R is the resonant surface

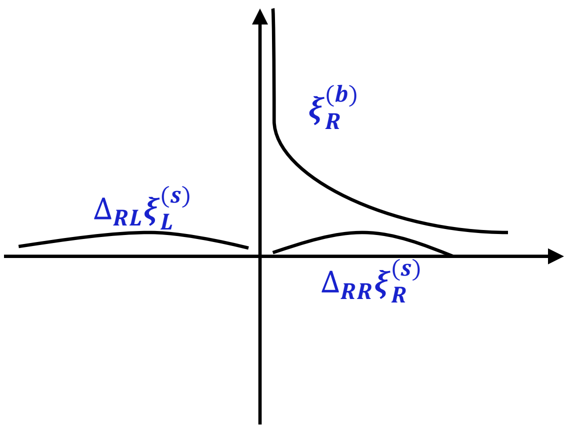
Idea of Solving Singular Outer Region Solution

- **Outer region allows the non-vanishing large solutions on left and right of resonant surfaces.**

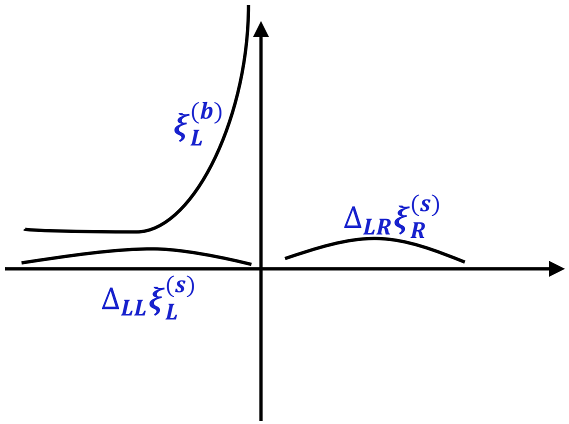
A simple case: plasma with one resonant surface in toroidal geometry

$$\xi_R = \xi_R^{(b)} + \Delta_{RR}\xi_R^{(s)} + \Delta_{RL}\xi_L^{(s)} + \xi_{reg,R}$$

$$\xi_L = \xi_L^{(b)} + \Delta_{LR}\xi_R^{(s)} + \Delta_{LL}\xi_L^{(s)} + \xi_{reg,L}$$



$$\xi = C_R\xi_R + C_L\xi_L$$



Large solution drives the response of small solution

(also the regular solution)

Δ_{LL}, Δ_{LR}, Δ_{RL} Δ_{RR} are the coefficients to match with inner layer solution.

C_R and C_L are also be determined by the matching.

Pletzer and Dewar, J. Plasma Physics (1991)

Idea of Solving Singular Outer Region Solution

Δ_{LL}, Δ_{LR}, Δ_{RL} Δ_{RR} can be easily converted to the conventional matching data.

$$\xi_R = \xi_R^{(b)} + \Delta_{RR}\xi_R^{(s)} + \Delta_{RL}\xi_L^{(s)} + \xi_{reg,R}$$

$$\xi_L = \xi_L^{(b)} + \Delta_{LR}\xi_R^{(s)} + \Delta_{LL}\xi_L^{(s)} + \xi_{reg,L}$$

→

$$\xi_+ = \xi_R + \xi_L = \xi_+^{(b)} + \frac{A'}{2}\xi_+^{(s)} + \frac{B'}{2}\xi_-^{(s)} + \xi_{reg,+}$$ **(even parity)**

$$\xi_- = \xi_R - \xi_L = \xi_-^{(b)} + \frac{\Gamma'}{2}\xi_+^{(s)} + \frac{\Delta'}{2}\xi_-^{(s)} + \xi_{reg,-}$$ **(odd parity)**

$$A' = \Delta_{RR} + \Delta_{RL} + \Delta_{LR} + \Delta_{LL}$$

$$B' = \Delta_{RR} - \Delta_{RL} + \Delta_{LR} - \Delta_{LL}$$

$$\Gamma' = \Delta_{RR} + \Delta_{RL} - \Delta_{LR} - \Delta_{LL}$$

$$\Delta' = \Delta_{RR} - \Delta_{RL} - \Delta_{LR} + \Delta_{LL}$$

← *Pletzer and Dewar, J. Plasma Physics (1991)* **PEST 3**

→ **Solved by DCON**

Solution with multiple resonant surface in toroidal geometry

$$\xi = \xi_R + \xi_L$$

$$\xi_R = \sum_i \left\{ \xi_R^{i(b)} + \sum_j \left[\xi_R^{j(s)}\Delta_{RR}^{ij} + \xi_L^{j(s)}\Delta_{RL}^{ij} \right] + \xi_{reg,R}^i \right\} C_R^i \qquad \xi_L = \sum_i \left\{ \xi_L^{i(b)} + \sum_j \left[\xi_R^{j(s)}\Delta_{LR}^{ij} + \xi_L^{j(s)}\Delta_{LL}^{ij} \right] + \xi_{reg,L}^i \right\} C_L^i$$

Numerical Formulation of DCON in Outer Region

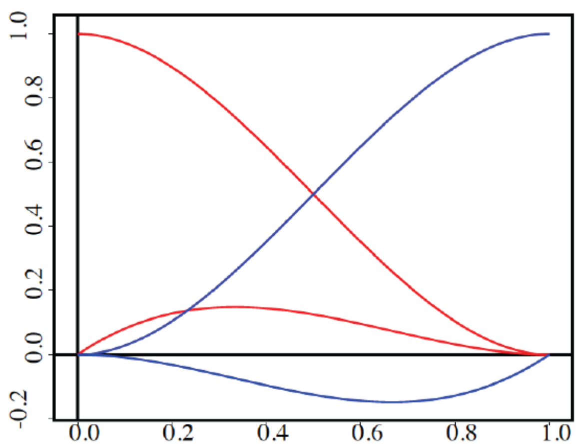
The plasma response driven by the large solution $u_l^{i(b)}$:

$$\mathbf{L\bar{u}} = -\left(\mathbf{F\bar{u}}' + \mathbf{K\bar{u}}\right)' + \left(\mathbf{K}^{\dagger}\mathbf{\bar{u}}' + \mathbf{G\bar{u}}\right) = \mathbf{r}$$

Driving term: $\mathbf{r} = -\mathbf{Lu}_l^{i(b)}$ $\quad$ $\mathrm{u}_\mathrm{l}^{\mathrm{i(b)}}$ is infinite at the resonant surface. $\mathrm{Lu}_\mathrm{l}^{\mathrm{i(b)}}$ can be finite.

Two-point boundary problem needs to be solved.

C[1] Hermite Cubics



- Cubic polynomials on (0,1)
- C^1 continuity: function values and first derivatives
- **Used for non-resonant solutions across the singular surface**.

Convergent Power Series Expansion

$$\mathbf{u} = z^p \sum_{n=0}^{N} \mathbf{u}^{(n)} z^n \qquad z = \begin{cases} \psi - \psi_r & (\psi \geq \psi_r) \\ \psi_r - \psi & (\psi < \psi_r) \end{cases}$$

$p = -\frac{1}{2} \pm \sqrt{-D_I}$ denotes the large $\boldsymbol{u}^{i(b)}$ and small $\boldsymbol{u}^{i(s)}$ resonant solutions at ψ_r^i.

$p = 0$, non-resonant powers.

Solved to arbitrarily high order N;

Automated using matrix formulation;

Essential for larger values of $|D_I|$;

Generally improves convergence.

Numerical Formulation of DCON in Outer Region

The plasma response driven by the large solution $u_l^{i(b)}$:

$$\mathbf{L}\bar{\mathbf{u}} = -\left(\mathbf{F}\bar{\mathbf{u}}' + \mathbf{K}\bar{\mathbf{u}}\right)' + \left(\mathbf{K}^{\dagger}\bar{\mathbf{u}}' + \mathbf{G}\bar{\mathbf{u}}\right) = \mathbf{r}$$

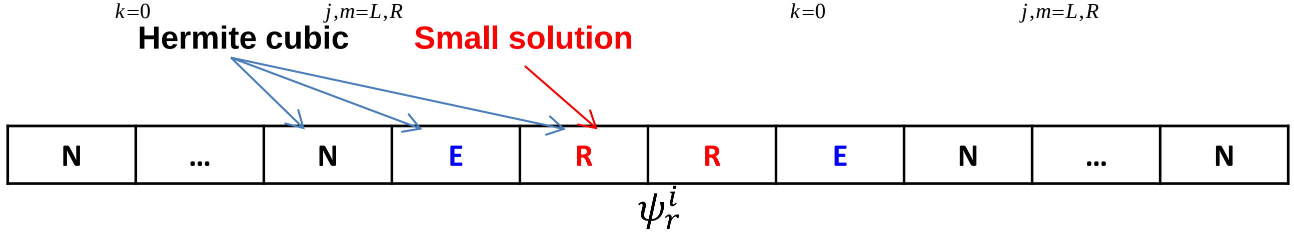
Variational Principle

$$W = \frac{1}{2}\left(\bar{\mathbf{u}}, \mathbf{L}\bar{\mathbf{u}}\right) - \left(\bar{\mathbf{u}}, \mathbf{r}\right)$$

$$\delta W = \left(\delta\bar{\mathbf{u}}, \mathbf{L}\bar{\mathbf{u}}\right) - \left(\delta\bar{\mathbf{u}}, \mathbf{r}\right) = 0$$

Resonant-Galerkin expansion

$$\mathbf{u}(\psi) = \sum_{k=0}^{p} \bar{\mathbf{u}}_k \alpha_k(\psi) + \sum_{j,m=L,R} \mathbf{u}_m^{j(s)}(\psi)\Delta_{lm}^{ij} \qquad \delta\mathbf{u}(\psi) = \sum_{k=0}^{p} \alpha_k(\psi)\delta\bar{\mathbf{u}}_k + \sum_{j,m=L,R} \mathbf{u}_m^{j(s)}(\psi)\delta\Delta_{lm}^{ij}$$



Extension element (E) connecting Resonant element (R) and Normal element (N) allows the resonant small solution smoothly vanishes.

Adjustable Grid Packing is applied to the interval between each two adjacent resonant surfaces.

Summary and Discussion

➢ **The resonant Galerkin method used in DCON for singular MHD calculation is developed by extending the method of Pletzer & Dewar (1991).**

- The method allows the non-vanishing large solutions on left and right of resonant surfaces.
- The Hermite cubic basis is used to impose the required C^1 continuity on the non-resonant solutions across the singular surfaces.
- The grid packing is carried out between each two adjacent resonant surfaces.
- The resonant small solutions are introduced as the extra basis on left and right of resonant surfaces. The solved coefficients give the matching data Δ value.

Discussion of resonant elements

- Weierstrass Convergence Theorem:

  Polynomial approximation uniformly convergent for analytic function.
- Large and small resonant solutions are non-analytic near the singular surface.
- Supplement polynomial basis with small resonant solution near singular surface.
- DCON fits equilibrium data to Fourier series and cubic splines, computes resonant power series to arbitrarily high order.
- Convergence requires that the large solution be computed to at least n=2*sqrt(di) terms. PEST 3 is limited to n=1.

Summary and Discussion

- Resistive DCON with the fixed boundary condition has been developed
- The code shows an excellent convergence of matching data by varying various parameters in high beta NSTX case.
- The eigenfunction computed by DCON and MARS-F shows a very good agreement.

Next step of work

Match the outer region and inner layer solution (GGJ, DELTAR) to get the mode growth rate.

Further benchmark with MARS-F and PEST 3.

Include the vacuum and coils for free boundary calculation to study resistive stability and resistive perturbed equilibrium.

Backup Slides

Numerical Formulation of DCON in Outer Region

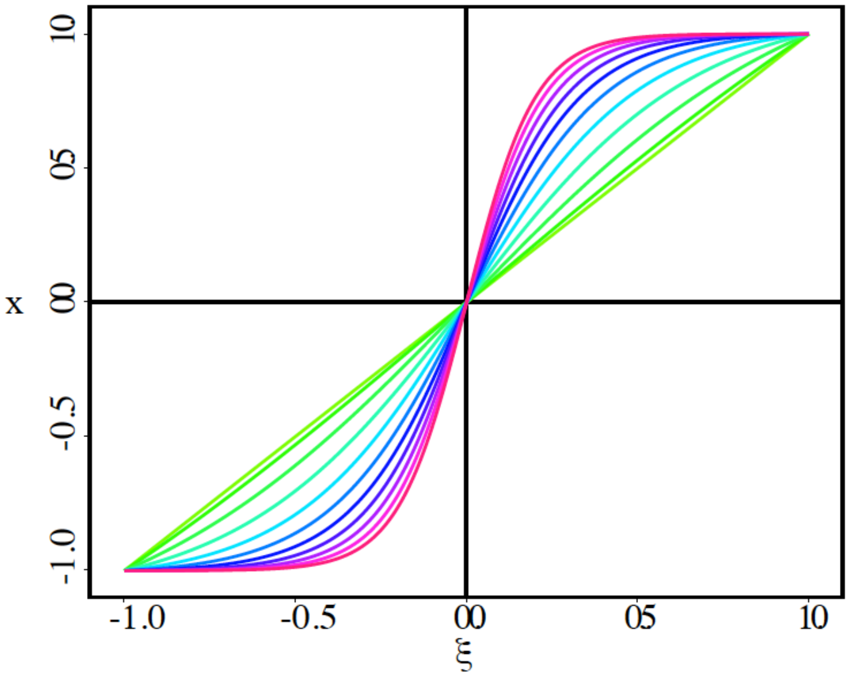
Adjustable Grid Packing is applied to each interval between two adjacent resonant surfaces.

Grid Packing Function

$$\lambda(a) = \coth a = \frac{e^a + 1}{e^a - 1}, \quad a(\lambda) = \text{acoth}\,\lambda = \ln\left(\frac{1+\lambda}{1-\lambda}\right)$$

$$x(\xi, \lambda) = \frac{\tanh a\xi}{\lambda} = \frac{1}{\lambda}\left(\frac{e^{a\xi} - 1}{e^{a\xi} + 1}\right)$$

$$\lim_{\lambda \to 0} a(\lambda) = 2\lambda, \quad \lim_{\lambda \to 0} x(\xi, \lambda) = \xi$$



Center and Edge Grid Densities

$$\frac{\partial x}{\partial \xi} = \frac{1}{\lambda}\frac{2ae^{a\xi}}{(e^{a\xi}+1)^2} = \frac{1}{\lambda}\frac{2ae^{-a\xi}}{(e^{-a\xi}+1)^2}$$

$$\left.\frac{\partial x}{\partial \xi}\right|_{\xi=0} = \frac{a}{2\lambda}$$

$$\left.\frac{\partial x}{\partial \xi}\right|_{\xi=\pm 1} = \frac{a}{2\lambda}(1-\lambda^2)$$

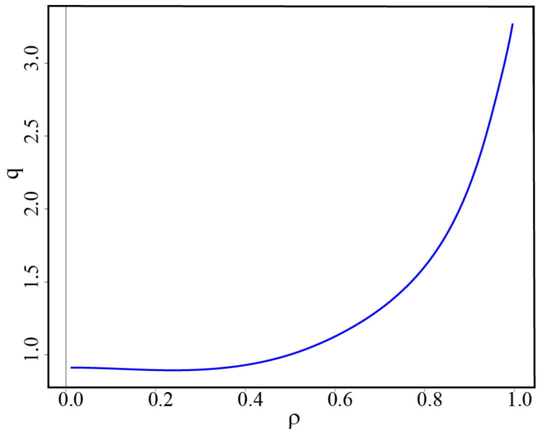
Packing Ratio

$$P(\lambda) \equiv \frac{\partial x/\partial \xi|_{\xi=\pm 1}}{\partial x/\partial \xi|_{\xi=0}} = 1 - \lambda^2$$

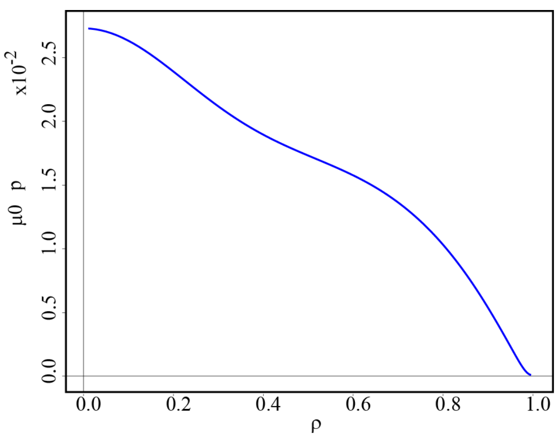
The Convergence Issue in Shooting Method

D-IIID Equilibrium
Dylan Brennan

Safety Factor



Plasma Pressure



ucrit	*tol*	*Δ'(2)*	*Δ'(3)*	*Δ'(4)*
1.000E+03	1.000E-10	-3.659E+03	-3.976E-01	-3.841E+01
1.000E+03	1.000E-11	-3.659E+03	-3.978E-01	-3.841E+01
1.000E+03	1.000E-12	-3.659E+03	-3.976E-01	-3.841E+01
1.000E+03	1.000E-13	-3.659E+03	-3.976E-01	-3.841E+01
1.000E+03	1.000E-14	-3.659E+03	-3.975E-01	-3.841E+01
1.000E+04	1.000E-10	-2.548E+03	-1.851E+01	-3.904E+01
1.000E+04	1.000E-11	-3.664E+03	-9.442E-02	-3.843E+01
1.000E+04	1.000E-12	-3.575E+03	-1.670E+00	-3.843E+01
1.000E+04	1.000E-13	-3.676E+03	-7.744E-01	-3.843E+01
1.000E+04	1.000E-14	-3.485E+03	-5.534E+00	-3.844E+01
1.000E+05	1.000E-10	-3.764E+03	8.039E+00	-3.827E+01
1.000E+05	1.000E-11	-3.483E+03	-4.151E+00	-3.856E+01
1.000E+05	1.000E-12	-3.906E+03	1.155E+01	-3.815E+01
1.000E+05	1.000E-13	-3.863E+03	4.782E+00	-3.822E+01
1.000E+05	1.000E-14	-3.802E+03	8.489E+00	-3.830E+01
1.000E+06	1.000E-10	-2.792E+03	-3.093E+00	-3.847E+01
1.000E+06	1.000E-11	-2.031E+03	-1.841E+00	-3.845E+01
1.000E+06	1.000E-12	-2.808E+03	-3.347E+00	-3.845E+01
1.000E+06	1.000E-13	-2.674E+03	-2.794E+00	-3.844E+01
1.000E+06	1.000E-14	-2.825E+03	-3.240E+00	-3.846E+01
1.000E+07	1.000E-10	-2.595E+03	-2.305E+01	2.569E+02
1.000E+07	1.000E-11	-2.740E+03	-4.291E+01	-3.330E+01
1.000E+07	1.000E-12	-2.706E+03	-2.044E+01	-4.040E+01
1.000E+07	1.000E-13	-3.003E+03	-1.384E+01	-7.113E+01
1.000E+07	1.000E-14	-3.043E+03	-4.272E+01	-7.463E+01

Bad convergence

Integral Representation

$$\mathbf{u} = \check{\mathbf{u}} + \hat{\mathbf{u}}$$

$$\mathsf{L}\mathbf{u} = \mathsf{L}(\check{\mathbf{u}} + \hat{\mathbf{u}}) = 0$$

$$(\mathbf{u}, \mathsf{L}\mathbf{u}) = (\check{\mathbf{u}}, \mathsf{L}\check{\mathbf{u}}) + (\check{\mathbf{u}}, \mathsf{L}\hat{\mathbf{u}}) + (\hat{\mathbf{u}}, \mathsf{L}\check{\mathbf{u}}) + (\hat{\mathbf{u}}, \mathsf{L}\hat{\mathbf{u}}) = 0$$

$$\begin{aligned}(\hat{\mathbf{u}}, \mathsf{L}\check{\mathbf{u}}) - (\check{\mathbf{u}}, \mathsf{L}\hat{\mathbf{u}}) &= [\![P(\hat{\mathbf{u}}, \check{\mathbf{u}}|\psi)]\!] \\ &= [\![\hat{\mathbf{u}}'^{\dagger}\mathsf{F}\check{\mathbf{u}} - \hat{\mathbf{u}}^{\dagger}\mathsf{F}\check{\mathbf{u}}']\!] \\ &= -2\mu n^2 q_0'^2 \bar{F}_{0rr}\Delta\end{aligned}$$

$$\begin{aligned}2\mu n^2 q_0'^2 \bar{F}_{0rr}\Delta &= (\check{\mathbf{u}}, \mathsf{L}\hat{\mathbf{u}}) - (\hat{\mathbf{u}}, \mathsf{L}\check{\mathbf{u}}) \\ &= (\check{\mathbf{u}}, \mathsf{L}\check{\mathbf{u}}) + 2(\check{\mathbf{u}}, \mathsf{L}\hat{\mathbf{u}}) + (\hat{\mathbf{u}}, \mathsf{L}\hat{\mathbf{u}})\end{aligned}$$

Agree with Pletzer & Dewar result (1991)

Uniqueness of Solution

$$\hat{\mathbf{u}} \rightarrow \hat{\mathbf{u}} + \mathbf{v}, \quad \check{\mathbf{u}} \rightarrow \check{\mathbf{u}} - \mathbf{v}, \quad \mathbf{v} \sim o(\mathbf{u}_s) \in \mathcal{H}$$

$$\begin{aligned} 2\mu n^2 q_0'^2 \bar{F}_{0rr}\Delta =& (\check{\mathbf{u}} - \mathbf{v}, \mathsf{L}\check{\mathbf{u}} - \mathbf{v}) + 2(\check{\mathbf{u}} - \mathbf{v}, \mathsf{L}\hat{\mathbf{u}} + \mathbf{v}) \\ &+ (\hat{\mathbf{u}} + \mathbf{v}, \mathsf{L}\hat{\mathbf{u}} + \mathbf{v}) \\ =& (\check{\mathbf{u}}, \mathsf{L}\check{\mathbf{u}}) + 2(\check{\mathbf{u}}, \mathsf{L}\hat{\mathbf{u}}) + (\hat{\mathbf{u}}, \mathsf{L}\hat{\mathbf{u}}) \\ &- 2(\check{\mathbf{u}}, \mathsf{L}\mathbf{v}) + 2(\check{\mathbf{u}}, \mathsf{L}\mathbf{v}) \\ &- 2(\mathbf{v}, \mathsf{L}\hat{\mathbf{u}}) + (\mathbf{v}, \mathsf{L}\hat{\mathbf{u}}) + (\hat{\mathbf{u}}, \mathsf{L}\mathbf{v}) \\ &+ (\mathbf{v}, \mathsf{L}\mathbf{v}) - 2(\mathbf{v}, \mathsf{L}\mathbf{v}) + (\mathbf{v}, \mathsf{L}\mathbf{v}) \end{aligned}$$

$$(\hat{\mathbf{u}}, \mathsf{L}\mathbf{v}) - (\mathbf{v}, \mathsf{L}\hat{\mathbf{u}}) = [\![P(\hat{\mathbf{u}}, \mathbf{v}|\psi)]\!] = 0$$

$$2\mu n^2 q_0'^2 \bar{F}_{0rr}\Delta = (\check{\mathbf{u}}, \mathsf{L}\check{\mathbf{u}}) + 2(\check{\mathbf{u}}, \mathsf{L}\hat{\mathbf{u}}) + (\hat{\mathbf{u}}, \mathsf{L}\hat{\mathbf{u}})$$

Convergence of Resonant Galerkin Method

$$P'(\mathbf{u},\mathbf{v}\,|\,\psi) = \mathbf{u}^\dagger(\mathbf{L}\mathbf{v}) - (\mathbf{L}\mathbf{u})^\dagger\,\mathbf{v}$$

$$P(\mathbf{u},\mathbf{v}\,|\,\psi) = \left(\mathbf{u}^\dagger{}'\mathbf{F}\mathbf{v} - \mathbf{u}^\dagger\mathbf{F}\mathbf{v}'\right) + \mathbf{u}^\dagger\left(\mathbf{K}^\dagger - \mathbf{K}\right)\mathbf{v} + const$$

$$(\mathbf{a},\mathbf{b}) = \int \mathbf{a}^\dagger\mathbf{b}\,dx$$

Integral over one interval.

$$\int P'(\mathbf{u}_b,\bar{\mathbf{u}}\,|\,\psi)\,d\psi = \int d\psi\left[\mathbf{u}_b^\dagger(\mathbf{L}\bar{\mathbf{u}}) - (\mathbf{L}\mathbf{u}_b)^\dagger\,\bar{\mathbf{u}}\right]$$

$$(\mathbf{u}_b,\mathbf{L}\bar{\mathbf{u}}) - (\mathbf{L}\mathbf{u}_b,\bar{\mathbf{u}}) = \int P'(\mathbf{u}_b,\bar{\mathbf{u}}\,|\,\psi)\,d\psi$$

$$(\mathbf{u}_b,\mathbf{L}\bar{\mathbf{u}}) - (\bar{\mathbf{u}},\mathbf{L}\mathbf{u}_b)^\dagger = \int P'(\mathbf{u}_b,\bar{\mathbf{u}}\,|\,\psi)\,d\psi$$

$$(\mathbf{u}_b,\mathbf{L}\bar{\mathbf{u}}) - (\bar{\mathbf{u}},\mathbf{L}\mathbf{u}_b)^\dagger = \left(\mathbf{u}_b^\dagger{}'\mathbf{F}\bar{\mathbf{u}} - \mathbf{u}_b^\dagger\mathbf{F}\bar{\mathbf{u}}'\right)|_{\psi_s} + \mathbf{u}_b^\dagger\left(\mathbf{K}^\dagger - \mathbf{K}\right)\bar{\mathbf{u}}\,|_{\psi_s} + const - const$$

$$(\mathbf{u}_b,\mathbf{L}\bar{\mathbf{u}}) - (\bar{\mathbf{u}},\mathbf{L}\mathbf{u}_b)^\dagger = \left(\mathbf{u}_b^\dagger{}'\mathbf{F}\bar{\mathbf{u}} - \mathbf{u}_b^\dagger\mathbf{F}\bar{\mathbf{u}}'\right)|_{\psi_s} + \mathbf{u}_b^\dagger\left(\mathbf{K}^\dagger - \mathbf{K}\right)\bar{\mathbf{u}}\,|_{\psi_s}$$

$$\bar{\mathbf{u}} = \Delta\mathbf{u}_s + \mathbf{u}_h$$

$$(\mathbf{u}_b,\mathbf{L}\bar{\mathbf{u}}) - (\bar{\mathbf{u}},\mathbf{L}\mathbf{u}_b)^\dagger = \left[\mathbf{u}_b^\dagger{}'\mathbf{F}(\Delta\mathbf{u}_s + \mathbf{u}_h) - \mathbf{u}_b^\dagger\mathbf{F}(\Delta\mathbf{u}_s + \mathbf{u}_h)'\right]|_{\psi_s} + \mathbf{u}_b^\dagger\left(\mathbf{K}^\dagger - \mathbf{K}\right)(\Delta\mathbf{u}_s + \mathbf{u}_h)|_{\psi_s}$$

$$(\mathbf{u}_b,\mathbf{L}\bar{\mathbf{u}}) - (\bar{\mathbf{u}},\mathbf{L}\mathbf{u}_b)^\dagger = \Delta\left(\mathbf{u}_b^\dagger{}'\mathbf{F}\mathbf{u}_s - \mathbf{u}_b^\dagger\mathbf{F}\mathbf{u}_s' + \mathbf{u}_b^\dagger\mathbf{K}^\dagger\mathbf{u}_s - \mathbf{u}_b^\dagger\mathbf{K}\mathbf{u}_s\right)|_{\psi_s} + \left(\mathbf{u}_b^\dagger{}'\mathbf{F}\mathbf{u}_h - \mathbf{u}_b^\dagger\mathbf{F}\mathbf{u}_h' + \mathbf{u}_b^\dagger\mathbf{K}^\dagger\mathbf{u}_h - \mathbf{u}_b^\dagger\mathbf{K}\mathbf{u}_h\right)|_{\psi_s}$$

MARS-F Formulation

$$(\frac{\partial}{\partial t} + in\Omega)\vec{\xi} = \vec{v} + (\vec{\xi} \cdot \nabla\Omega)R^2\nabla\phi,$$

$$\rho(\frac{\partial}{\partial t} + in\Omega)\vec{v} = -\nabla p + \vec{j} \times \vec{B} + \vec{J} \times \vec{b} - \rho[2\Omega\vec{Z} \times \vec{v} + (\vec{v} \cdot \nabla\Omega)R\hat{\phi}] - \nabla \cdot (\rho\vec{\xi})\Omega\vec{Z} \times \vec{V}_0,$$

$$(\frac{\partial}{\partial t} + in\Omega)\vec{b} = \nabla \times (\vec{v} \times \vec{B}) + (\vec{b} \cdot \nabla\Omega)R\hat{\phi} - \nabla \times (\eta\vec{j}),$$

$$(\frac{\partial}{\partial t} + in\Omega)p = -\vec{v} \cdot \nabla P - \Gamma P \nabla \cdot \vec{v},$$

$$\vec{j} = \nabla \times \vec{b},$$