# Analysis and implementation of nonlinear radio-frequency sheaths

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**Special thanks to N. Bertelli, S. Shiraiwa, E. H. Kim, and M. Ono**

PPPL

# Motivations

- Ion Cyclotron Radio Frequency (ICRF) antennas are planned to play roles in future fusion experiments including ITER and SPARC

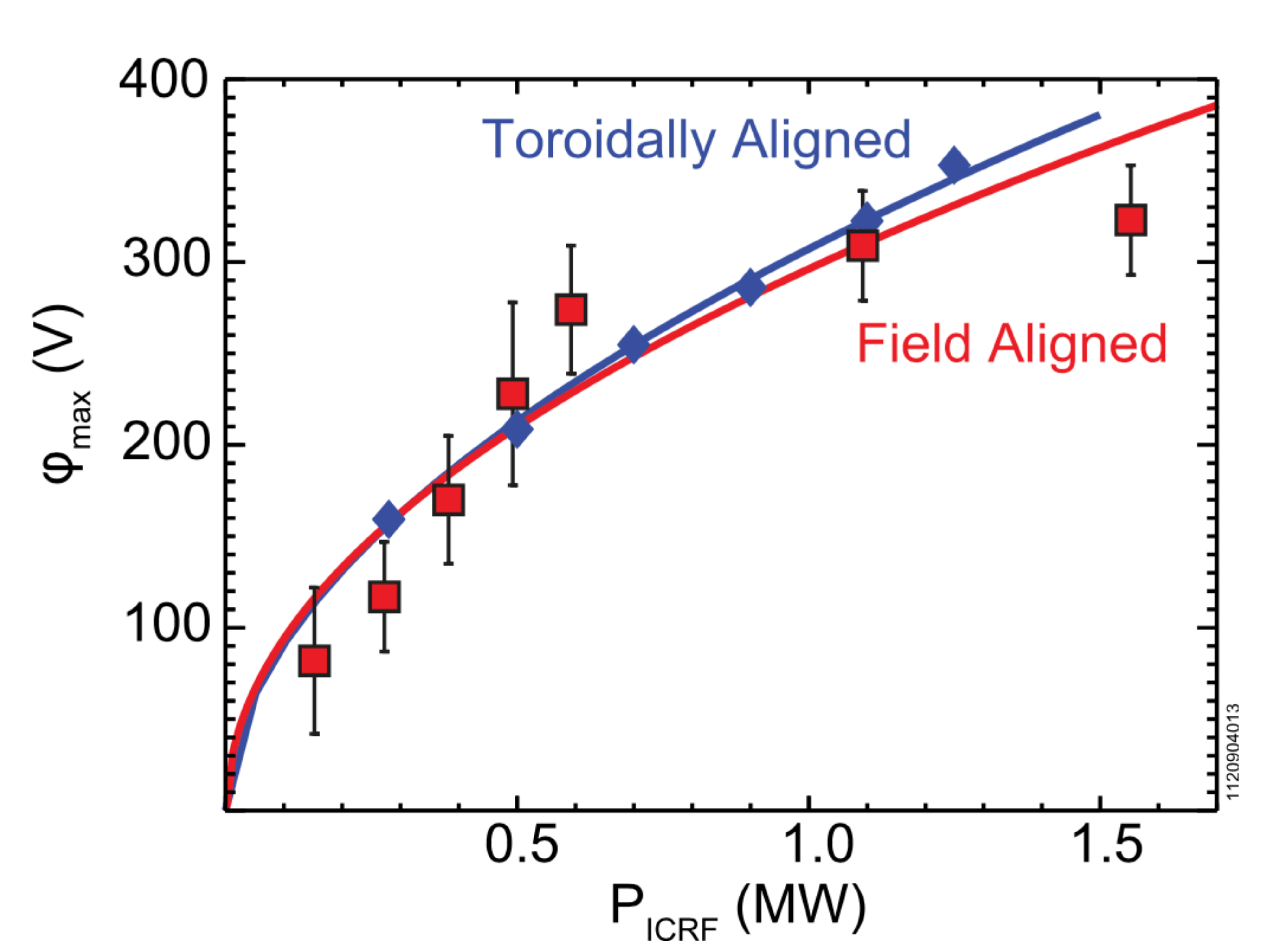
- It has been observed that ICRF power leads to the generation of impurities via RF sheath rectification; data is available from previous and ongoing experiments: C-Mod and WEST

- First principle understanding of RF sheath rectification and its implications are topics of active research

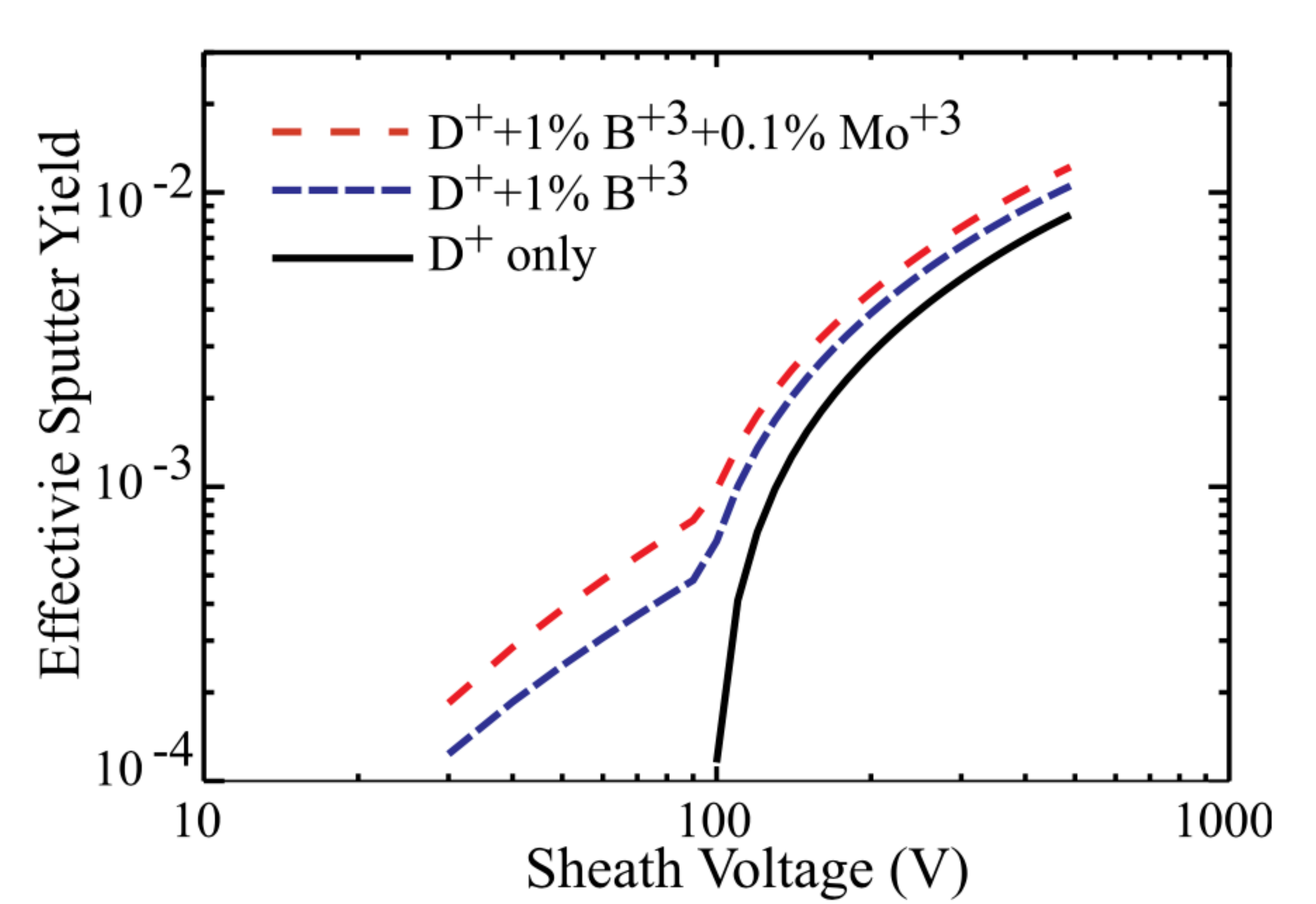
- Goal of this talk: to elucidate the basic physics underlying RF sheath—plasma interactions and to develop the tools necessary to predict and model RF sheath rectification and the generation of impurities due to ICRF antennas

# Motivations

Measured sheath voltage rectification due to ICRF antenna in C-Mod




Molybdenum sputtering yield



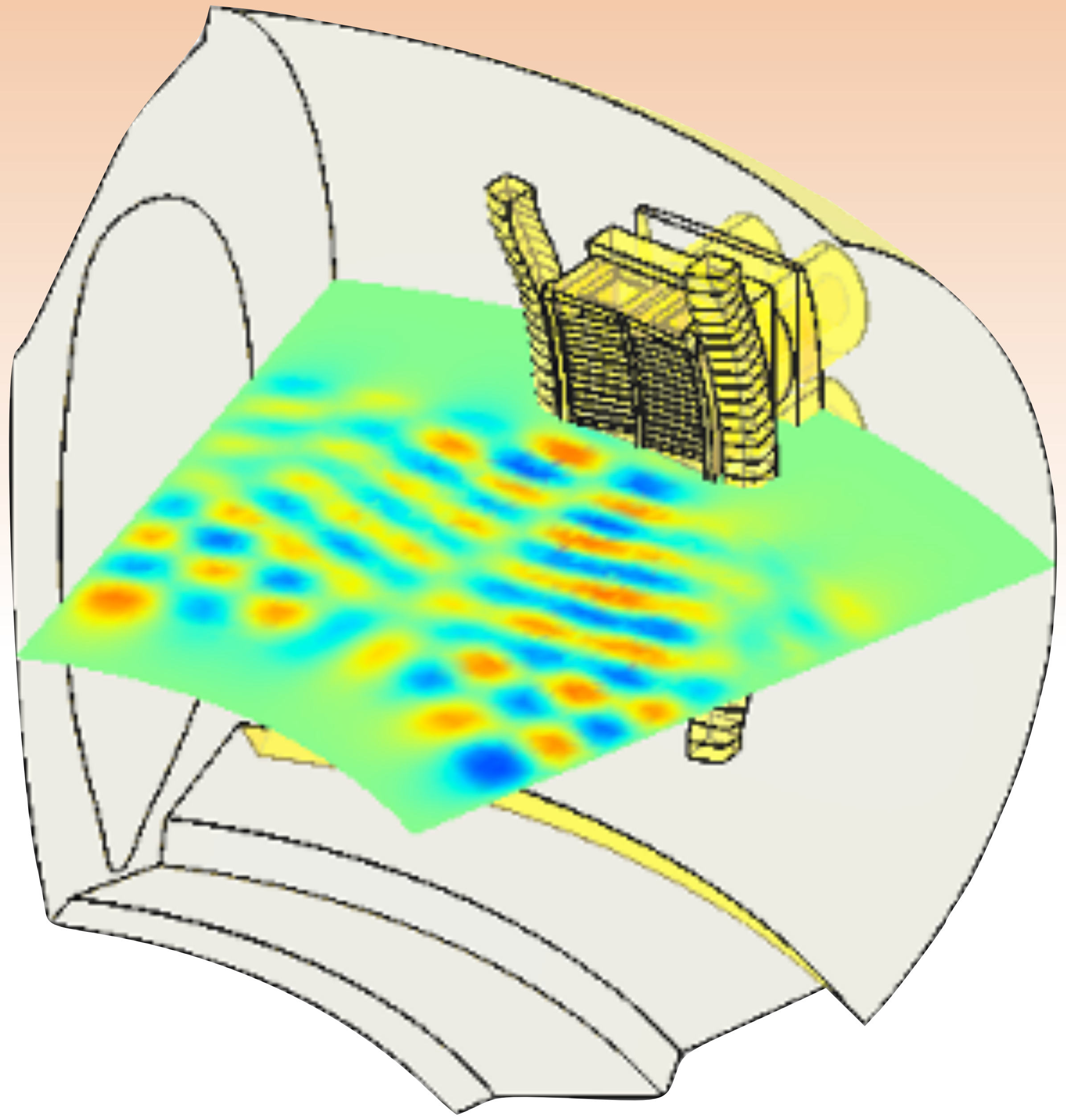

ICRF power leads to sheath rectification → Rectified sheath voltages lead to sputtering

S. J. Wukitch, M. L. Garrett, R. Ochoukov, et al. *Phys. Plasmas* **20**, 056117 (2013); https://doi.org/10.1063/1.4803882

# Motivations

RF actuator modeling should include

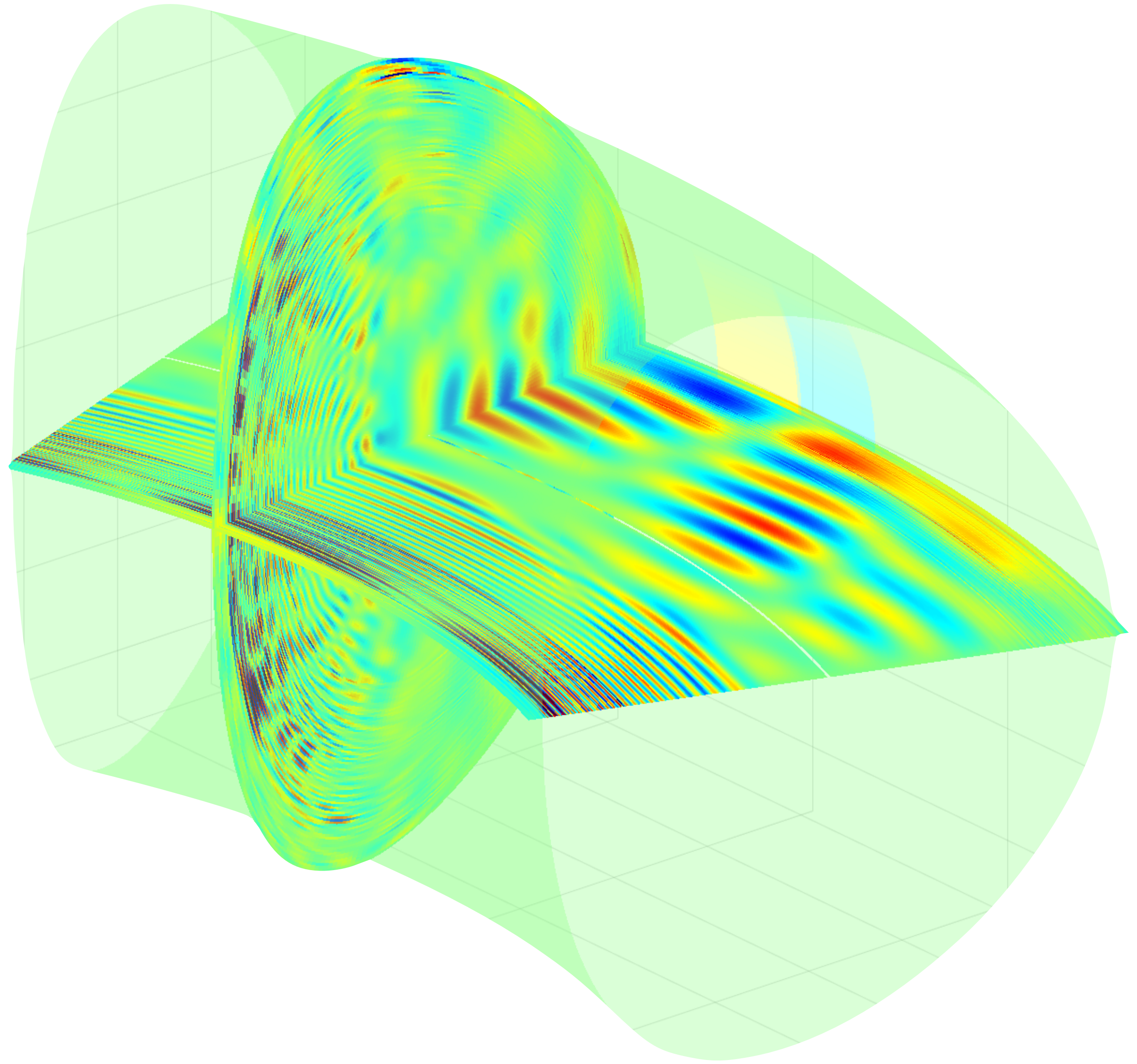
- **High-fidelity modeling of realistic antenna/wall geometry**
  - Local FEM elements
  - CAD antenna geometry
  - Assumes cold plasma in SOL



Petra-M
FEM code

WEST ICRH

- **Nonlocal kinetic response in core plasma**
  - Flux surface geometry
  - Toroidal/poloidal Fourier decomposition
  - Nonlocal Landau/Cyclotron damping



Toric
Kinetic solver
in flux geometry

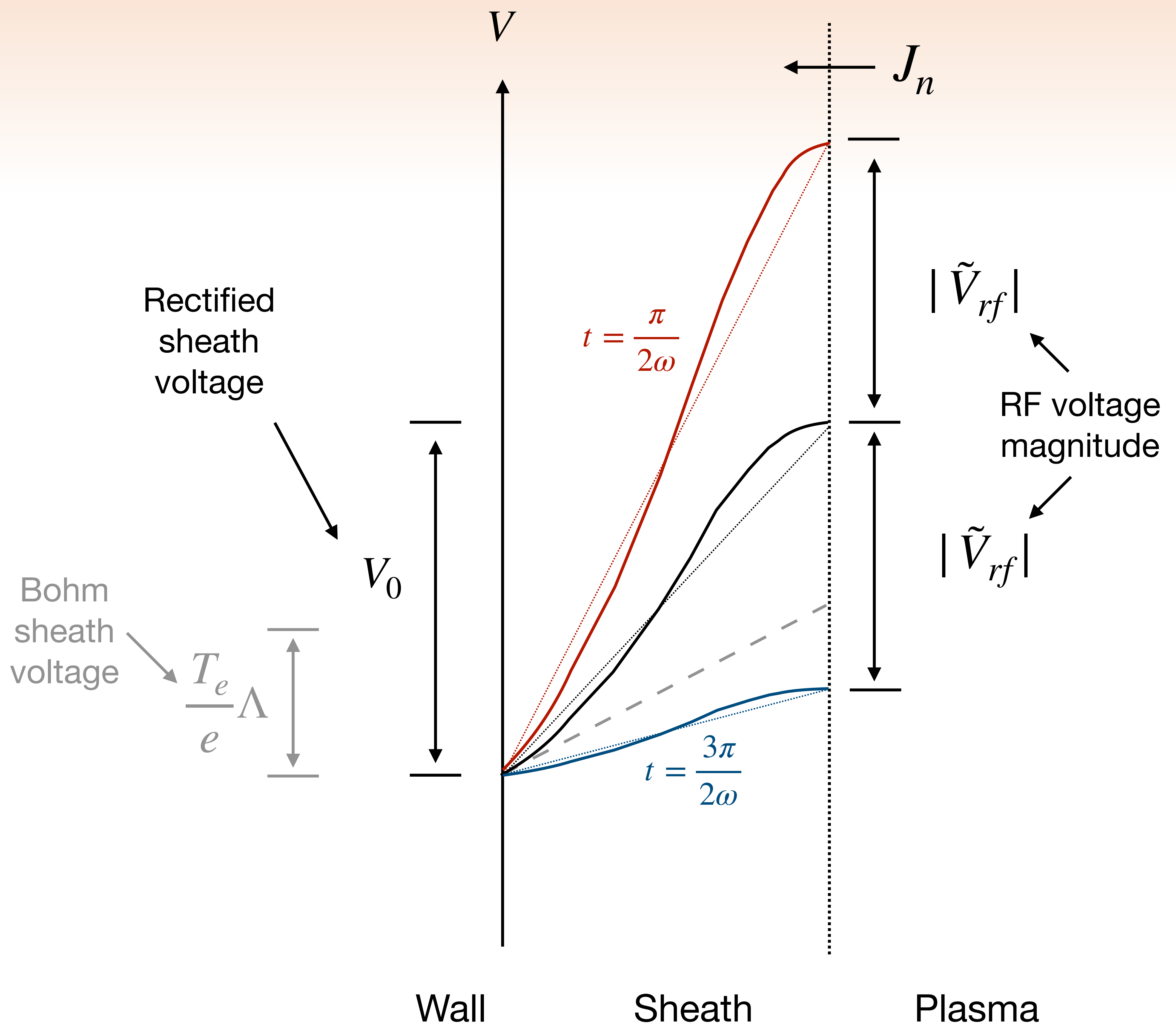
- **Incorporation of nonlinear RF sheath boundary physics**

# Outline

- Motivations
- **RF sheath physics and boundary conditions**
- Three-dimensional implementation in plane-stratified geometry and emergent problems with fixed point iteration schemes
- Analytic results and nonlinear features
  - Unbounded half-space: multivaluedness, stability, and hysteresis
  - Bounded domain: spontaneous symmetry breaking
- Conclusions

# RF sheath rectification

1D schematic of RF sheath




The total current normal to the sheath takes the form

$$J_n = j_i + j_e + \frac{1}{4\pi}\frac{\partial E_n}{\partial t}$$

$$\approx enc_s\left(1 - \exp\left(\Lambda - e(V_0 + V_{rf}(t))/T_e\right)\right) + \frac{1}{4\pi}\frac{\partial E_n}{\partial t}$$

Requiring the DC current to vanish yields the rectified voltage

$$\langle J_n \rangle = 0 \quad \Rightarrow \quad V_0 = \frac{T_e}{e}\left\{\Lambda + \ln\left[I_0\left(e|\tilde{V}_{rf}|/T_e\right)\right]\right\} \approx \Lambda T_e/e + |\tilde{V}_{rf}|$$

The Fourier component of the current density corresponding with the RF frequency is approximately given by

$$\tilde{J}_n = y_{sh}\tilde{V}_{rf}$$

where $y_{sh}$ is a nonlinear complex RF sheath admittance.

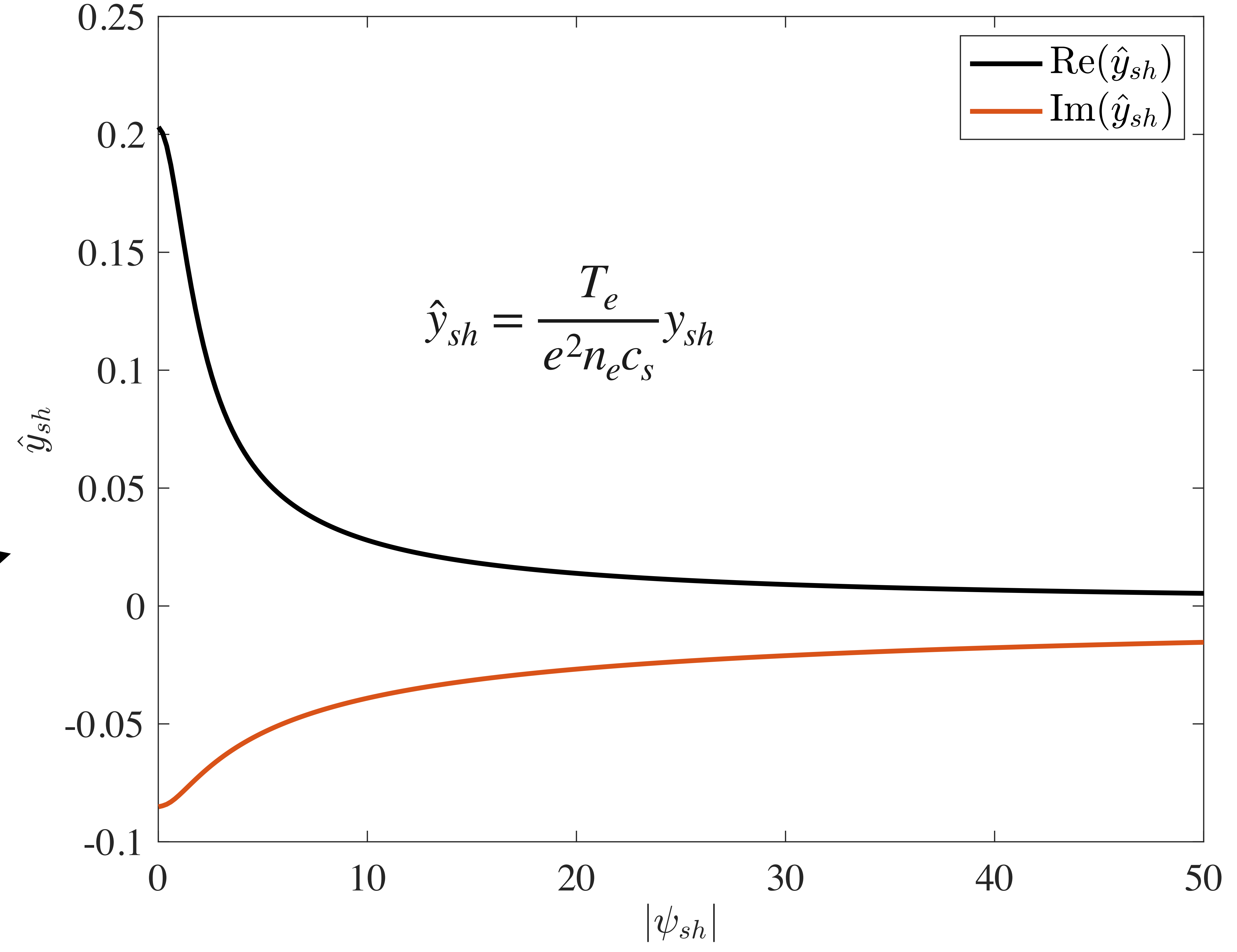
# Nonlinear RF sheath admittance

RF sheath admittance relates the local values of the oscillating total current $J_n$ (particle + displacement) normal to the boundary and the RF sheath potential $V_{sh}$

$$J_n = y_{sh} V_{sh}$$

| | | |
|---|---|---|
| $B_x = 0.3\ T$ | $B_y = 0\ T$ | $B_z = 2\ T$ |
| $n_e = 1 \times 10^{18}\ m^{-3}$ | $T_e = 15\ eV$ | $f = 80\ MHz$ |

Real and imaginary parts of nonlinear sheath admittance for varying sheath potential




$\psi_{sh} = eV_{sh}/T_e$

JR Myra, Phys. Plasmas **24**, 072507 (2017)

# RF sheath boundary conditions

In frequency space, Maxwell's equation (c.g.s. units) imply the wave equation

$$\nabla \times \nabla \times \vec{E} - k_0^2 \overleftrightarrow{\epsilon} \cdot \vec{E} = i k_0 \frac{4\pi}{c} \vec{j}_{ext}$$

On the sheath boundaries, the tangential components of the electric field satisfy

$$\vec{E}_t = - \nabla_t V_{sh}$$

The complex sheath potential $V_{sh}$ is locally constrained by the RF sheath boundary condition

$$y_{sh} V_{sh} = J_n = \frac{\omega}{4\pi i} \hat{n} \cdot \overleftrightarrow{\epsilon} \cdot \vec{E}$$

($\hat{n}$ is the unit vector pointing out of the plasma)

- $J_n$ : the total RF current density (particle + displacement) normal to the boundary
- $y_{sh}$ : the sheath admittance, a nonlinear function of the absolute value of the local sheath potential $|V_{sh}|$, the RF frequency $\omega$, and the local boundary plasma parameters (i.e., $y_{sh} = y_{sh}(|V_{sh}|, \omega, \vec{B}_0, n_e, T_e)$).

# Consequences of linearity

Maxwell's equations provide a <u>linear</u> relationship between the boundary values of the RF sheath potential and the total RF current density normal to the boundary

$$J_n(\vec{r}_t) = \int_{\partial\mathscr{D}} d^2r'_t \; y_p(\vec{r}_t, \vec{r}'_t) V_{sh}(\vec{r}'_t) + J_n^{ext}(\vec{r}_t)$$

- the source $J_n^{ext}(\vec{r}_t)$ is the total RF current density normal to the boundary when conducting walls are imposed
    - scales linearly with the external antenna current

- the integral kernel $y_p(\vec{r}_t, \vec{r}'_t)$ is termed the plasma wave admittance
    - depends only on zeroth-order plasma and geometry

Both $y_p(\vec{r}_t, \vec{r}'_t)$ and $J_n^{ext}(\vec{r}_t)$ can be obtained in arbitrary geometry using a FEM code.
Analytic solutions exist in special cases.

# Self-consistent equation for the sheath potential

- The RF sheath boundary condition gives nonlinear, local condition

$$J_n(\vec{r}_t) = y_{sh}(|V_{sh}(\vec{r}_t)|)V_{sh}(\vec{r}_t)\,. \tag{1}$$

- Maxwell's equations yield the linear, nonlocal condition

$$J_n(\vec{r}_t) = \int_{\partial\mathscr{D}} d^2r_t'\; y_p(\vec{r}_t, \vec{r}_t')V_{sh}(\vec{r}_t') + J_n^{ext}(\vec{r}_t)\,. \tag{2}$$

- Eliminating the total current $J_n$ gives a nonlinear integral equation for $V_{sh}$ on the boundary

$$y_{sh}(|V_{sh}(\vec{r}_t)|)V_{sh}(\vec{r}_t) - \int_{\partial\mathscr{D}} d^2r_t'\; y_p(\vec{r}_t, \vec{r}_t')V_{sh}(\vec{r}_t') = J_n^{ext}(\vec{r}_t)\,.$$

# Outline

- Motivations
- RF sheath physics and boundary conditions
- **Three-dimensional implementation in plane-stratified geometry and emergent problems with fixed point iteration schemes**
- Analytic results and nonlinear features
  - Unbounded half-space: multivaluedness, stability, and hysteresis
  - Bounded domain: spontaneous symmetry breaking
- Conclusions

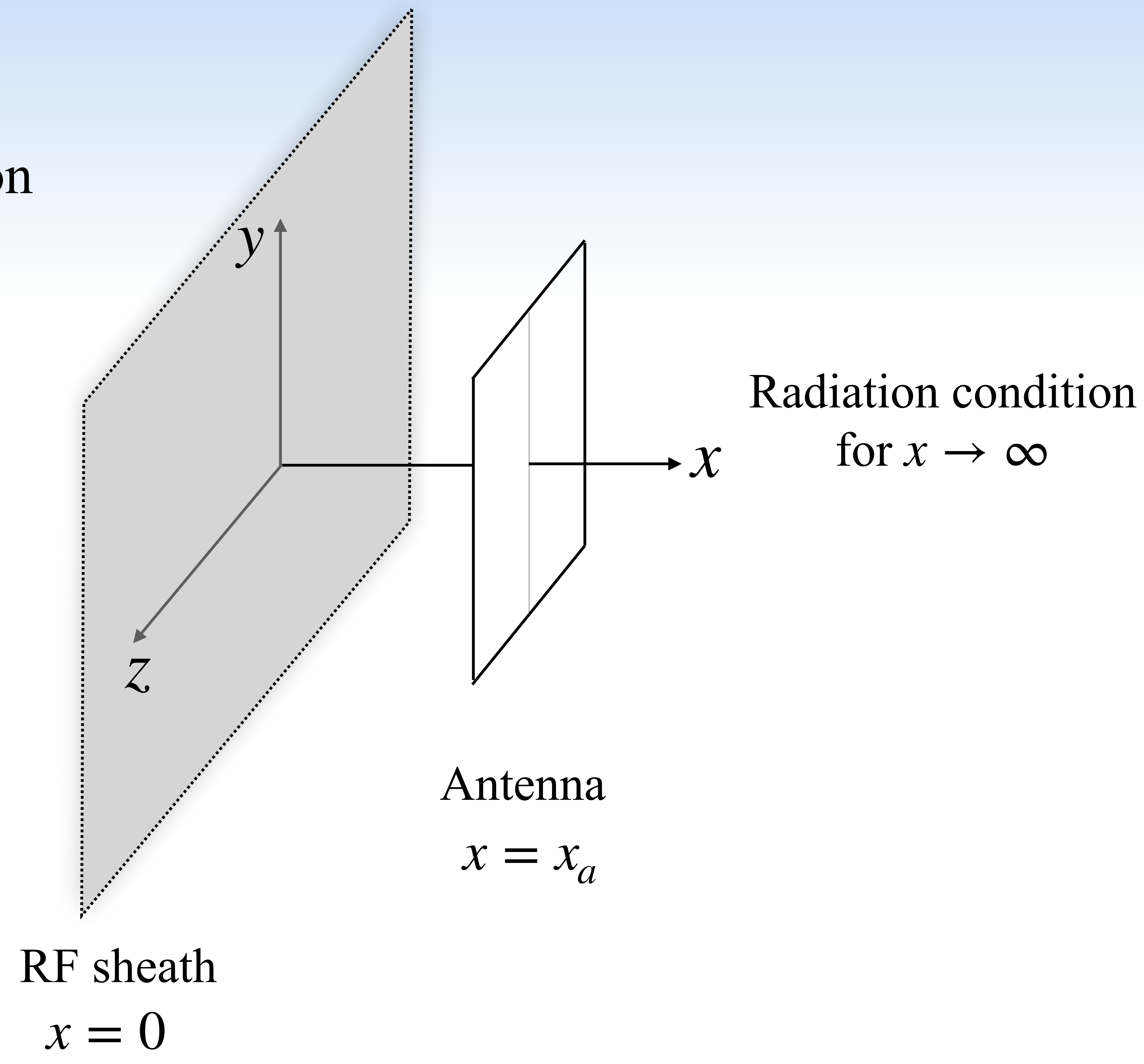
# 3D implementation in planar geometry

In planar geometry, the integral kernel simplifies to a convolution

$$y_{sh}(|V_{sh}(\vec{r}_t)|)V_{sh}(\vec{r}_t) - \int_{\partial\mathcal{D}} d^2r'_t\, y_p(\vec{r}_t - \vec{r}'_t)V_{sh}(\vec{r}'_t) = J_n^{ext}(\vec{r}_t),$$

which yields a straight-forward fixed point iteration scheme leveraging fast Fourier transforms.

The Fourier transformed plasma wave admittance $\tilde{y}_p(\overrightarrow{k}_t)$ is obtained analytically from Maxwell's equations.

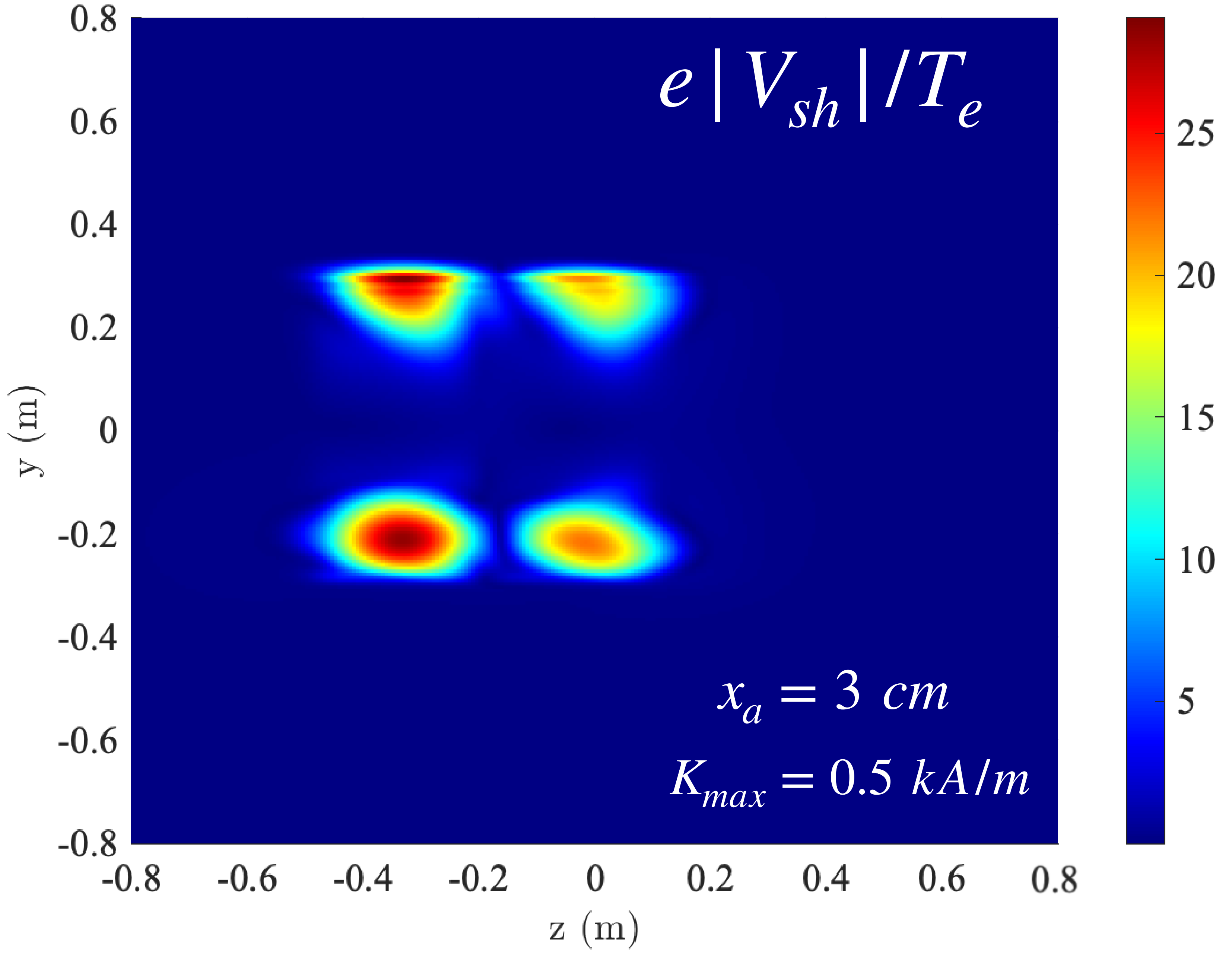


# 3D implementation in planar geometry

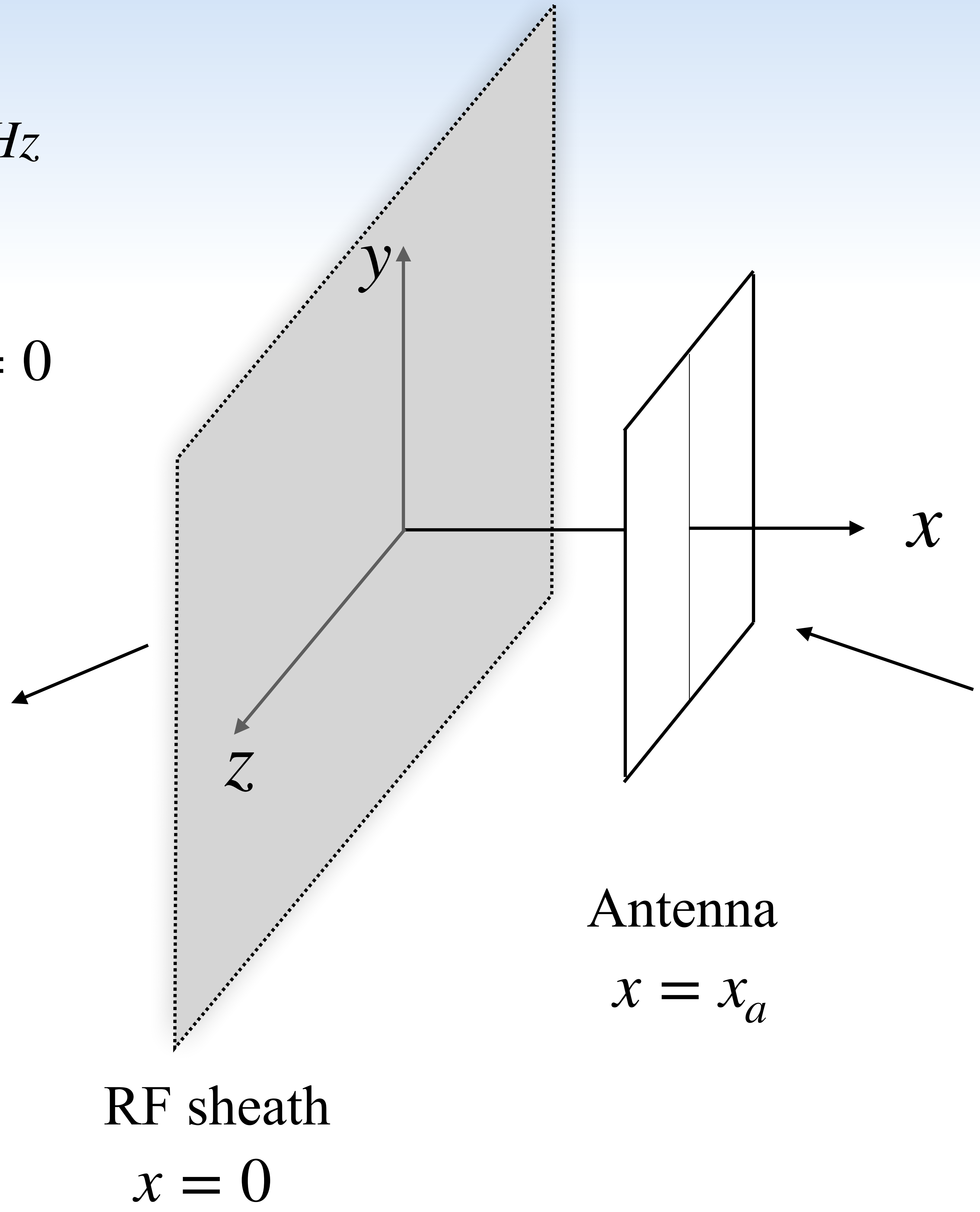
$B_x = 0.3\ T$ $B_y = 0\ T$ $B_z = 2\ T$

$n_e = 1 \times 10^{18}\ m^{-3}$ $T_e = 15\ eV$ $f = 80\ MHz$

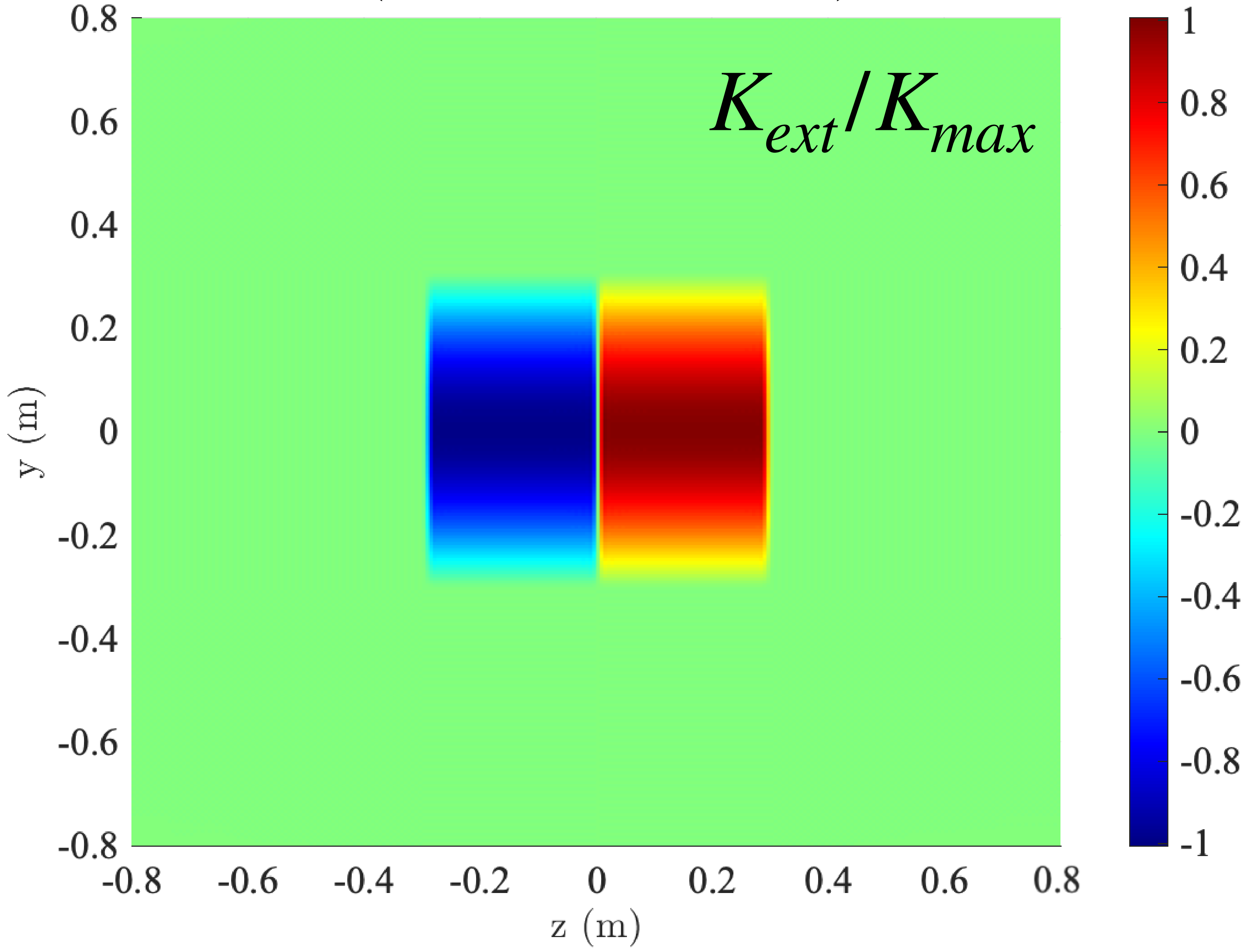
RF sheath potential magnitude on plane $x = 0$




$I_{ant} = 145\ A$

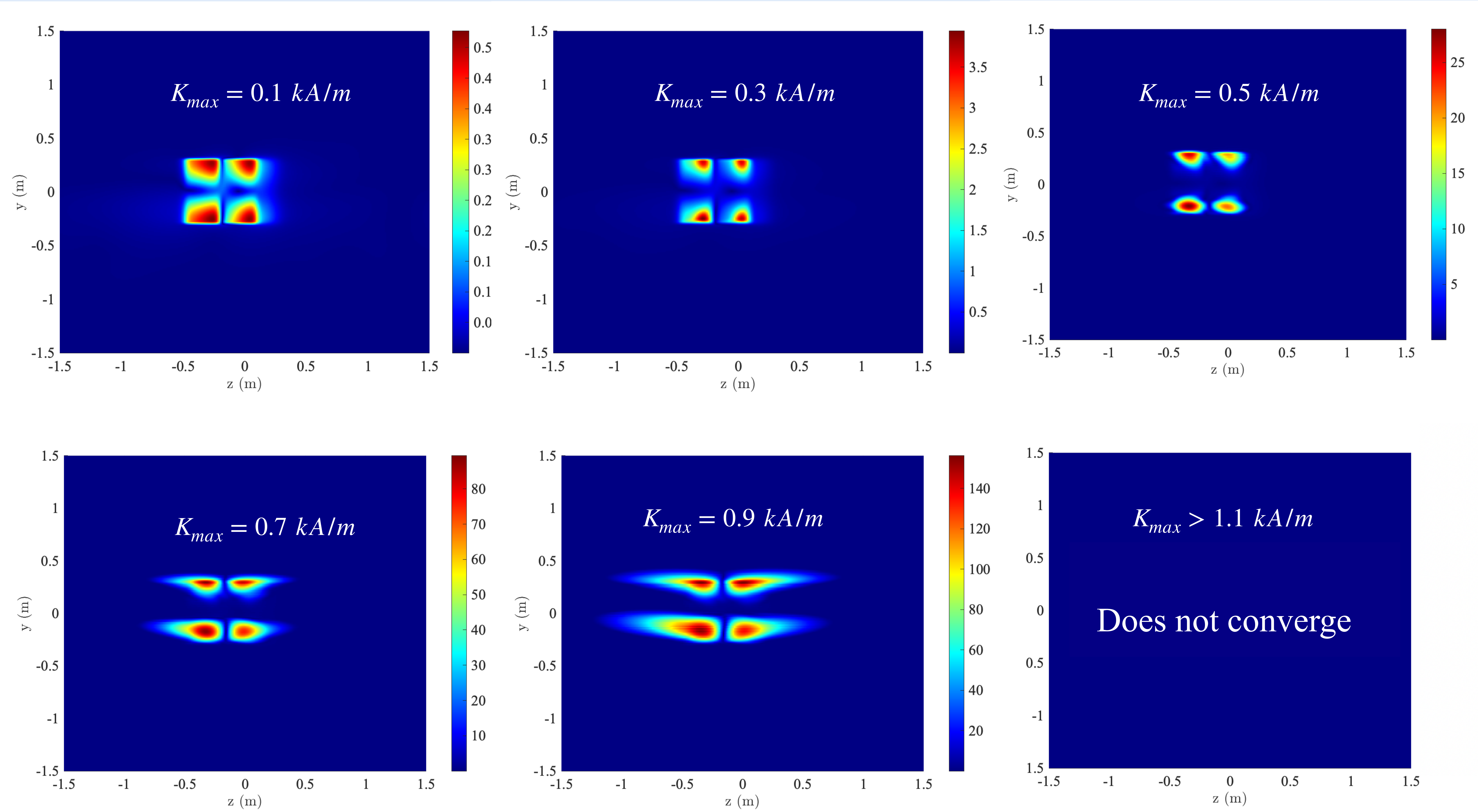



Antenna surface current on the plane $x = x_a$
(scaled to maximum)



# 3D implementation in planar geometry

Sheath potential magnitudes for varying antenna current

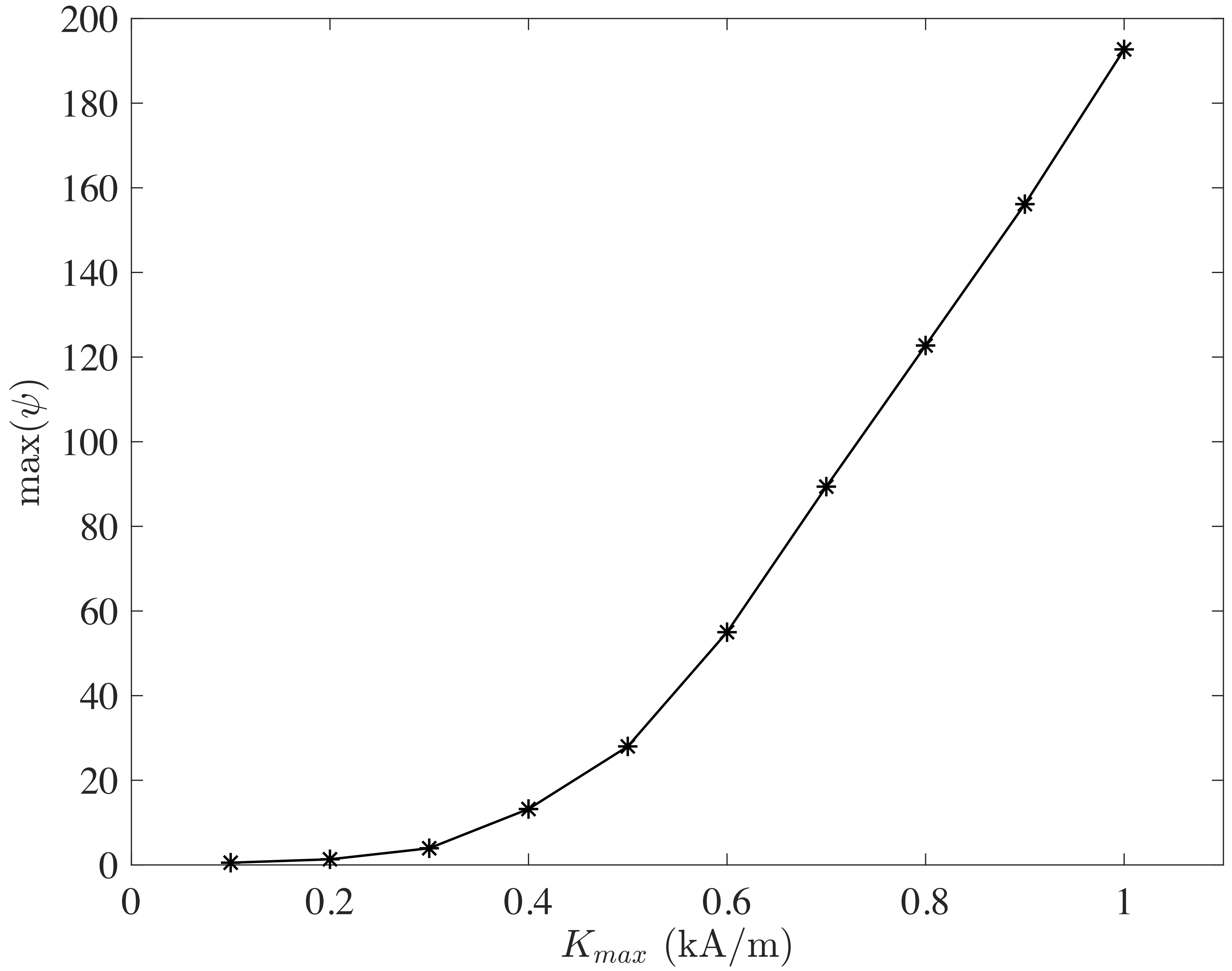



Fixed point iteration does not converge for $K_{max} > 1.1\ kA/m$

Solutions exhibit oscillatory behavior before going numerically unstable

# 3D implementation in planar geometry

Maximum sheath potential for varying antenna surface current




Fixed point iteration convergence is very slow when entering the nonlinear regime and exhibits clear converge issues for large antenna currents

Newton-Ralphson method is prohibitively expensive, as each iteration requires the inversion of a large, dense Jacobian matrix with ~ O(N^4) elements

Are converged solutions unique? Bounded? Stable?

Analytic results lead to insight regarding these questions

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# Analytic solution for plane wave sourcing

For a plane wave surface current imposed at $x = x_a$

$$\vec{j}_{ext}(x, y, z) = \hat{y}\delta(x - x_a)K_{ext}e^{ik_z z}$$

Linearity and symmetry imply at $x = 0$

$$J_n^{ext}(y, z) = \tilde{J}_n^{ext}e^{ik_z z} \qquad V_{sh}(y, z) = \tilde{V}_{sh}e^{ik_z z}$$

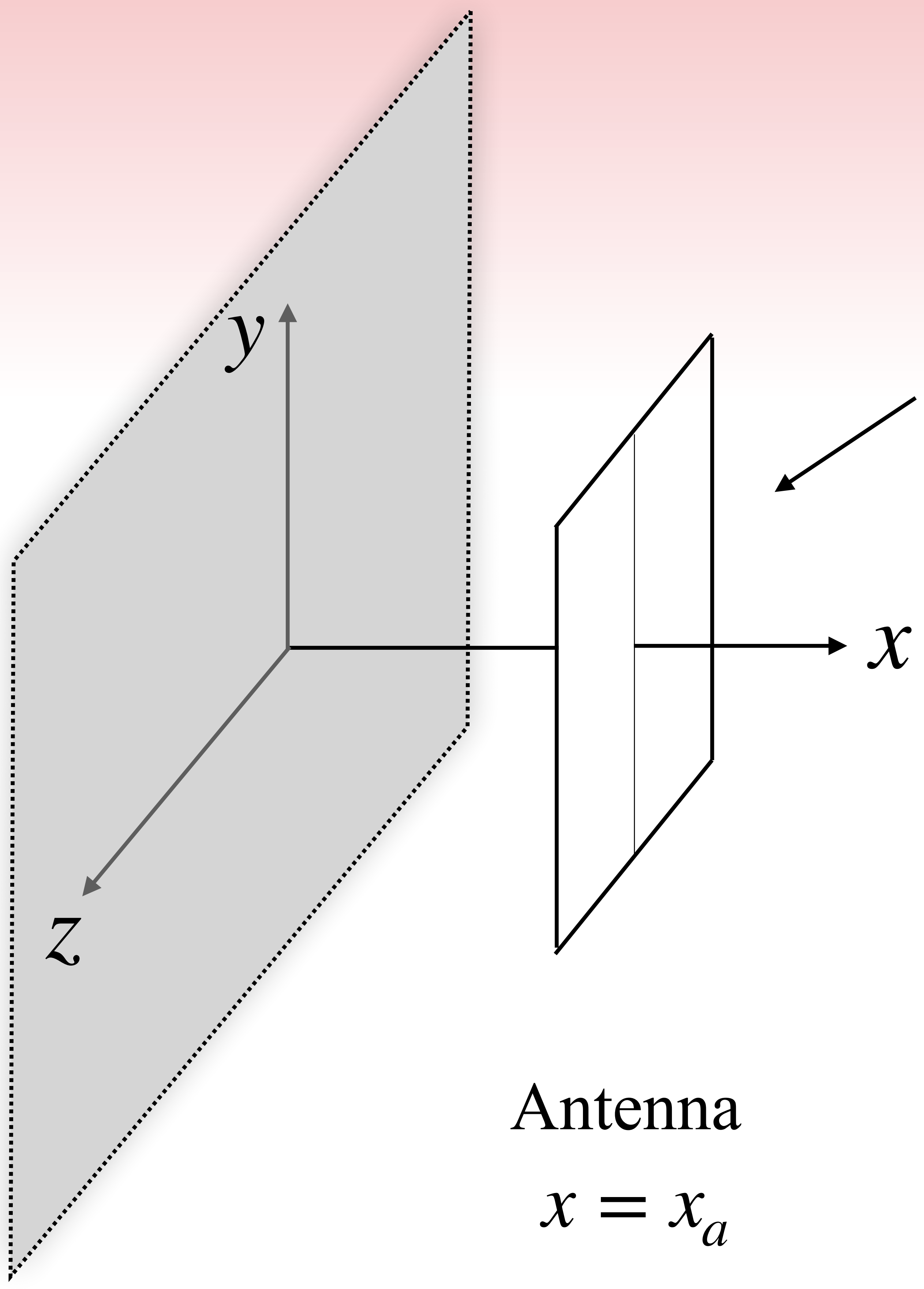
where $\tilde{J}_n^{ext}$ and $\tilde{V}_{sh}$ are complex constants.

The nonlinear integral equation

$$y_{sh}(|V_{sh}(\vec{r}_t)|)V_{sh}(\vec{r}_t) - \int_{\partial\mathscr{D}} d^2r_t'\; y_p(\vec{r}_t - \vec{r}_t')V_{sh}(\vec{r}_t') = J_n^{ext}(\vec{r}_t)$$

Reduces to a nonlinear algebraic equation for $\tilde{V}_{sh}$

$$(y_{sh}(|\tilde{V}_{sh}|) - \tilde{y}_p(k_z))\tilde{V}_{sh} = \tilde{J}_n^{ext}$$



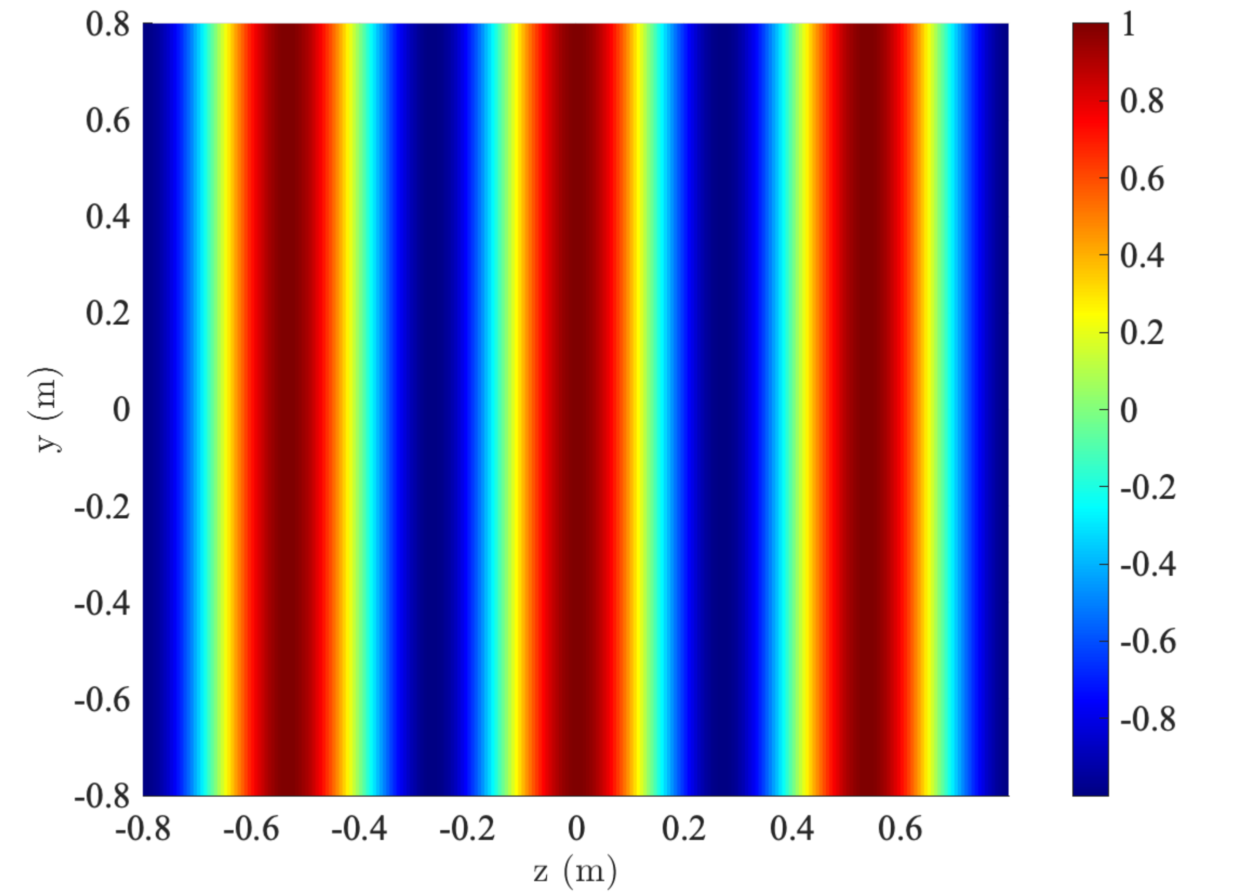

RF sheath

$x = 0$

Antenna

$x = x_a$

$\Re(K_{ext})$




Antenna surface current on the plane $x = x_a$

(scaled to maximum)

# Analytic result reveals all solutions for varying antenna current

Rescaling the nonlinear equation

$$|\tilde{J}_n^{ext}| = |y_{sh}(|\tilde{V}_{sh}|) - \tilde{y}_p(k_z)||\tilde{V}_{sh}|$$

gives

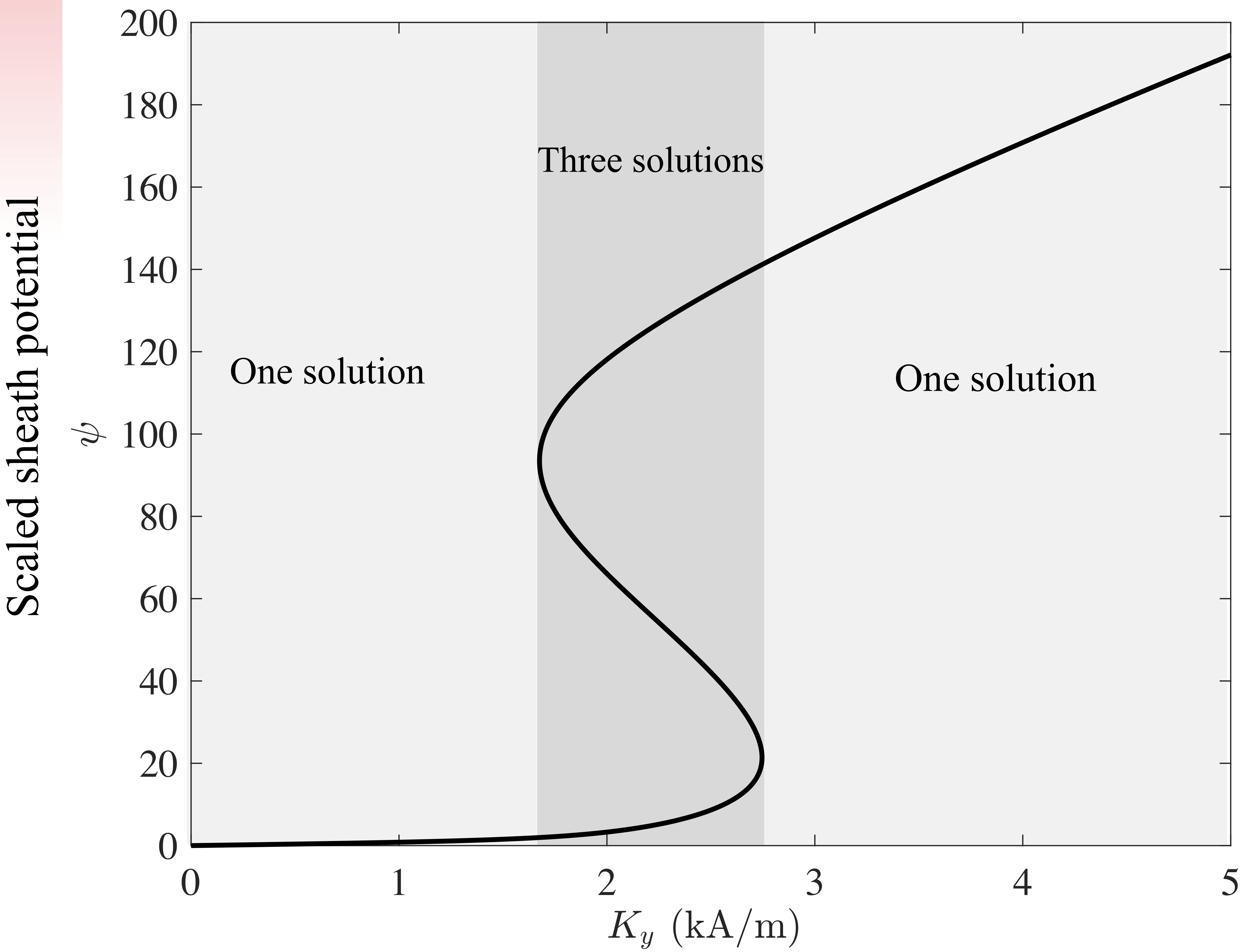
$$K_{ext} = \frac{|y_{sh}(|\tilde{V}_{sh}|) - \tilde{y}_p(k_z)||\tilde{V}_{sh}|}{|\tilde{J}_n^{ext}/K_{ext}|}$$

Antenna surface current

$B_x = 0.3\ T$ $B_y = 0\ T$ $B_z = 2\ T$

$n_e = 1 \times 10^{18}\ m^{-3}$ $T_e = 15\ eV$

$f = 80\ MHz$ $k_z = 10.8\ m^{-1}$




Antenna surface current

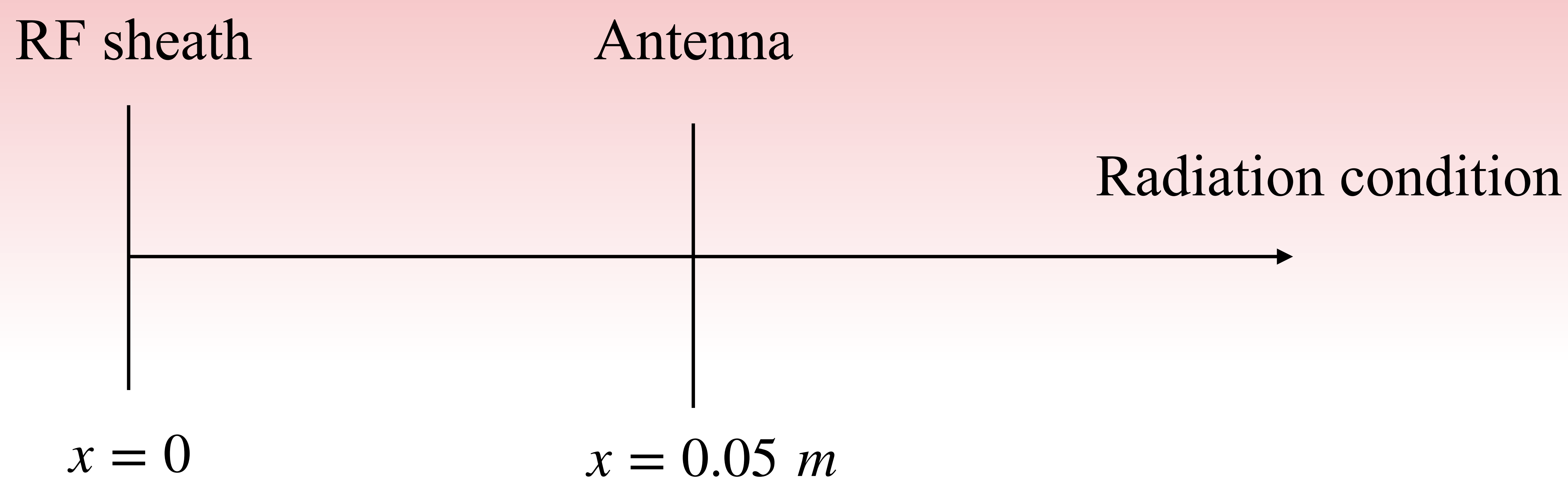
# Three solutions for a fixed antenna current

$K_y = 2\ kA/m$

$B_x = 0.3\ T \quad B_y = 0\ T \quad B_z = 2\ T$

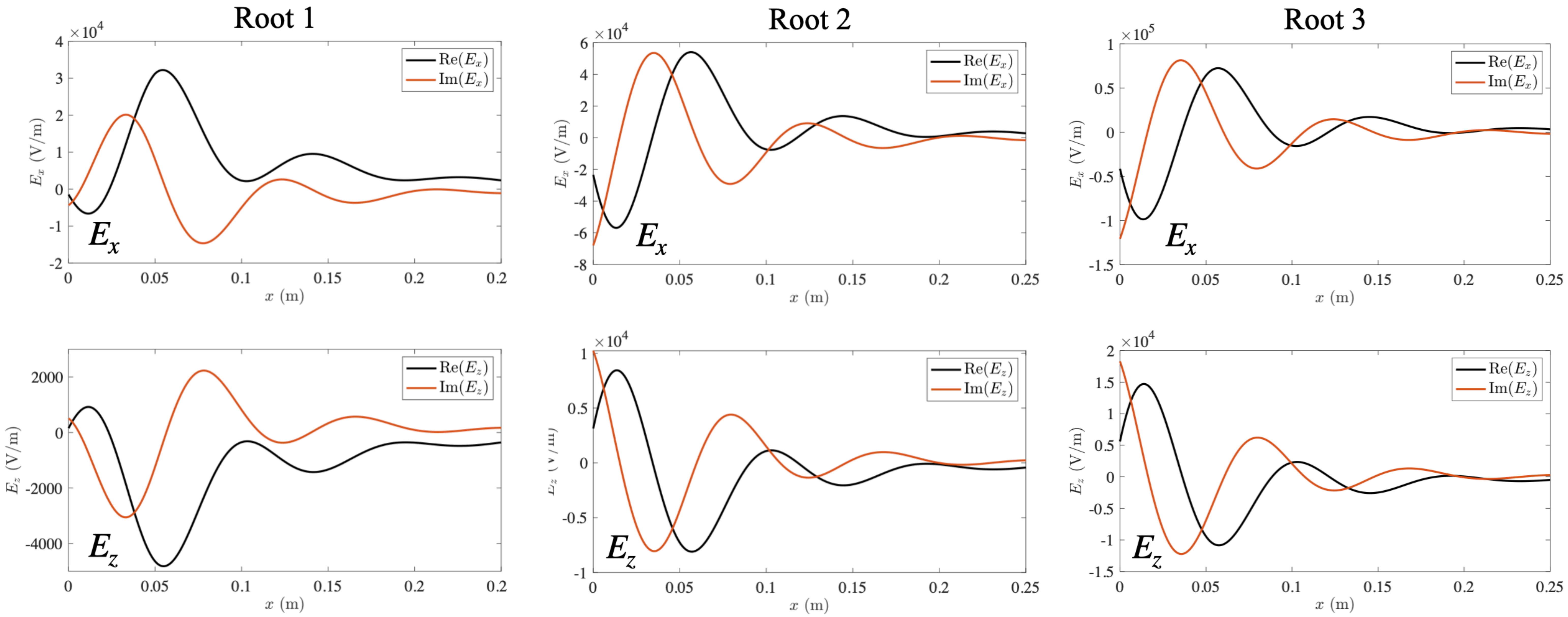
$n_e = 1 \times 10^{18}\ m^{-3} \quad T_e = 15\ eV$

$f = 80\ MHz \quad k_z = 10.8\ m^{-1}$




For these parameters and geometry, three solutions exist that are consistent with Maxwell's equations and the RF sheath boundary condition

| Root # | $e\|V_{sh}\|/T_e$ |
|---|---|
| 1 | 3.3 |
| 2 | 66.1 |
| 3 | 118.1 |



# RF sheath instability in multi-valued cases

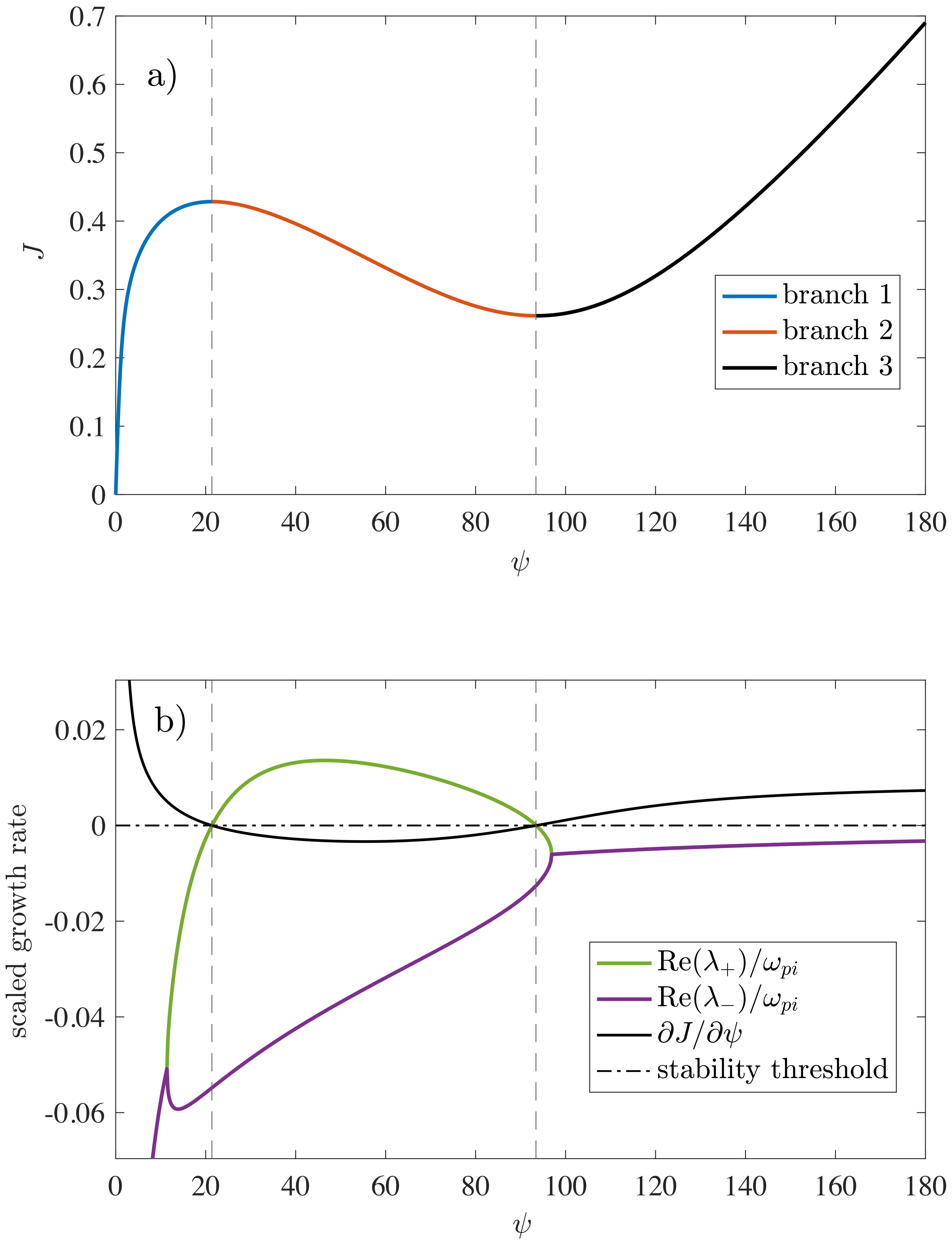
For slow time variation of the external sourcing, a nonlinear equation describing the evolution of the sheath potential is obtained as

$$\frac{d}{dt_s}V_{sh} = \frac{J_n^{ext} - (y_{sh}(|V_{sh}|) - \tilde{y}_p(k_z))V_{sh}}{i\frac{\partial}{\partial\omega}(y_{sh}(|V_{sh}|) - \tilde{y}_p(k_z))},$$

The equilibrium solutions to this equation are the self-consistent solutions to the RF sheath-plasma problem.

<u>Not all equilibrium solutions are stable to small perturbations.</u>

Instability criterion: $\frac{\partial K_{ext}}{\partial \psi} < 0$



# Hysteresis formation for slow time variation

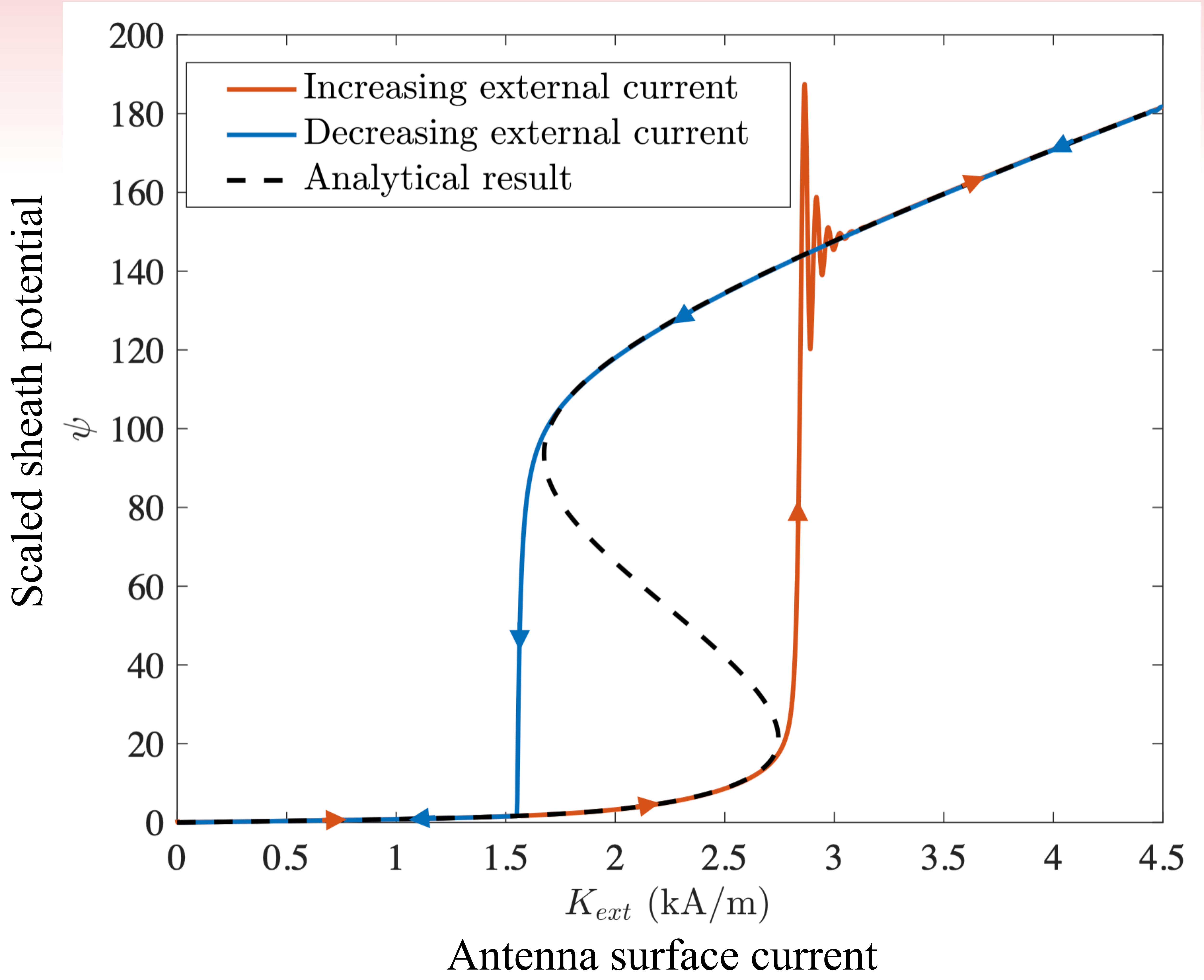
The nonlinear equation

$$\frac{d}{dt_s}V_{sh} = \frac{J_n^{ext} - (y_{sh}(|V_{sh}|) - \tilde{y}_p(k_z))V_{sh}}{i\frac{\partial}{\partial\omega}(y_{sh}(|V_{sh}|) - \tilde{y}_p(k_z))},$$

is evolved with an external current that is slowly ramped up and down in time.

Behavior of the system is different for increasing current vs. decreasing current.

RF sheath instability leads to jumping from branch to branch.



# Lessons from the analysis of an unbounded half-space

- Multiple solutions can exist to the self-consistent RF sheath plasma problem
- An instability is always associated with the occurrence of multiple solutions
- Hysteresis behavior in the time domain is a generic consequence of the RF sheath instability
- Complicated function topology is the culprit behind non-convergence of fixed-point iterations in three-dimensional sheath implementation

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  - **Bounded domain: spontaneous symmetry breaking**
- Conclusions

# Analytic solution in a bounded domain

Maxwell's equations imply the nonlocal linear relation

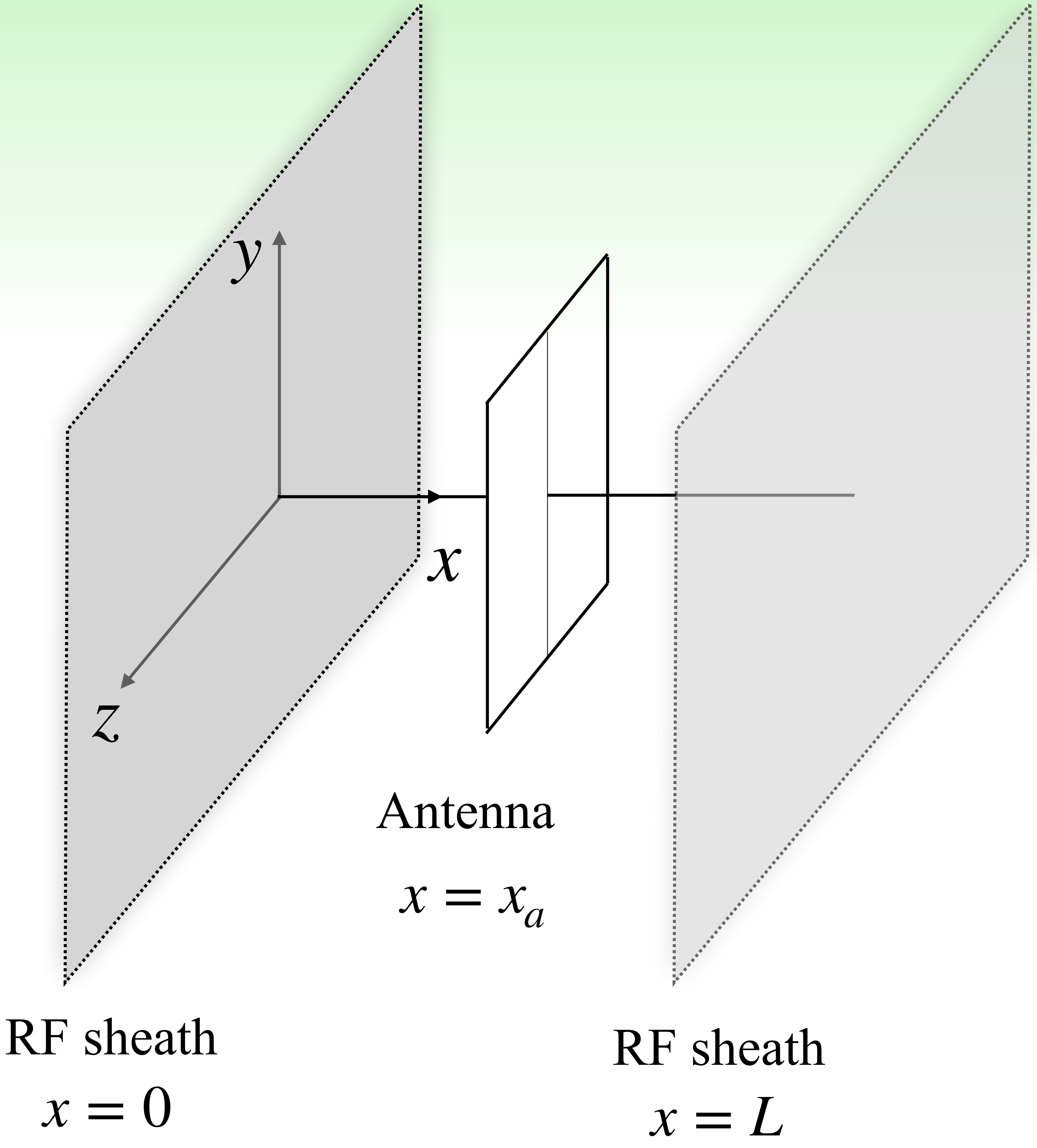
$$\begin{bmatrix} J_n^0 \\ J_n^L \end{bmatrix} = \begin{bmatrix} y_p^{00} & y_p^{0L} \\ y_p^{L0} & y_p^{LL} \end{bmatrix} \begin{bmatrix} V_{sh}^0 \\ V_{sh}^L \end{bmatrix} + \begin{bmatrix} J_n^{cw,0} \\ J_n^{cw,L} \end{bmatrix}$$

The RF sheath boundary condition gives the local nonlinear relation

$$\begin{bmatrix} J_n^0 \\ J_n^L \end{bmatrix} = \begin{bmatrix} y_{sh}(|V_{sh}^0|) & 0 \\ 0 & y_{sh}(|V_{sh}^L|) \end{bmatrix} \begin{bmatrix} V_{sh}^0 \\ V_{sh}^L \end{bmatrix}$$

Eliminating $J_n^0$ and $J_n^L$ leads to the nonlinear matrix equation

$$\begin{bmatrix} y_{sh}(|V_{sh}^0|) - y_p^{00} & -y_p^{0L} \\ -y_p^{L0} & y_{sh}(|V_{sh}^L|) - y_p^{LL} \end{bmatrix} \begin{bmatrix} V_{sh}^0 \\ V_{sh}^L \end{bmatrix} = \begin{bmatrix} J_n^{cw,0}/K_{ext} \\ J_n^{cw,L}/K_{ext} \end{bmatrix} K_{ext},$$



# Analytic solution in a bounded domain

To view the solution space of the nonlinear matrix equation

$$\begin{bmatrix} y_{sh}^0 - y_p^{00} & -y_p^{0L} \\ -y_p^{L0} & y_{sh}^L - y_p^{LL} \end{bmatrix} \begin{bmatrix} V_{sh}^0 \\ V_{sh}^L \end{bmatrix} = \begin{bmatrix} J_n^{cw,0}/K_{ext} \\ J_n^{cw,L}/K_{ext} \end{bmatrix} K_{ext},$$

define the two functions

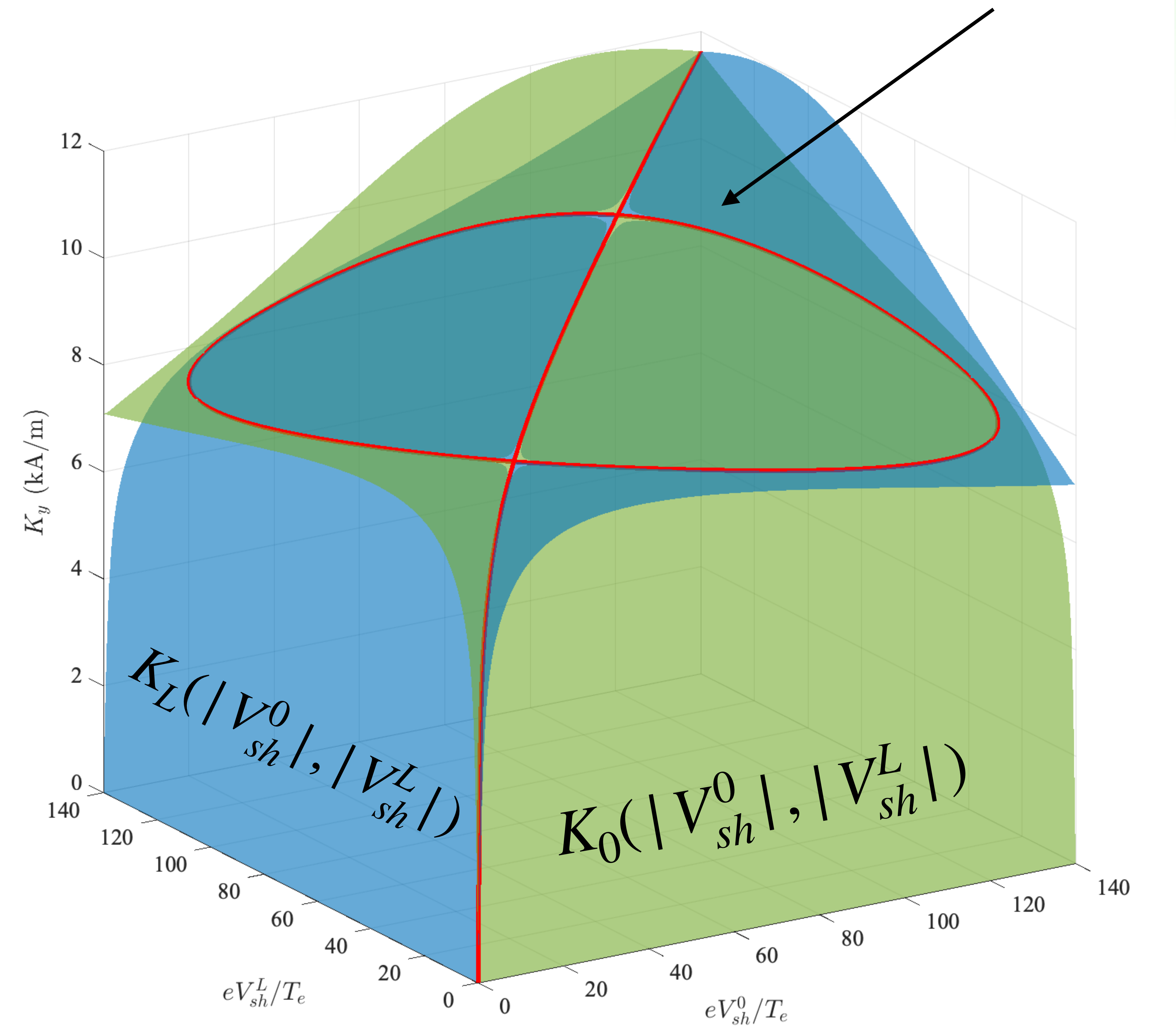
$$K_0(|V_{sh}^0|,|V_{sh}^L|) = \left| \frac{(y_{sh}^0 - y_p^{00})(y_{sh}^L - y_p^{LL}) - y_p^{0L} y_p^{L0}}{(y_{sh}^L - y_p^{LL}) J_n^{cw,0}/K_{ext} + y_p^{0L} J_n^{cw,L}/K_{ext}} \right| |V_{sh}^0|,$$

$$K_L(|V_{sh}^0|,|V_{sh}^L|) = \left| \frac{(y_{sh}^0 - y_p^{00})(y_{sh}^L - y_p^{LL}) - y_p^{0L} y_p^{L0}}{(y_{sh}^0 - y_p^{00}) J_n^{cw,L}/K_{ext} + y_p^{L0} J_n^{cw,0}/K_{ext}} \right| |V_{sh}^L|,$$

and note that solutions satisfy

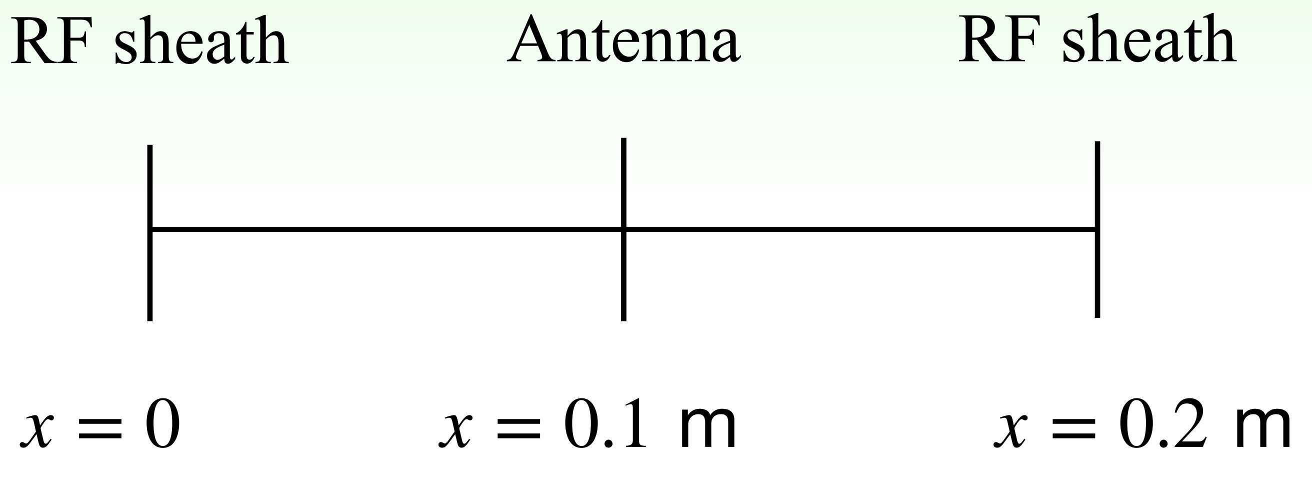
$$K_0(|V_{sh}^0|,|V_{sh}^L|) = K_L(|V_{sh}^0|,|V_{sh}^L|) = K_{ext}$$

The surface intersection denoted by the red curve gives all possible solution triples $(|V_{sh}^0|, |V_{sh}^L|, K_{ext})$




The functions $K_0$ and $K_L$ are thought of as surfaces over the two-dimensional domain spanned by $|V_{sh}^0|$ and $|V_{sh}^L|$. The intersection of these two surfaces gives all the allowable values for the triplets $(|V_{sh}^0|, |V_{sh}^L|, K_{ext})$.

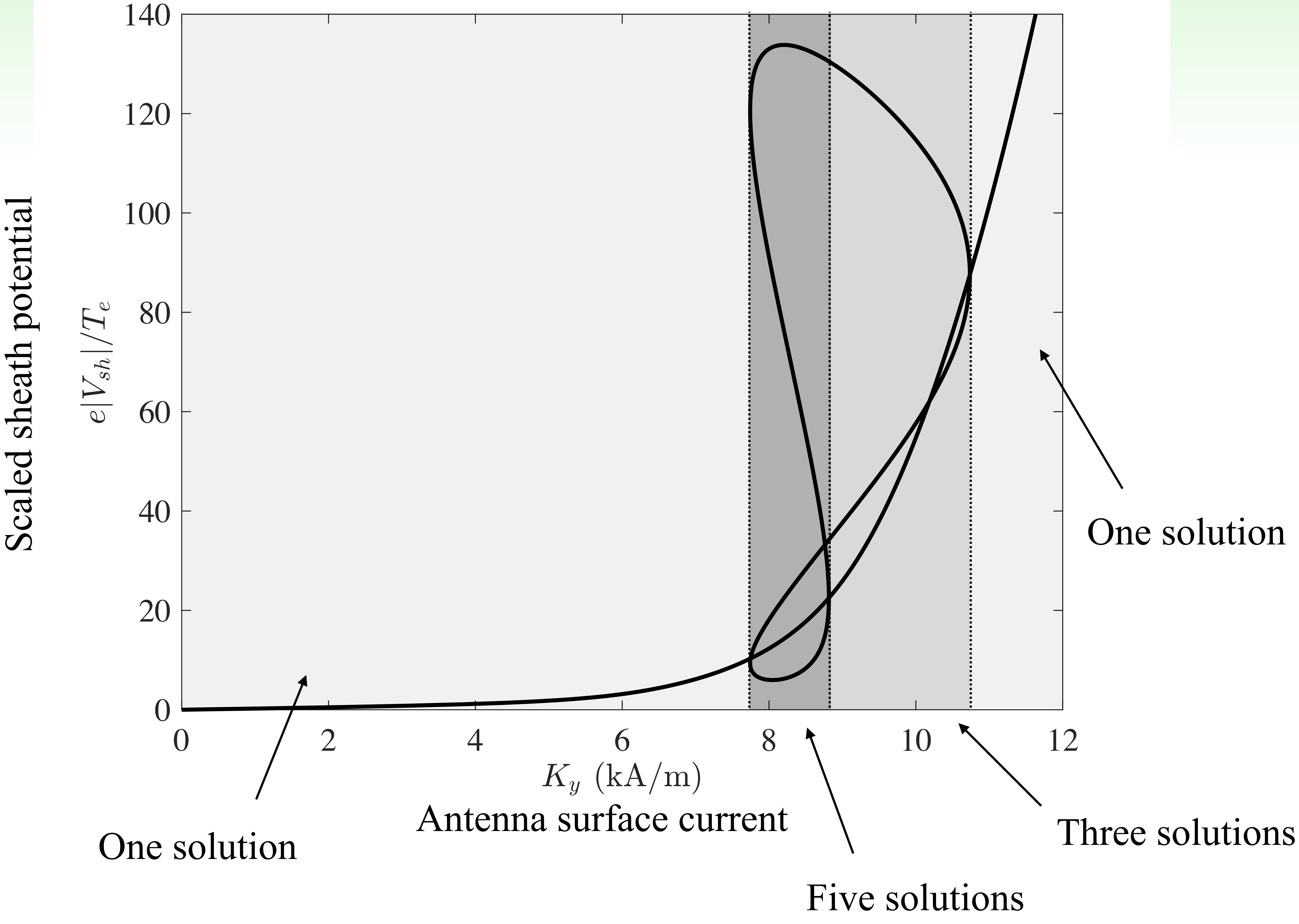
# Analytic result shows all solutions for varying antenna current




$B_x = 1\ T$ $\quad$ $B_y = B_z = 0$

$n_e = 3 \times 10^{17}\ m^{-3}$ $\quad$ $T_e = 15\ eV$

$f = 56\ MHz$ $\quad$ $k_z = 10.8\ m^{-1}$



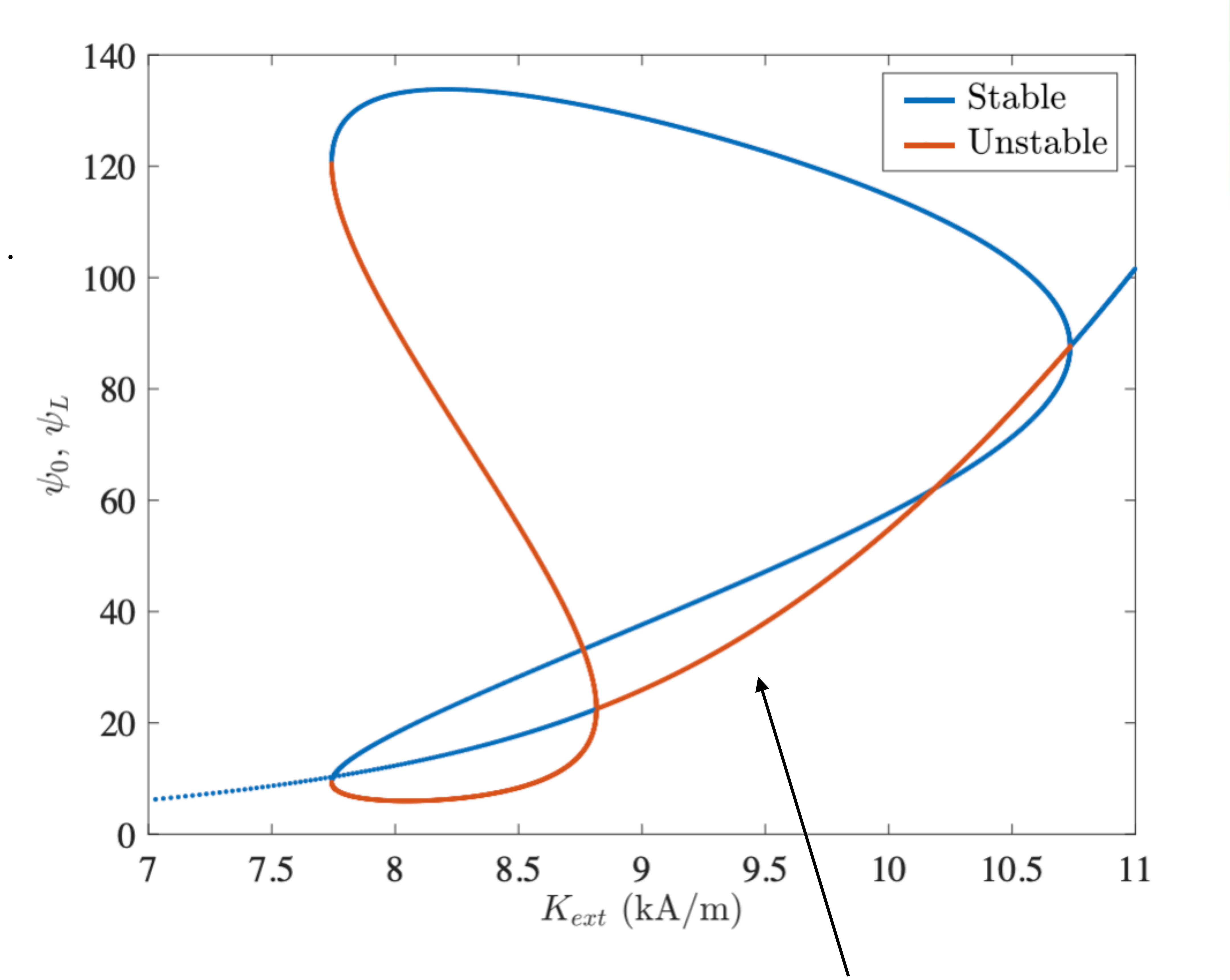
# RF sheath instability in multi-valued cases

For slow time variation of the external current

$$\frac{d}{dt_s}\begin{bmatrix}\psi_{sh}^{0}\\ \psi_{sh}^{L}\end{bmatrix} = \left(i\frac{\partial}{\partial\omega}\begin{bmatrix}\hat{y}_{sh}^{0}-\hat{y}_{p}^{00} & -\hat{y}_{p}^{0L}\\ -\hat{y}_{p}^{L0} & \hat{y}_{sh}^{L}-\hat{y}_{p}^{LL}\end{bmatrix}\right)^{-1}\cdot\left(\begin{bmatrix}J_n^{cw,0}\\ J_n^{cw,L}\end{bmatrix}-\begin{bmatrix}\hat{y}_{sh}^{0}-\hat{y}_{p}^{00} & -\hat{y}_{p}^{0L}\\ -\hat{y}_{p}^{L0} & \hat{y}_{sh}^{L}-\hat{y}_{p}^{LL}\end{bmatrix}\begin{bmatrix}\psi_{sh}^{0}\\ \psi_{sh}^{L}\end{bmatrix}\right).$$

Analyzing the linear stability of equilibrium solutions leads to the instability criterion:

$$\frac{\partial K_0}{\partial \psi_0}\frac{\partial K_L}{\partial \psi_L}-\frac{\partial K_0}{\partial \psi_L}\frac{\partial K_L}{\partial \psi_0}<0$$




Intermediate branch composed of symmetric solutions is unstable to any asymmetric perturbation

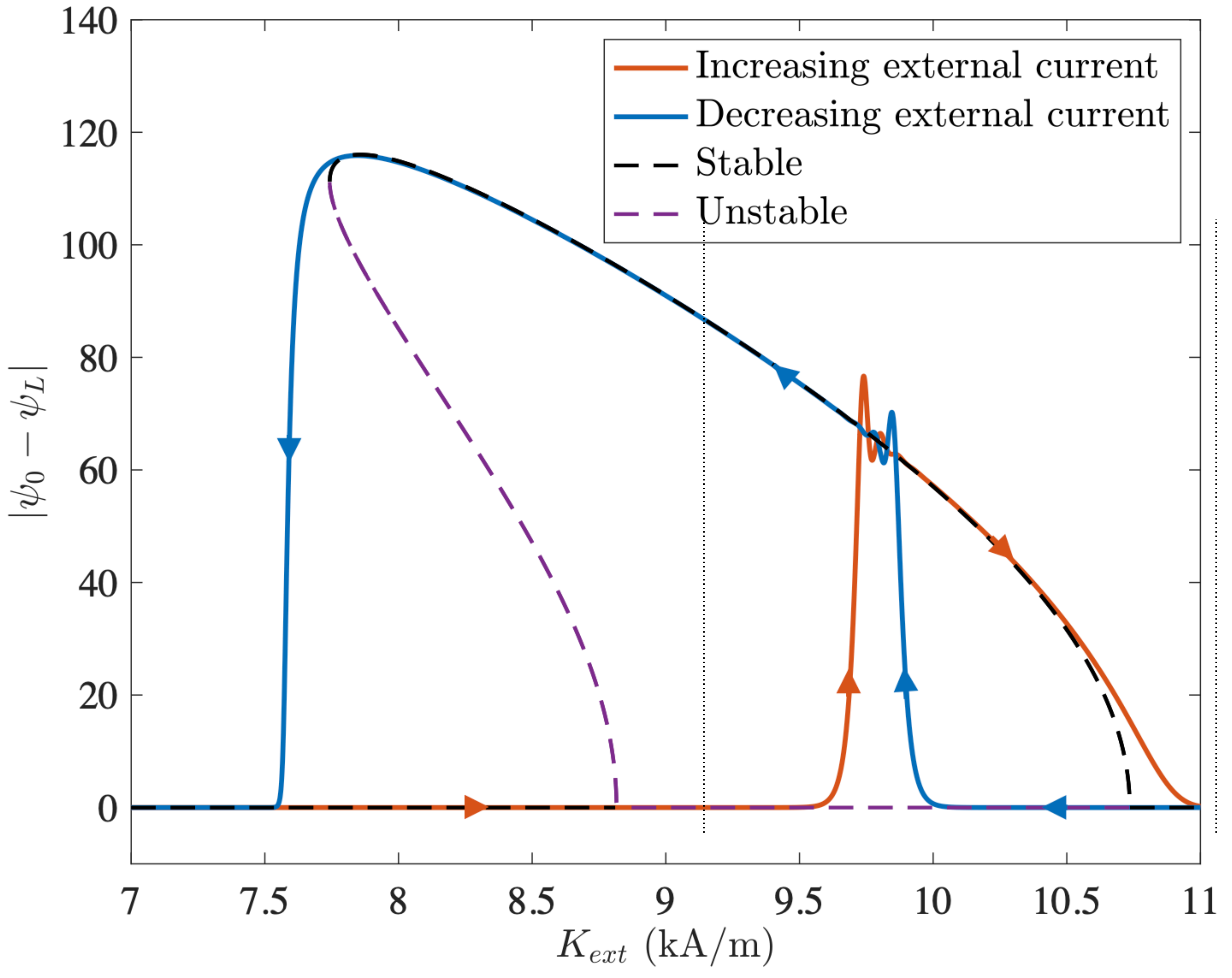
Implication: spontaneous symmetry breaking

# Spontaneous symmetry breaking

Slowly increasing and decreasing the antenna current leads to the time evolution shown

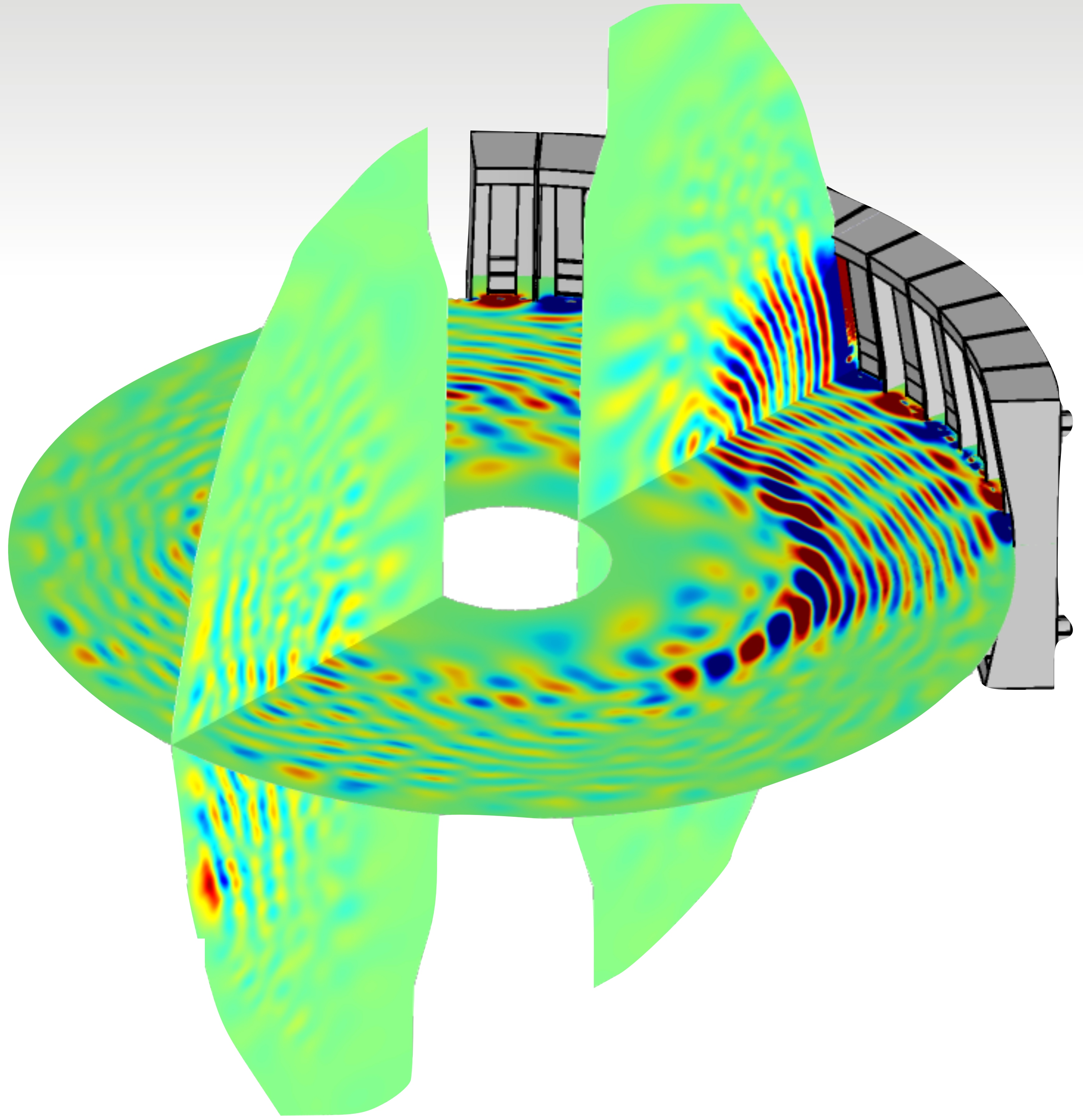
Note that the symmetry of the problem is spontaneously broken by the RF sheath instability

Clear hysteresis behavior is observed



# Directions for future research

- Incorporation of nonlinear RF sheath boundaries in high-fidelity 3D simulations (i.e., Petra-M)
- Development of 'smart' iterative solvers that anticipate nonlinear sheath effects
- Coupling to secondary emission model to evaluate impurity production due to RF sheath rectification
- Coupling to transport model to evaluate the transport of impurities



NSTX-U HHFW Antenna in Petra-M

# Conclusions

- Analytical solutions exist for two classes of one-dimensional problems (unbounded domain/bounded domain). Precise conditions for the appearance of multiple roots are obtained.

- Multiple roots and sheath plasma resonance are linked to the presence of an instability that alleviates the ambiguity in the frequency domain and gives a physical mechanism for hysteresis.

- Non-intuitive spontaneous symmetry breaking is demonstrated in an otherwise completely symmetric problem, highlighting the nontrivial features of the physics underpinning RF sheath—plasma interactions.

- Attempts to model and implement nonlinear RF sheaths in numerical codes should be aware of these nonlinearities and the exotic features they present.