

LETTER

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## Letter

# Analytical nonlinear collisional dynamics of near-threshold eigenmodes

V.N. Duarte<sup>✉</sup> and N.N. Gorelenkov

Princeton Plasma Physics Laboratory, Princeton University, Princeton, NJ 08543, United States of America

E-mail: [vduarte@pppl.gov](mailto:vduarte@pppl.gov)

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## Abstract

The nonlinear dynamics of isolated, near-threshold waves is studied in the presence of strong resonant particle scattering. In this limit, a simple closed-form analytical solution is found and compared with numerical simulations, which shows reasonable agreement when the wave evolves quasi-steadily to saturation, even for moderate tokamak-relevant collisionality levels. The solution can be useful in verifying codes that deal with Alfvénic instabilities and thermal plasma turbulence in fusion plasmas, as well as a rapid means for predicting and analyzing experimental outcome.

Keywords: nonlinear dynamics, Alfvén wave, wave instability, mode saturation

(Some figures may appear in colour only in the online journal)

The obtention of reliable bounds for the nonlinear instability of waves is an outstanding problem in kinetic systems of fusion interest [1, 2]. The burning plasma sustainment in ITER imposes severe constraints on the amount of fast ions ejected through their resonant interaction with Alfvénic waves [3]. Therefore, procedures to anticipate the nonlinear evolution of waves destabilized by the sub-population of highly energetic particles are needed for establishing limits for wave growth in ITER as well as in present tokamaks. In this work, we show that, for situations in which an isolated mode quasi-steadily reaches its saturation, a simple analytical expression provides a reasonable approximation for its nonlinear behavior. It can be a rapid means for experimental prediction and interpretation, as well as for the verification of codes. The analytical solution is especially useful for waves that evolve just above their marginal stability, for which simulations need to be extended to long times.

The nonlinear dynamics of a non-overlapping wave near marginal stability has been found to be governed by a

time-delayed, integro-differential cubic equation<sup>1</sup> which, in the presence of diffusive processes, reads [5, 6]

$$\frac{dA(t)}{dt} = A(t) - \frac{1}{2} \sum_I \int d\Gamma \mathcal{H} \left\{ \int_0^{t/2} dz z^2 A(t-z) \times \int_0^{t-2z} dy e^{-\hat{\nu}_{\text{eff}}^3 z^2 (2z/3+y)} A(t-z-y) A^*(t-2z-y) \right\} \quad (1)$$

where  $\hat{\nu}_{\text{eff}}$  represents the effective scattering frequency  $\nu_{\text{eff}}$  normalized with  $\gamma_L - \gamma_d$  ( $\gamma_L$  is the linear growth rate in the absence of damping,  $\gamma_L = \frac{\omega\pi}{\Lambda} \sum_I \int d\Gamma |V_{n,p}|^2 \delta(\Omega_{n,p} - \omega) \frac{\partial F}{\partial I}$ , and  $\gamma_d$  is the sum of a wave background damping rates due to several mechanisms). Time is normalized with  $(\gamma_L - \gamma_d)^{-1}$ . In  $\gamma_L$ , the sum is over all resonances of a mode. Considering that resonant particle dynamics is

<sup>1</sup> Within the context of fusion plasmas, the same equation can be recovered for the evolution of a mode in a turbulent plasma under a geometric optics approximation, i.e. when the turbulent modes can be treated as quasi-particles [4]. In that case,  $\hat{\nu}_{\text{eff}}$  plays the role of the damping rates of turbulent modes.

computed in terms of the invariants of the unperturbed motion  $\mathcal{E}$  (energy),  $P_\varphi$  (canonical toroidal momentum) and  $\mu$  (magnetic moment), the phase-space element is given by  $d\Gamma = (2\pi)^3 \sum_{\sigma_\parallel} dP_\varphi d\mu m_{\text{EP}} c / q_{\text{EP}} d\mathcal{E} / \omega_\theta$ , where  $m_{\text{EP}}$  is the mass of EPs,  $c$  is the light speed and  $\sigma_\parallel$  accounts for the sign of the parallel velocity.  $\mathcal{H}$  is a phase-space weighting defined as in [7, 8]:

$\mathcal{H} = 2\pi\omega\delta(\Omega_{n,p} - \omega) |V_{n,p}|^4 \left(\frac{\partial\Omega_{n,p}}{\partial I}\right)^3 \frac{\partial F}{\partial\Omega}$ .  $F$  is the distribution function at  $t = 0$ ,  $I = -P_\varphi/n$  at constant  $\mathcal{E}' = \mathcal{E} + \omega P_\varphi/n$  is the relevant variable describing the dynamics at a given resonance (with  $\partial/\partial I \equiv -n\partial/\partial P_\varphi + \omega\partial/\partial\mathcal{E}$ ). The mode frequency is  $\omega$  and the resonance condition is  $\Omega_{n,p} = n\omega_\varphi - p\omega_\theta = \omega$ , where  $\varphi$  and  $\theta$  refer to the toroidal and poloidal angles,  $n$  is the toroidal mode number and  $p$  is an integer. The matrix elements of wave-particle interaction are defined as  $V_{n,p}(I) = \frac{iq_{\text{EP}}}{\omega} \int \frac{d\varphi d\theta}{(2\pi)^2} e^{-i(p\theta - n\varphi)} \mathbf{v} \cdot \mathbf{e}$ , where  $\mathbf{e}$  is the electric field eigenstructure,  $\mathbf{v}$  is the velocity of a resonant particle and  $q_{\text{EP}}$  is the EP charge. The full energy of the mode is given by  $\delta K = \Lambda C^2 = C^2\omega^2 \int d\mathbf{r}\rho |\xi|^2$ , where  $\xi$  is the fluid displacement and  $\rho$  is the plasma density. In (1), the amplitude has the normalization  $|A| = \sqrt{2}|C|\Lambda^{-1/2}(\gamma_L - \gamma_d)^{-5/2}$ ,  $\omega_b = \left\langle \left| 2C(t)V_{n,p}(I_r) \frac{\partial\Omega_{n,p}}{\partial I}|_{I=I_r} \right| \right\rangle^{1/2}$  is the bounce (or trapping) frequency of the most deeply trapped resonant particles<sup>2</sup>, the subscript  $r$  denotes the resonance location,  $C$  is the wave amplitude and  $\langle \dots \rangle$  denote phase-space averaging.

In principle,  $\hat{\nu}_{\text{eff}}$  is an effective frequency due a combination of stochastic processes experienced by the resonant population, e.g. collisional pitch-angle scattering, collisionless turbulent scattering and diffusion due to RF heating waves. Previous numerical analysis for Alfvénic modes in DIII-D, NSTX and TFTR [8, 9] have shown that the phase average, over multiple mode resonance surfaces, leads to typical effective collisional scattering frequency of order  $10^4\text{s}^{-1}$ . Anomalous scattering [10] as well as diffusion due to radio-frequency heating [11] contribute to increase the effective scattering rate. The net growth rate is typically of order up to a percent of the wave frequency (the frequency of toroidicity-induced and reversed-shear Alfvénic eigenmodes is typically of order  $10^5\text{s}^{-1}$ ). Therefore, regimes with  $\hat{\nu}_{\text{eff}} \gg 1$  are relevant for experiments, especially when the modes are close to threshold and when diffusive mechanisms, in addition to collisions, are taken into consideration.

For large scattering frequency, memory effects are easily destroyed as resonant particles receive frequent random kicks, and only the very recent history dictates the wave dynamics. For  $\hat{\nu}_{\text{eff}} \gg 1$ , the integral kernel makes the nonlinear term be zero at all integration times except when both  $y$  and  $z$  are close to zero. For very small  $y$  and  $z$ , the kernel of equation (1) changes much faster than the arguments of the amplitudes in the cubic term and the term in the curly brackets can be written as

$$\frac{A(t)|A(t)|^2}{\hat{\nu}_{\text{eff}}^3} \int_0^{t/2} dz \left[ e^{-(2/3)\hat{\nu}_{\text{eff}}^3 z^3} - e^{-\hat{\nu}_{\text{eff}}^3 z^2(3t-4z)/3} \right]. \quad \text{This allows}$$

<sup>2</sup> For a simplified bump-on-tail electrostatic case,  $\omega_b$  is given by  $\sqrt{qEk/m}$  with  $k$  and  $E$  being the absolute value of the wave number vector and the amplitude of the electric field.

equation (1) to be formally cast in a Landau–Stuart [12, 13] form. The argument of the first exponential approaches zero faster than the one of the second exponential, therefore it is the term that gives the most important contribution. By redefining the variable of integration as  $x = \hat{\nu}_{\text{eff}} z$ , the resulting integral can be written as  $\frac{1}{\hat{\nu}_{\text{eff}}} \int_0^\infty dx e^{-(2/3)x^3} = \frac{1}{3\hat{\nu}_{\text{eff}}} \left(\frac{3}{2}\right)^{1/3} \Gamma\left(\frac{1}{3}\right)$ . We can then seek an analytical solution of the resulting equation,

$$\frac{dA(t)}{dt} = A(t) - bA(t)|A(t)|^2 \quad (2)$$

by dividing it by  $A(t)$  and defining an auxiliary variable  $u = \log A$ . Assuming  $A(t) \in \mathbb{R}$ , a closed-form result is<sup>3</sup>

$$A(t) = \frac{A(0)e^t}{\sqrt{1 - bA^2(0)(1 - e^{2t})}} \quad (3)$$

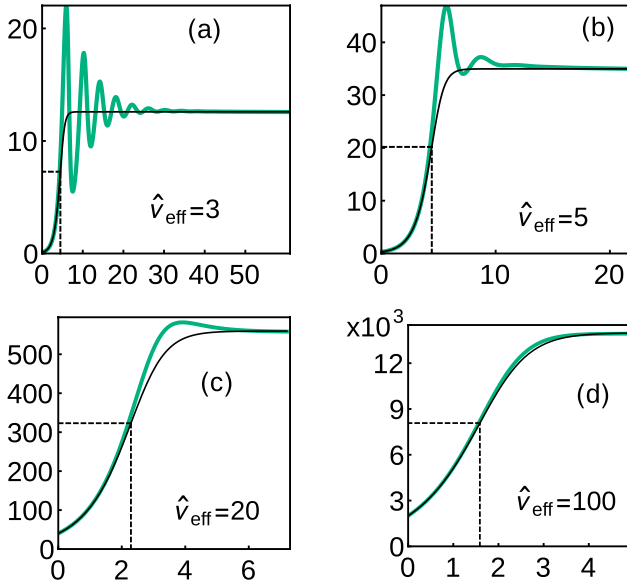
where  $A(0)$  is the initial amplitude and  $b \equiv \int d\Gamma \mathcal{H} \frac{\Gamma(1/3)}{6\nu_{\text{eff}}^4} \left(\frac{3}{2}\right)^{1/3}$ . Equation (3) is consistent with its expected asymptotic behaviors since (i) for  $t \rightarrow 0$ , when the cubic term is unimportant, the mode grows linearly, i.e.  $A(t) = A(0)e^t$  provided that  $bA^2(0) \ll 1$  and (ii) for  $t \rightarrow \infty$ , the saturation level is  $A_{\text{sat}} = \pm 1/\sqrt{b} \simeq \pm 1.4/\sqrt{\langle \nu_{\text{eff}}^{-4} \rangle}$  (the sign depends on whether  $A(0)$  is positive or negative). Using the amplitude normalization adopted for equation (1) we find that, under a bump-on-tail simplification, i.e. when the multi-dimensional resonance structure collapses to one point and the eigenstructure is taken as uniform, this corresponds to the saturation level  $\omega_{b,\text{sat}} \simeq \pm 1.18 \left(1 - \frac{\gamma_d}{\gamma_L}\right)^{1/4} \nu_{\text{eff}}$ , which agrees with the one previously reported in [7, 15]. To the best of our knowledge, equation (3) is the first analytical solution for the mode amplitude evolution, from a seed level up to saturation, in the presence of collisions. An explosive solution [5] for the cubic equation (1) has been obtained for the situation in which the linear term is disregarded and the kernel can be replaced by the unity. In this case, the finite-time amplitude blow-up signals the breakdown of the theory validity.

The nonlinear growth rate  $\gamma_{\text{NL}}(t)$  associated with equation (3) can be calculated from  $A(t) = A(0) \exp\left[\int_0^t \frac{\gamma_{\text{NL}}(t') - \gamma_d}{\gamma_L - \gamma_d} dt'\right]$ , which gives

$$\frac{\gamma_{\text{NL}}(t) - \gamma_d}{\gamma_L - \gamma_d} = \frac{1 - bA^2(0)}{1 - bA^2(0)(1 - e^{2t})}. \quad (4)$$

For experimental purposes, it can be useful to anticipate the timescale for mode saturation, as a function of  $\hat{\nu}_{\text{eff}}$  and the initial amplitude  $A(0)$ . For that purpose, one can gain insights by analyzing the inflection time point of the solution (3), which is

<sup>3</sup> In terms of the trapping frequency, the solution is  $\omega_b(t) = \omega_b(0)e^{t/2} / [1 - c\omega_b^4(0)(1 - e^{2t})]^{1/4}$ , where  $c = [\gamma_{L,0}/(\gamma_{L,0} - \gamma_d)] \Gamma(1/3) (3/2)^{1/3} \langle \nu_{\text{eff}}^{-4} \rangle / 6$ . In this expression the time variable  $t$  is the actual time multiplied by  $\gamma_L - \gamma_d$ . The average over the resonance surfaces is defined by  $\langle \dots \rangle = \int d\Gamma Q \dots / \int d\Gamma Q$ , where  $Q = |\mathbf{e} \cdot \mathbf{E}|^2 \partial F / \partial \mathcal{E}|_{\mathcal{E}}$ ,  $\delta(\Omega - \omega)$ , as defined in [14].



**Figure 1.** Mode amplitude  $A$  versus time  $t$  (normalized with  $\gamma_L - \gamma_d$ ) for (a)  $\hat{\nu}_{\text{eff}} = 3$ , (b)  $\hat{\nu}_{\text{eff}} = 5$ , (c)  $\hat{\nu}_{\text{eff}} = 20$  and (d)  $\hat{\nu}_{\text{eff}} = 100$ . In green is the numerical solution of the full cubic equation (1) and in black is the analytical solution (3). The dashed lines indicate the characteristic inflection time for (3), given by equation (5), which can vary depending on the choice for  $A(0)$  but always happens at  $A_{\text{sat}}/\sqrt{3}$ .

$$t_{\text{infl}} = \frac{1}{2} \log \left[ \frac{1 - bA^2(0)}{2bA^2(0)} \right] \quad (5)$$

and corresponds to a characteristic amplitude of  $A(t_{\text{infl}}) = A_{\text{sat}}/\sqrt{3}$ . The inflection is indicated by the dashed lines in figure 1.

In figure 1, we compare the solution for  $\hat{\nu}_{\text{eff}} \gg 1$ , equation (3), with the full time-delayed cubic equation, equation (1), for different values of  $\hat{\nu}_{\text{eff}}$ . The difference between numerical results of equation (1) and the analytic solution (3) can be understood in terms of the high diffusivity approximation. The extrinsic stochasticity coming from  $\hat{\nu}_{\text{eff}}$  acts to destroy phase-space correlations and to produce entropy. The lack of coherence in the orbital motion means that resonant particles very easily forget their phases and decorrelate from the resonance, i.e. they transition between the trapped and the de-trapped regimes within a characteristic time much smaller than the bounce time. The loss of phase information implied by the approximation  $\hat{\nu}_{\text{eff}} \gg 1$  translates into the system's inability to resolve the fine oscillations predicted by equation (1). Interestingly, however, equation (3) can still describe the general trend of a wave quasi-steady evolution.

We observe that equation (3) describes the trace of the wave amplitude reasonably well for  $\hat{\nu}_{\text{eff}} \gtrsim 2$ , which is when the full cubic equation admits a steady solution [6, 16]. The assumption of high  $\hat{\nu}_{\text{eff}}$  used to derive the analytical solution therefore turns out to be less restrictive than anticipated when one is only interested in the essence of the amplitude evolution. For getting the general trend, in fact,  $\hat{\nu}_{\text{eff}}$  simply needs to be high enough to ensure steady saturation (i.e. to prevent the emergence of wave chirping as well as other higher-order nonlinear bifurcations). We note that close to marginal

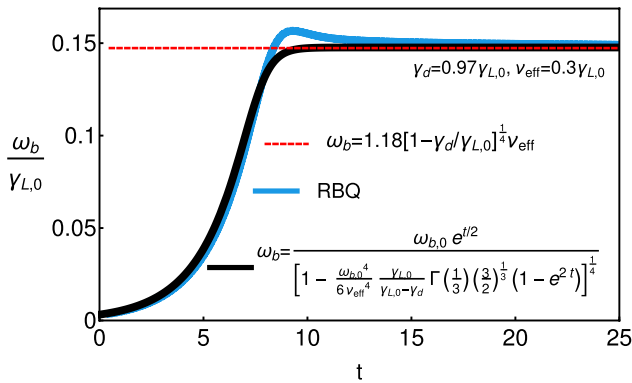
stability, simulations usually can get very costly as it takes longer to saturate, making equation (3) particularly useful in that regime. Regarding comparison with experiments, we note that in order to pull out the mode signal from the background noise, a Fourier time window has to be employed, which very much limits the capability of experimentally resolving the very fine oscillations (e.g., in figure 1a). Therefore the fact that equation (3) does not reproduce the oscillations around the mean amplitude is not too stringent when one has in mind experimental applications. In any event, the assumption  $\hat{\nu}_{\text{eff}} \gg 1$  falls into a typical tokamak operation scenario, as previously discussed.

The existence of a steady solution is always allowed in equation (2) since the linear term can in principle balance the cubic term. The stability of solution (3) can be addressed via eigenvalue analysis by substituting in equation (2) a perturbed solution in the form  $A_{\text{sat}} + \delta A e^{(\lambda_R + i\lambda_I)t}$ , with  $\lambda_R, \lambda_I \in \mathbb{R}$ . The result is  $\lambda_R = -2$  and  $\lambda_I = 0$ , which means that the saturated solution is intrinsically stable: any linear perturbation will exponentially asymptote to the saturation level, without the possibility of oscillations, which are suppressed by strong scattering processes.

We note that if the collisional scattering kernel of equation (1),  $e^{-\hat{\nu}_{\text{eff}}^2 z^2 (2z/3+y)}$ , were substituted by a Krook-type kernel  $e^{-\hat{\nu}_K (2z+y)}$  ( $\hat{\nu}_K$  is the Krook collisional frequency normalized with  $\gamma_L - \gamma_d$ ), then solutions of the same type of equations (3)–(5) are admitted, with the transformation  $b \mapsto \int \frac{d\Gamma \mathcal{H}}{8\hat{\nu}_K^4}$ . For the Krook case, the saturation level implied by the analytical solution is  $A_{\text{sat}} = 2\sqrt{2}\hat{\nu}_K^2$ , in agreement with [5].

Equations (3)–(5) can be used as a verification for simulations for the situation in which the amplitude of a marginally unstable wave evolves towards a quasi-steady saturation. Another possibility to explore the analytical solution (3) is to study its implications on the distribution function folding within the cubic equation framework, as recently numerically demonstrated [20]. A high scattering frequency used in this work destroys phase-space correlations and therefore prevents the emergence of self-organized scenarios, such as wave chirping and avalanching. Quasilinear theory employs a similar reasoning since it neglects the ballistic fast-oscillating term in its derivation, thereby also not capturing fully nonlinear wave behavior. An example of the comparison between equation (3) and the RBQ code [19] is shown in figure 2, which show fair agreement for regions of parameters where RBQ does not admit intermittent solutions.

If collisionality is moderate, we note that an amplitude overshoot occurs following the linear phase, as can be seen from figure 1(a). This can lead to instantaneous wide resonance islands (the resonance width is roughly proportional to  $\omega_b$  [21] and therefore proportional to  $\sqrt{A}$ ). The overshoot can be several times the saturated amplitude, as shown in [22]. This may lead to instantaneous overlap of distinct resonances and invalidate the analysis within the cubic equation framework. Therefore, for purposes of code verification, the expression (3) best applies when collisions are high enough to ensure a near-monotonic saturation, in addition to the near threshold, isolated regime. As a final remark, we point out that



**Figure 2.** Comparison between the resonance broadened quasilinear (RBQ) model [17–19] in its bump-on-tail formulation (blue) and the analytical solution (3) (black). The expected saturation level near marginal stability [6] is shown by the red dashed line. The parameters used in the simulation are  $\gamma_d = 0.97\gamma_{L,0}$  and  $\nu_{\text{eff}} = 0.3\gamma_{L,0}$ . The broadened resonance frequency  $\Delta\Omega = (\pi/2)(1.18)^4\nu_{\text{eff}}\gamma_d/\gamma_{L,0}$  used in RBQ ensures that the expected saturation level is achieved.

higher-order nonlinear effects not considered in this work, such as MHD nonlinearities and wave-wave coupling [23–25] can establish further bounds on the saturation level.

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## ORCID iDs

V.N. Duarte  <https://orcid.org/0000-0001-8096-7518>

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