The Theory of Kinetic Effects on Resistive Wall Mode Stability

J.W. Berkery,[1] R. Betti,[2] and S.A. Sabbagh[1]

[1)] *Department of Applied Physics and Applied Mathematics, Columbia University, New York, New York 10027*

[2)] *Laboratory for Laser Energetics, University of Rochester, Rochester, New York 14623*

(Dated: 12 March 2012)

1
2
3
4
5
Put in reference for Hao2011PRL

I. INTRODUCTION

Tokamak fusion plasmas generate energy most efficiently when the ratio of plasma stored energy to magnetic confining field energy is high. This ratio can be characterized by the quantity beta-normal: β_N. When a plasma reaches high β_N, an MHD kink-ballooning mode of instability can begin to grow. This can lead to a disruption of the plasma current and a loss of confinement on the relatively short Alfvén time scale. However, the growth rate of this mode can be slowed quite considerably by the presence of a close-fitting wall around the plasma. This forces the magnetic perturbations penetrate the wall in order to grow, and the time scale for that penetration is much longer than the Alfvén time scale. In fact, if the wall is perfectly conducting the time is infinite and the plasma is completely stabilized. If the wall has some resistance then the time scale is characterized by a wall time, τ_w. When the mode is converted to the more slowly growing mode in this way, it is called the resistive wall mode (RWM).

The plasma is stable up to a value of $\beta_N^{no-wall}$. Above this "no-wall" limit the plasma is unstable to kink-ballooning modes when no wall is present, or the resistive wall mode when a wall is present. The RWM grows on a much slower time scale, but it is still fast compared to the duration of the plasma shot. Therefore it is necessary to stabilize this mode as well. Originally it was thought that the presence of a resistive wall could slow down the kink-ballooning mode, but that the RWM itself could not be stabilized. Experiments soon found, however, that tokamaks could be stably operated above $\beta_N^{no-wall}$[1,2]. It was then postulated theoretically that the RWM can be stabilized by a combination of plasma rotational inertia and an energy dissipation mechanism[3–5]. Simple models proved to be insufficient to explain experimental results[2,6–8], however, and recently theoretical investigation has turned to the kinetic effects on plasma stability[9? ? –24]. Here we will derive the theoretical model for those kinetic effects in detail.

Let us consider all plasma quantities to be perturbed in time by a mode of instability from their equilibrium states with the following form[25]: $x = x_0 + \tilde{x}e^{-i\omega t - in\phi}$. Here, x is any quantity such as position, velocity, pressure, etc... We use the notation $\omega = \omega_r + i\gamma$ for the complex mode frequency, so that ω_r is the real mode rotation frequency, and γ is the growth rate. Also, ω_ϕ is the plasma toroidal rotation frequency, and n is the toroidal mode number. Now, we consider what happens when the plasma is displaced perpendicular to the magnetic field lines a small distance from its equilibrium position of $\boldsymbol{\xi_0} = 0$, so, $\boldsymbol{\xi}_\perp = \tilde{\boldsymbol{\xi}}_\perp e^{-i\omega t - in\phi}$. The goal is to find out whether this small displacement is stable or unstable, i.e. whether it will damp ($\gamma < 0$) or grow exponentially in time ($\gamma > 0$).

We will outline two general approaches to calculating stability by determining ω. The first approach, outlined briefly in Sec. II is to write a self-consistent set of equations for ω in terms of known quantities and then to solve the system. This approach has the advantage of self-consistency between the calculation of the mode frequency ω and the mode displacement $\boldsymbol{\xi}_\perp$. The second approach, which is the subject of the rest of this work, is to write an expression for ω in terms of changes in potential energy (δW) called a dispersion relation, and then solve for the δW terms. This approach has the advantage of clarity in distinguishing the various stabilizing and destabilizing effects. In Sec. III we begin with a conservation of energy equation and decompose it into constituent kinetic and potential energy terms. This equation becomes the basis of the dispersion relation for the RWM that we outline in Sec. IV. Although the concentration here is on the RWM, the physics involved is also directly applicable to internal kink modes[26] or neoclassical toroidal viscosity[?] . In this work we will concentrate on the change of potential energy that arises from the perturbed kinetic pressure and is written in terms of the perturbed distribution function of the various species of particles in the plasma. This perturbed distribution function is derived from the drift kinetic equation in Sec. V without any assumption of pressure isotropy. This results in anisotropy corrections to the fluid terms in Sec. VI, and a general form for δW_K and the electrostatic effect, that depend upon the distribution function of the particles considered, in Sec. VII. In Sec. VIII the δW equations are reformulated for easy calculation. Finally, in

Sec. IX the kinetic effects of particles with four specific distribution functions: Maxwellian, bi-Maxwellian, isotropic slowing-down, and anisotropic slowing-down, are considered.

II. A SELF-CONSISTENT SET OF EQUATIONS FOR ω

It is possible to derive a self-consistent set of equations for ω[25,27]. To begin we will consider the perturbed velocity of the plasma due to the perpendicular displacement $\boldsymbol{\xi}_\perp$ discussed in the introduction (it can be shown that a parallel displacement $\xi_\parallel$ is akin to reorientation of the frame of reference, and does not contribute to the kinetic effects that we are concerned with in this work[?]). The perturbed velocity is given by

$$\tilde{\mathbf{v}} = \frac{d\boldsymbol{\xi}_\perp}{dt} = \frac{\partial \boldsymbol{\xi}_\perp}{\partial t} + \boldsymbol{\nabla}\cdot\left(\boldsymbol{\xi}_\perp \mathbf{v}\right), \tag{1}$$

so that

$$\left(-i\omega\right)\boldsymbol{\xi}_\perp = \tilde{\mathbf{v}} - \mathbf{v}\cdot\boldsymbol{\nabla}\boldsymbol{\xi}_\perp - \boldsymbol{\xi}_\perp\boldsymbol{\nabla}\cdot\mathbf{v} \tag{2}$$

$$\left(-i\omega\right)\boldsymbol{\xi}_\perp = \tilde{\mathbf{v}} - \left(\mathbf{v_0} + \tilde{\mathbf{v}}e^{-i\omega t - in\phi}\right)\cdot\boldsymbol{\nabla}\tilde{\boldsymbol{\xi}}_\perp e^{-i\omega t - in\phi} - \tilde{\boldsymbol{\xi}}_\perp e^{-i\omega t - in\phi}\boldsymbol{\nabla}\cdot\left(\mathbf{v_0} + \tilde{\mathbf{v}}e^{-i\omega t - in\phi}\right). \tag{3}$$

Neglecting the quantities of second order in $e^{-i\omega t - in\phi}$, we have,

$$-i\omega\boldsymbol{\xi}_\perp = \tilde{\mathbf{v}} - \left(\mathbf{v_0}\cdot\boldsymbol{\nabla}\right)\boldsymbol{\xi}_\perp - \boldsymbol{\xi}_\perp\left(\boldsymbol{\nabla}\cdot\mathbf{v_0}\right). \tag{4}$$

Note that

$$\mathbf{v_0} = R^2\Omega_j\boldsymbol{\nabla}\hat{\boldsymbol{\phi}} \tag{5}$$

is the toroidal velocity[28], with R the major radius and Ω_j the toroidal rotation frequency for particles j. For thermal ions and electrons, $\Omega_j = \omega_\phi$, the bulk plasma frequency, but for energetic particles $\Omega_j \approx \omega_{*j}$, the diamagnetic frequency, which can be seen from radial force balance (see appendix A). One possible simplification is to use a rigid rotor assumption, in which case $\mathbf{v_0} = R\Omega_j\hat{\boldsymbol{\phi}}$ and Ω_j is a constant, rather than a function of the magnetic flux coordinate, Ψ.

Now, we can see that

$$\mathbf{v_0}\cdot\boldsymbol{\nabla} = -in\Omega_j, \tag{6}$$

and therefore for thermal particles Eq. 7 can be written:

$$\left(\gamma + i(n\omega_\phi - \omega_r)\right)\boldsymbol{\xi}_\perp = \tilde{\mathbf{v}} - \boldsymbol{\xi}_\perp\boldsymbol{\nabla}\cdot\mathbf{v_0}. \tag{7}$$

This is the first equation in the desired set of self-consistent equations. These equations will be noted as we proceed, and then gathered together at the end of the section.

Next, we proceed with a force balance equation for the plasma:

$$\rho\frac{d\mathbf{v}}{dt} = \mathbf{j}\times\mathbf{B} - \boldsymbol{\nabla}\cdot\mathbb{P}. \tag{8}$$

where $\mathbb{P}$ is the pressure tensor. Expanding the total derivative and linearizing the equation (expanding the perturbed quantities in the same manner as above), for the left hand side we find:

$$\rho\frac{d\mathbf{v}}{dt} = \rho\frac{\partial\mathbf{v}}{\partial t} + \rho\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} \tag{9}$$

$$= \left(\rho_0 + \tilde{\rho}e^{-i\omega t - in\phi}\right)\left(-i\omega\right)\left(\tilde{\mathbf{v}}e^{-i\omega t - in\phi}\right) + \left(\rho_0 + \tilde{\rho}e^{-i\omega t - in\phi}\right)\left(\mathbf{v_0} + \tilde{\mathbf{v}}e^{-i\omega t - in\phi}\right)\cdot\boldsymbol{\nabla}\left(\mathbf{v_0} + \tilde{\mathbf{v}}e^{-i\omega t - in\phi}\right). \tag{10}$$

Retaining only terms of first order in $e^{-i\omega t - in\phi}$, we have

$$\rho\frac{d\mathbf{v}}{dt} = (-i\omega)\,\rho_0\tilde{\mathbf{v}}e^{-i\omega t-in\phi} + \rho_0\tilde{\mathbf{v}}e^{-i\omega t-in\phi}\cdot\boldsymbol{\nabla}\mathbf{v_0} + \rho_0\mathbf{v_0}\cdot\boldsymbol{\nabla}\tilde{\mathbf{v}}e^{-i\omega t-in\phi} + \tilde{\rho}e^{-i\omega t-in\phi}\mathbf{v_0}\cdot\boldsymbol{\nabla}\mathbf{v_0}. \tag{11}$$

Now dividing both sides of Eq. 8 by $e^{-i\omega t-in\phi}$, we have,

$$(-i\omega)\left(\rho_0\tilde{\mathbf{v}}\right) + \rho_0\left(\mathbf{v_0}\cdot\nabla\tilde{\mathbf{v}} + \tilde{\mathbf{v}}\cdot\nabla\mathbf{v_0}\right) + \tilde{\rho}\mathbf{v_0}\cdot\nabla\mathbf{v_0} = \mathbf{j_0}\times\tilde{\mathbf{B}} + \tilde{\mathbf{j}}\times\mathbf{B_0} - \boldsymbol{\nabla}\cdot\tilde{\mathbb{P}}, \tag{12}$$

or,

$$\rho_0(\gamma + i(n\omega_\phi - \omega_r))\tilde{\mathbf{v}} = \mathbf{j_0}\times\tilde{\mathbf{B}} + \tilde{\mathbf{j}}\times\mathbf{B_0} - \boldsymbol{\nabla}\cdot\tilde{\mathbb{P}} - \rho_0\tilde{\mathbf{v}}\cdot\nabla\mathbf{v_0} - \tilde{\rho}\mathbf{v_0}\cdot\nabla\mathbf{v_0}. \tag{13}$$

This is the second equation in the self-consistent set. The perturbed current, $\tilde{\mathbf{j}}$ is given by Ampere's Law:

$$\tilde{\mathbf{j}} = \frac{1}{\mu_0}\boldsymbol{\nabla}\times\tilde{\mathbf{B}}, \tag{14}$$

which is also an equation in the self-consistent set. The perturbed magnetic field, $\tilde{\mathbf{B}}$, is found through Faraday's induction equation

$$\frac{\partial\mathbf{B}}{\partial t} = -\boldsymbol{\nabla}\times\mathbf{E}, \tag{15}$$

and Ohm's law

$$\mathbf{E} + \mathbf{v}\times\mathbf{B} = \eta\mathbf{j}. \tag{16}$$

Together these two equations result in:

$$\frac{\partial\mathbf{B}}{\partial t} = \boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{B}) - \boldsymbol{\nabla}\times(\eta\mathbf{j}). \tag{17}$$

Linearizing, we find

$$(-i\omega)\tilde{\mathbf{B}}e^{-i\omega t-in\phi} = \boldsymbol{\nabla}\times\left(\left(\mathbf{v_0}+\tilde{\mathbf{v}}e^{-i\omega t-in\phi}\right)\times\left(\mathbf{B_0}+\tilde{\mathbf{B}}e^{-i\omega t-in\phi}\right)\right) - \boldsymbol{\nabla}\times\left(\eta\left(\mathbf{j_0}+\tilde{\mathbf{j}}e^{-i\omega t-in\phi}\right)\right). \tag{18}$$

Now keeping terms of first order in $e^{-i\omega t-in\phi}$,

$$\begin{aligned}(-i\omega)\tilde{\mathbf{B}} &= \boldsymbol{\nabla}\times\left(\mathbf{v_0}\times\tilde{\mathbf{B}}\right) + \boldsymbol{\nabla}\times\left(\tilde{\mathbf{v}}\times\mathbf{B_0}\right) - \boldsymbol{\nabla}\times\left(\eta\tilde{\mathbf{j}}\right) &(19)\\ &= \boldsymbol{\nabla}\times\left(\tilde{\mathbf{v}}\times\mathbf{B_0}\right) - \boldsymbol{\nabla}\times\left(\eta\tilde{\mathbf{j}}\right) - \tilde{\mathbf{B}}\left(\boldsymbol{\nabla}\cdot\mathbf{v_0}\right). &(20)\end{aligned}$$

which is the fourth equation in the set. Not the same as Eq. 48! (Check also Ref. **?** for the Lagrangian form.) We also note that if $\eta = 0$, the ideal assumption, then

$$\begin{aligned}(-i\omega + \boldsymbol{\nabla}\cdot\mathbf{v_0})\tilde{\mathbf{B}} &= \boldsymbol{\nabla}\times\left(\tilde{\mathbf{v}}\times\mathbf{B_0}\right) &(21)\\ &= \boldsymbol{\nabla}\times\left(\left((-i\omega)\boldsymbol{\xi}_\perp + \mathbf{v_0}\cdot\nabla\boldsymbol{\xi}_\perp\right)\times\mathbf{B_0}\right), &(22)\end{aligned}$$

or,

$$\tilde{\mathbf{B}} = \boldsymbol{\nabla} \times (\boldsymbol{\xi}_\perp \times \mathbf{B_0}) \tag{23}$$
$$= \boldsymbol{\xi}_\perp \cancel{(\boldsymbol{\nabla} \cdot \mathbf{B_0})} - \mathbf{B_0} (\boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp) \tag{24}$$
$$= -\mathbf{B_0} (\boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp) . \tag{25}$$

Note that $\tilde{\mathbf{B}}$ is often denoted as $\mathbf{Q}$ (in Refs. 29–31, for example).

Additionally, Eq. 16 can be used to find an expression for $\tilde{\mathbf{E}}$:

$$\tilde{\mathbf{E}} = \eta\tilde{\mathbf{j}} - \tilde{\mathbf{v}} \times \mathbf{B_0} - \mathbf{v_0} \times \tilde{\mathbf{B}} \tag{26}$$
$$= \eta\tilde{\mathbf{j}} - (-i\omega\boldsymbol{\xi}_\perp + \mathbf{v_0} \cdot \boldsymbol{\nabla}\boldsymbol{\xi}_\perp) \times \mathbf{B_0} - \mathbf{v_0} \times (-\mathbf{B_0} (\boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp)) \tag{27}$$
$$= \eta\tilde{\mathbf{j}} + i\omega\boldsymbol{\xi}_\perp \times \mathbf{B_0} - \mathbf{v_0} \cdot \boldsymbol{\nabla}\boldsymbol{\xi}_\perp \times \mathbf{B_0} + \mathbf{v_0} \times \mathbf{B_0} (\boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp) \tag{28}$$
$$= i\omega\boldsymbol{\xi}_\perp \times \mathbf{B_0} - \boldsymbol{\nabla}\tilde{\boldsymbol{\Phi}}. \tag{29}$$

We have taken perpendicular current to be zero, and parallel current to be due to the gradient of a perturbed potential. Such a potential can arise from a perturbation of quasineutrality in the parallel direction. (Note, look at Ref. 32 in the appendix).

The perturbed density is found through conservation of mass:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0 \tag{30}$$
$$(-i\omega)\tilde{\rho}e^{-i\omega t - in\phi} = -\left(\rho_0 + \tilde{\rho}e^{-i\omega t - in\phi}\right) \boldsymbol{\nabla} \cdot \left(\mathbf{v_0} + \tilde{\mathbf{v}}e^{-i\omega t - in\phi}\right) - \left(\mathbf{v_0} + \tilde{\mathbf{v}}e^{-i\omega t - in\phi}\right) \boldsymbol{\nabla} \cdot \left(\rho_0 + \tilde{\rho}e^{-i\omega t - in\phi}\right) \tag{31}$$
$$(-i\omega + \boldsymbol{\nabla} \cdot \mathbf{v_0})\tilde{\rho} = -\mathbf{v_0}\boldsymbol{\nabla} \cdot \tilde{\rho} - \rho_0 \boldsymbol{\nabla} \cdot \tilde{\mathbf{v}} - \tilde{\mathbf{v}}\boldsymbol{\nabla} \cdot \rho_0 \tag{32}$$
$$(-i\omega + \boldsymbol{\nabla} \cdot \mathbf{v_0} + \mathbf{v_0} \cdot \boldsymbol{\nabla})\tilde{\rho} = -\rho_0 \boldsymbol{\nabla} \cdot \tilde{\mathbf{v}} - \tilde{\mathbf{v}} \cdot \boldsymbol{\nabla}\rho_0. \tag{33}$$

This is the fifth equation in the set. We can also continue further, by substituting for $\tilde{\mathbf{v}}$ from Eq. 4:

$$\tilde{\rho} = -\rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp - (\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla})\, \rho_0. \tag{34}$$

The well known problem of closure of the set of equations requires us to make specification for the equilibrium and perturbed pressures. We will consider a pressure tensor with components in the directions parallel and perpendicular to the magnetic field,

$$\mathbb{P} = p_\| \hat{\mathbf{b}}\hat{\mathbf{b}} + p_\perp \left(\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}\right), \tag{35}$$

where $\hat{\mathbf{I}}$ is the identity tensor. One must now be careful in linearizing the above equation, remembering that $\hat{\mathbf{b}} = \mathbf{B}/B$ can also be perturbed[29]. Therefore,

$$\tilde{\mathbb{P}} = \tilde{p}_\| \hat{\mathbf{b}}\hat{\mathbf{b}} + \tilde{p}_\perp \left(\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}\right) + (p_\| - p_\perp)B^{-2} \left(\tilde{\mathbf{B}}\mathbf{B} + \mathbf{B}\tilde{\mathbf{B}}\right). \tag{36}$$

At this point the problem naturally separates into fluid and kinetic approaches. In the fluid approach the perturbed pressures are given in terms of macroscopic quantities. In the kinetic approach, $\tilde{p}_\perp$ and $\tilde{p}_\|$ are defined by using the perturbed distribution function $\tilde{f}$. There are two common fluid approximations. The first is to assume the equilibrium pressure and the perturbed pressure are isotropic so, $\boldsymbol{\nabla} \cdot \tilde{\mathbb{P}} = \nabla\tilde{p}$. Then the adiabatic equation is used to find $\tilde{p}$ in a method outlined below. In the second common fluid approach, two adiabatic equations are used to find the two CGL perturbed pressures, $\tilde{p}_\perp$ and $\tilde{p}_\|$. This method is outlined in appendix M.

In the kinetic approach[29,32–35] one can separate the perturbed pressure components into fluid and kinetic parts, $\tilde{p}_\| = \tilde{p}_\|^F + \tilde{p}_\|^K$ and $\tilde{p}_\perp = \tilde{p}_\perp^F + \tilde{p}_\perp^K$. This way, taking the divergence of the perturbed pressure tensor from Eq. 36, we finally have

$$\nabla \cdot \tilde{\mathbb{P}} = \nabla \cdot \left(\left(\tilde{p}_{\parallel}^{F} + \tilde{p}_{\parallel}^{K} \right) \hat{\mathbf{b}}\hat{\mathbf{b}} \right) + \nabla \cdot \left(\left(\tilde{p}_{\perp}^{F} + \tilde{p}_{\perp}^{K} \right) \left(\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}} \right) \right) + \nabla \cdot \left[(p_{\parallel} - p_{\perp}) B^{-2} \left(\tilde{\mathbf{B}}\mathbf{B} + \mathbf{B}\tilde{\mathbf{B}} \right) \right], \tag{37}$$

which is the sixth equation in the set.

Finally, let us make an assumption, for the moment, that the perturbed fluid pressure is isotropic, so that $\tilde{p}_{\parallel}^F = \tilde{p}_{\perp}^F = \tilde{p}$, which will now be given by the adiabatic equation. The adiabatic equation conserves entropy density, $p\rho^{-\gamma_j}$ (with γ_j as the ratio of specific heats)[33,36,37]. Note that using $\gamma_j = 0$ conserves pressure, and $\gamma_j = 1$ conserves temperature (isothermal).

$$\frac{d}{dt}\left(p\rho^{-\gamma_j}\right) = 0 \tag{38}$$

$$\rho^{-\gamma_j}\frac{dp}{dt} + p\frac{d\rho^{-\gamma_j}}{dt} = 0 \tag{39}$$

$$\frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p = \gamma_j p \rho^{\gamma_j}\rho^{-\gamma_j - 1}\frac{\partial \rho}{\partial t} \tag{40}$$

$$(-i\omega)\tilde{p}e^{-i\omega t - in\phi} + \left(\mathbf{v_0} + \tilde{\mathbf{v}}e^{-i\omega t - in\phi}\right)\cdot\nabla\left(p_0 + \tilde{p}e^{-i\omega t - in\phi}\right) = -\gamma_j\left(p_0 + \tilde{p}e^{-i\omega t - in\phi}\right)\frac{1}{\rho}\left(\nabla\cdot\rho\left(\mathbf{v_0} + \tilde{\mathbf{v}}e^{-i\omega t - in\phi}\right)\right) \tag{41}$$

$$(-i\omega + \mathbf{v_0}\cdot\nabla)\tilde{p} = -\left(\tilde{\mathbf{v}}\cdot\nabla\right)p_0 - \gamma_j p_0 \nabla\cdot\tilde{\mathbf{v}} - \gamma_j\tilde{p}\nabla\cdot\mathbf{v_0}. \tag{42}$$

which is the final equation in the self-consistent set. The full set of equations will now be relisted for convenience, but first as we did for the density, we can take one step further and eliminate $\tilde{\mathbf{v}}$: fix this:

$$(-i\omega + \mathbf{v_0}\cdot\nabla)\tilde{p} = -(-i\omega + \mathbf{v_0}\cdot\nabla)\boldsymbol{\xi}_{\perp}\cdot\nabla p_0 - \gamma_j p_0\left(\nabla\cdot(-i\omega) + \mathbf{v_0}\cdot\nabla)\boldsymbol{\xi}_{\perp}\right), \tag{43}$$

Look in Ref. [38].

$$\tilde{p} = -\boldsymbol{\xi}_{\perp}\cdot\nabla p_0 - \gamma_j p_0 \nabla\cdot\boldsymbol{\xi}_{\perp}. \tag{44}$$

Equations 7, 13, 14, 20, 33, 37, and 42 form a full set of seven self-consistent equations for the perturbed quantities $\boldsymbol{\xi}_{\perp}$, $\tilde{\mathbf{v}}$, $\tilde{\mathbf{j}}$, $\tilde{\mathbf{B}}$, $\tilde{\rho}$, $\tilde{p}$ and the ultimate goal, ω. The quantities ρ_0, $\mathbf{j_0}$, $\mathbf{B_0}$, $\mathbf{v_0}$, p_0, and η must be specified as measured input quantities, while specification of the perturbed kinetic pressures, $\tilde{p}_{\parallel}^K$ and $\tilde{p}_{\perp}^K$ forms the crux of the problem. They will be found from moments of the perturbed distribution function using the kinetic approach, as discussed in the next section. The set of equations is:

$$(\gamma + i(n\omega_\phi - \omega_r))\boldsymbol{\xi}_{\perp} = \tilde{\mathbf{v}} - \mathbf{v_0}\cdot\nabla\boldsymbol{\xi}_{\perp} \tag{45}$$

$$\rho_0(\gamma + i(n\omega_\phi - \omega_r))\tilde{\mathbf{v}} = \mathbf{j_0}\times\tilde{\mathbf{B}} + \tilde{\mathbf{j}}\times\mathbf{B_0} - \nabla\cdot\tilde{\mathbb{P}} - \rho_0\left(\mathbf{v_0}\cdot\nabla\tilde{\mathbf{v}} + \tilde{\mathbf{v}}\cdot\nabla\mathbf{v_0}\right) - \tilde{\rho}\mathbf{v_0}\cdot\nabla\mathbf{v_0} \tag{46}$$

$$\tilde{\mathbf{j}} = \frac{1}{\mu_0}\nabla\times\tilde{\mathbf{B}} \tag{47}$$

$$(\gamma + i(n\omega_\phi - \omega_r))\tilde{\mathbf{B}} = \nabla\times\left(\tilde{\mathbf{v}}\times\mathbf{B_0}\right) - \nabla\times\left(\eta\tilde{\mathbf{j}}\right) - \tilde{\mathbf{B}}\left(\nabla\cdot\mathbf{v_0}\right) \tag{48}$$

$$(\gamma + i(n\omega_\phi - \omega_r))\tilde{\rho} = -\left(\tilde{\mathbf{v}}\cdot\nabla\right)\rho_0 - \rho_0\nabla\cdot\tilde{\mathbf{v}} \tag{49}$$

$$\nabla\cdot\tilde{\mathbb{P}} = \nabla\tilde{p} + \nabla\cdot\left[\tilde{p}_{\perp}^K\mathbf{I} + (\tilde{p}_{\parallel}^K - \tilde{p}_{\perp}^K)\hat{\mathbf{b}}\hat{\mathbf{b}}\right] + \nabla\cdot\left[(p_{\parallel} - p_{\perp})B^{-2}\left(\tilde{\mathbf{B}}\mathbf{B} + \mathbf{B}\tilde{\mathbf{B}}\right)\right] \tag{50}$$

$$(\gamma + i(n\omega_\phi - \omega_r))\tilde{p} = -\left(\tilde{\mathbf{v}}\cdot\nabla\right)p_0. \tag{51}$$

If the following assumptions are made: zero resistivity (the ideal assumption $\eta = 0$), isotropic equilibrium pressure ($p_{\parallel} - p_{\perp} = 0$), and $\mathbf{v_0}\cdot\nabla\mathbf{v_0} = 0$ (which eliminates the necessity for an equation for $\tilde{\rho}$)[12], and we substitute Eqs. 5 and 6 for $\mathbf{v_0}$ and use $\mathbf{v_0}\cdot\nabla\tilde{\mathbf{v}} = \tilde{\mathbf{v}}\times\left(\mathbf{v_0}/R\times\hat{\mathbf{R}}\right)$ in Eq. 46, then the result is the set of equations used in the `MARS-K` code[12,13,39,40] to self-consistently solve for the stability of the RWM:

$$(\gamma + i(n\omega_\phi - \omega_r))\boldsymbol{\xi}_\perp = \tilde{\mathbf{v}} - (\boldsymbol{\xi}_\perp \cdot \nabla\Omega_j)\, R^2 \nabla\hat{\phi} \tag{52}$$

$$\rho_0(\gamma + i(n\omega_\phi - \omega_r))\tilde{\mathbf{v}} = \mathbf{j_0} \times \tilde{\mathbf{B}} + \tilde{\mathbf{j}} \times \mathbf{B_0} - \boldsymbol{\nabla} \cdot \tilde{\mathbb{P}} - \rho_0 \left(2\Omega_j \hat{\mathbf{z}} \times \tilde{\mathbf{v}} + (\tilde{\mathbf{v}} \cdot \nabla\Omega_j)\, R^2 \nabla\hat{\phi}\right) \tag{53}$$

$$\tilde{\mathbf{j}} = \frac{1}{\mu_0} \boldsymbol{\nabla} \times \tilde{\mathbf{B}} \tag{54}$$

$$(\gamma + i(n\omega_\phi - \omega_r))\tilde{\mathbf{B}} = \boldsymbol{\nabla} \times (\tilde{\mathbf{v}} \times \mathbf{B_0}) - \left(\tilde{\mathbf{B}} \cdot \nabla\Omega_j\right) R^2 \nabla\hat{\phi} \tag{55}$$

$$\boldsymbol{\nabla} \cdot \tilde{\mathbb{P}} = \nabla\tilde{p} + \boldsymbol{\nabla} \cdot \left[\tilde{p}_\perp^K \mathbf{I} + (\tilde{p}_\parallel^K - \tilde{p}_\perp^K)\hat{\mathbf{b}}\hat{\mathbf{b}}\right] \tag{56}$$

$$(\gamma + i(n\omega_\phi - \omega_r))\tilde{p} = -\left(\tilde{\mathbf{v}} \cdot \boldsymbol{\nabla}\right) p_0. \tag{57}$$

Note that this set of equations makes no specific reference to the wall surrounding the plasma. The dependence of the RWM displacement, $\boldsymbol{\xi}_\perp$, on the geometry of the device arises self-consistently from the specification of $\mathbf{j_0}$ and $\mathbf{B_0}$.

III. THE ENERGY PRINCIPLE

A different approach is to change the force balance equation into an equation in terms of changes of kinetic and potential energies (δW), and then to write a dispersion relation for the complex mode frequency ω in terms of these δW terms[34,41]. This approach has been called an "energy principle"[?] - the principle being that if any small displacement, $\boldsymbol{\xi}_\perp$, from the equilibrium can be found that causes the potential energy to decrease, the kinetic energy to increase and the displacement to grow exponentially in time, then that equilibrium is unstable. Specifically applicable to the RWM is the so-called "low-frequency" energy principle[29,33,34,41,42], which requires the inclusion of the particle drift frequencies, as these can not be considered to be much lower than the mode frequency.

We start by rewriting Eq. 46 in a convenient form:

$$-i\left(\omega - n\omega_\phi\right)\rho_0\tilde{\mathbf{v}} + \rho_0\left(\mathbf{v_0} \cdot \nabla\right)\tilde{\mathbf{v}} + \rho_0\tilde{\mathbf{v}} \cdot \nabla\mathbf{v_0} = \mathbf{j_0} \times \tilde{\mathbf{B}} + \tilde{\mathbf{j}} \times \mathbf{B_0} - \boldsymbol{\nabla} \cdot \tilde{\mathbb{P}} - \tilde{\rho}\mathbf{v_0} \cdot \nabla\mathbf{v_0}, \tag{58}$$

and then substituting for $\tilde{\mathbf{v}}$ from Eq. 4 and for $\mathbf{v_0} \cdot \boldsymbol{\nabla}$ from Eq. 6 in the left hand side:

$$-\left(\omega - n\omega_\phi\right)^2 \rho_0\boldsymbol{\xi}_\perp - \rho_0\left(n\omega_\phi\right)\left(\omega - n\omega_\phi\right)\boldsymbol{\xi}_\perp + \rho_0\left(-i\right)\left(\omega - n\omega_\phi\right)\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla}\mathbf{v_0} = -\rho_0\boldsymbol{\xi}_\perp\left(\omega - n\omega_\phi\right)\left(\omega + i\boldsymbol{\nabla}\mathbf{v_0}\right), \tag{59}$$

so that we have (Missing something here):

$$-\rho_0\boldsymbol{\xi}_\perp\left(\omega - n\omega_\phi\right)\left(\omega - n\omega_\phi - \omega_{*j}\right) = \mathbf{j_0} \times \tilde{\mathbf{B}} + \tilde{\mathbf{j}} \times \mathbf{B_0} - \boldsymbol{\nabla} \cdot \tilde{\mathbb{P}} - \tilde{\rho}\mathbf{v_0} \cdot \nabla\mathbf{v_0}. \tag{60}$$

Now to convert the force balance to an energy balance, we multiply both sides of Eq. 60 by $\boldsymbol{\xi}_\perp^*/2$ (where $\boldsymbol{\xi}_\perp^*$ is the complex conjugate of $\boldsymbol{\xi}_\perp$), integrate with respect to volume, and sum over all species j:

$$\frac{1}{2}\sum_j \int \rho_0\left(\omega - n\omega_\phi\right)\left(\omega - n\omega_\phi - \omega_{*j}\right)\left|\boldsymbol{\xi}_\perp\right|^2 d\mathbf{V} = -\frac{1}{2}\sum_j \int \boldsymbol{\xi}_\perp^* \cdot \left[\mathbf{j_0} \times \tilde{\mathbf{B}} + \tilde{\mathbf{j}} \times \mathbf{B_0} - \boldsymbol{\nabla} \cdot \tilde{\mathbb{P}}_j - \tilde{\rho}\mathbf{v_0} \cdot \nabla\mathbf{v_0}\right] d\mathbf{V}. \tag{61}$$

(There is an imaginary part of the inertial term, as it is written here.) This equation can be written $\delta I = -\delta W$, where the left hand side is the kinetic energy, also known as the inertial term, and the right hand side is the negative of the change in the potential energy. Alternatively we can write $\mathcal{L} = \delta I + \delta W = 0$, where $\mathcal{L}$ is the MHD Lagrangian (a function summarizing the dynamics of the system)[36].

First, let us consider the kinetic energy integral, which we will henceforth call the inertial term to avoid confusion with the portion of the potential energy term, δW, arising from kinetic *effects*.

$$\delta I = \frac{1}{2}\sum_j \int \rho_0(\gamma + i(n\omega_\phi - \omega_r))(\gamma + i(n\omega_\phi - \omega_r) + i\omega_{*j})\left|\boldsymbol{\xi}_\perp\right|^2 d\mathbf{V} \tag{62}$$

Note that δI is often simplified to $\delta I = \omega^2 K_M$, where

$$K_M = \frac{1}{2}\sum_j \int \rho_0 \left|\boldsymbol{\xi}_\perp\right|^2 d\mathbf{V}, \tag{63}$$

is an MHD kinetic energy normalization term. This simplification is only truly valid for high frequency modes, when $|\omega_r| \gg |\omega_\phi|$, $|\omega_{*j}|$, however.

Now we will consider δW. Using Eqs. 14 for $\tilde{\mathbf{j}}$ and 25 for $\tilde{\mathbf{B}}$, we have

$$\begin{aligned}\delta W =& \frac{1}{2}\int \boldsymbol{\xi}_\perp^* \cdot \left[\mathbf{j_0} \times (\boldsymbol{\nabla} \times (\boldsymbol{\xi}_\perp \times \mathbf{B_0})) + \frac{1}{\mu_0}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times (\boldsymbol{\xi}_\perp \times \mathbf{B_0})) \times \mathbf{B_0}\right] d\mathbf{V} \\ &- \frac{1}{2}\sum_j \int \boldsymbol{\xi}_\perp^* \cdot \left[\tilde{\rho}\mathbf{v_0} \cdot \nabla \mathbf{v_0}\right] d\mathbf{V} - \frac{1}{2}\sum_j \int \boldsymbol{\xi}_\perp^* \cdot \left[\boldsymbol{\nabla} \cdot \tilde{\mathbb{P}}\right] d\mathbf{V}.\end{aligned} \tag{64}$$

The pressure term can now be evaluated with either an assumption of fluid or kinetic pressure.

A. The Isotropic Fluid Approach to δW

With the isotropic fluid assumption, $\boldsymbol{\nabla} \cdot \mathbb{P} = \nabla p$, and Eq. 44 can be written:

$$\boldsymbol{\nabla} \cdot \tilde{\mathbb{P}} = \nabla\left(-\gamma_j p_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp - \boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} p_0\right). \tag{65}$$

Then, using Eq. 65 in Eq. 64, δW is written

$$\delta W = \delta W_F + \delta W_C + \delta W_V \tag{66}$$

$$\begin{aligned}&= \frac{1}{2}\int_F \boldsymbol{\xi}_\perp^* \cdot \left[\mathbf{j_0} \times (\boldsymbol{\nabla} \times (\boldsymbol{\xi}_\perp \times \mathbf{B_0})) + \frac{1}{\mu_0}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times (\boldsymbol{\xi}_\perp \times \mathbf{B_0})) \times \mathbf{B_0} + \boldsymbol{\nabla}\left(\gamma_j p_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp - \boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} p_0\right)\right] d\mathbf{V} \\ &-\frac{1}{2}\int_F \boldsymbol{\xi}_\perp^* \cdot \left[\tilde{\rho}\mathbf{v_0} \cdot \nabla \mathbf{v_0}\right] d\mathbf{V} + \frac{1}{2}\int_V \boldsymbol{\xi}_\perp^* \cdot \left[\frac{1}{\mu_0}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times (\boldsymbol{\xi}_\perp \times \mathbf{B_0})) \times \mathbf{B_0}\right] d\mathbf{V},\end{aligned} \tag{67}$$

where F denotes the fluid, V the vacuum, C is for the centrifugal force, and we have made use of $\mathbf{j} = 0$, $\rho = 0$ and $p = 0$ in the vacuum region.

The above equation is solved by various numerical codes. For example, the `PEST` code[43] solves for the fluid δW_F, in the form of Eq. (17) of Ref. [30], and uses the `VACUUM` code[44] to solve for δW_V. It can also be written in various ways. It is useful to separate out the fluid and vacuum components as we have done (and for example in Eq. (6.4.7) in Ref. [45]), or to separate the various modes of instability, for example Eq. (39) in Ref. [37], Eq. (8) in Ref. [33], Eq. (58) in Ref. [29], or Eq. (1.18) in Ref. [46]. Then the various terms of the potential energy can be seen to be contributing to stabilizing shear Alfvén waves, compressional Alfvén waves, and sound waves, and the two terms that can drive instability by pressure driven modes or current driven modes.

One such useful solution is to use

$$\mathbf{j_0} \times \tilde{\mathbf{B}} + \tilde{\mathbf{j}} \times \mathbf{B_0} = \frac{1}{\mu_0}\left[-\boldsymbol{\nabla}\left(B\tilde{\mathbf{B}}_\parallel\right) + \mathbf{B} \cdot \boldsymbol{\nabla}\left(\tilde{\mathbf{B}}_\parallel \hat{\mathbf{b}} + \hat{\mathbf{b}}\tilde{\mathbf{B}}_\parallel\right) + \boldsymbol{\nabla} \cdot \left(\mathbf{B}\tilde{\mathbf{B}}_\perp + \tilde{\mathbf{B}}_\perp \mathbf{B}\right)\right]. \tag{68}$$

This form will be used in Sec. VI for the fluid term.

We will now turn our attention first briefly to the centrifugal term, which is typically not calculated by codes such as `PEST`. Then in the next subsection we will relax the assumption of an isotropic equilibrium and perturbed pressure tensor, and use the kinetic approach to find both the kinetic contribution to δW and the anisotropic corrections to the fluid term as well.

B. Centrifugal Force

Toroidal plasma rotation affects the plasma in various ways. First, equilibrium pressure profiles are shifted[47], and β_p is effectively enhanced [?] . These effects are important for the stability of the internal kink[48], as the greatest shift and enhancement occurs near the axis, where the rotation is high. Also, however, a centrifugal force arises from the $\tilde{\rho}\mathbf{v_0} \cdot \nabla \mathbf{v_0}$ term of Eq. 13. Since it always pushes outward, it is a destabilizing effect. If we consider the expression for δW_C in Eq. 67, and substitute for $\tilde{\rho}$ with Eq. 34 and for $\mathbf{v_0}$ with Eq. 5, then we find:

$$\delta W_C = \frac{1}{2} \sum_j \int \boldsymbol{\xi}_\perp^* \cdot \left[\boldsymbol{\nabla} \cdot (\rho_0 \xi_\perp) \, R^2 \Omega_j \boldsymbol{\nabla} \hat{\boldsymbol{\phi}} \cdot \boldsymbol{\nabla} \left(R^2 \Omega_j \boldsymbol{\nabla} \hat{\boldsymbol{\phi}} \right) \right] d\mathbf{V}. \tag{69}$$

This term is a fluid term, and is always real, but it is not calculated by a code such as `PEST`, for example, since `PEST` does not include plasma toroidal rotation.

Note that the contribution to the centrifugal force from energetic particles may be as important, or more important, than that of thermal particles[37]. Compare $\omega_\phi^2 \rho_i$ to $\omega_{*\alpha}^2 \rho_\alpha$.

C. The Kinetic Approach to δW

By using the kinetic approach and allowing for anisotropic pressure (Eq. 50), we will show that the Lagrangian is changed to:

$$\delta I + \delta W_F + \delta W_C + \delta W_V + \delta W_A + \delta W_\Phi + \delta W_K = 0, \tag{70}$$

where we have separated the kinetic term δW_K from all the other fluid terms. It will be shown that all of the fluid terms are strictly real, while δW_K is complex. Here δW_Φ is an electrostatic term and δW_A is an additional fluid term due to anisotropy of the plasma pressure. Part of it is due to a perturbation of the direction of the magnetic field in an anisotropic equilibrium pressure plasma[29], resulting from the last term of Eq. 50. This part is explicitly dependent on anisotropy due to the $p_\parallel - p_\perp$ dependence. There is also another piece to this term that is more subtle, which will be fleshed out in the following. Considering the fluid, anisotropic, electrostatic, and kinetic parts together then,

$$\delta W_F + \delta W_A + \delta W_\Phi + \delta W_K = \frac{1}{2} \int_F \boldsymbol{\xi}_\perp^* \cdot \left(\mathbf{j_0} \times \tilde{\mathbf{B}} + \tilde{\mathbf{j}} \times \mathbf{B_0} - \boldsymbol{\nabla} \cdot \tilde{\mathbb{P}} \right) d\mathbf{V}. \tag{71}$$

The total perturbed pressure tensor can be written:

$$\tilde{\mathbb{P}} = \sum_j m_j \int \mathbf{vv} \left(\tilde{f}_j + \frac{\partial f_j}{\partial B} \boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} \mathbf{B} + \frac{\partial f_j}{\partial \Phi} \boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} \boldsymbol{\Phi} \right) d^3\mathbf{v}. \tag{72}$$

We will turn our attention to the perturbed distribution function in Sec. V. With an expression for $\tilde{f}_j$ in hand, we will return to this point (Eq. 71) to complete the formulation of δW_F and δW_A in Sec. VI and δW_Φ and δW_K in Sec. VII. First, however, in the next section we will discuss dispersion relations for the RWM, using Eq. 70 as a starting point.

IV. DISPERSION RELATION AND STABILITY CONDITIONS

We wish to solve for the complex mode frequency of the RWM in terms of the δI and δW terms outlined in the previous section through a dispersion relation. In particular we are interested in the kinetic effects which appear in the δW_K term. In this section we will derive the dispersion relation and discuss the impact of the various δW terms on the mode's stability. We will then proceed to the derivation of δW_K in the rest of the paper.

When a resistive wall is placed in the vacuum region, δW_V takes the form:

$$\delta W_V = \delta W_V^i + \delta W_V^w + \delta W_V^o, \tag{73}$$

where i indicates the inner vacuum region between the plasma and the wall, w indicates the wall itself, and o indicates the outer region (from the wall to ∞). A lengthly calculation[49] can be used to recast this expression in terms of δW_V^b and δW_V^∞, which are the change in potential energies due to the mode in the vacuum region when a wall is placed at location b, or if there is no wall (wall at ∞). The δW_V term takes the form[4,11,28,49]:

$$\delta W_V = \frac{\delta W_V^b(-i\omega)\tau_w + \delta W_V^\infty}{(-i\omega)\tau_w + 1}, \tag{74}$$

where τ_w is related to the current decay time in the resistive wall, and is given by a specific formula (see appendix B). Sometimes this factor is written τ_w^* to distinguish it from the true wall time, τ_w, but we will drop the asterisk here for convenience. Note that as $b \to \infty$ the τ_w terms disappear. Equation 74 changes the Lagrangian from Eq. 70 to:

$$\delta I + \delta W_F + \delta W_K + \frac{\delta W_V^b(-i\omega)\tau_w + \delta W_V^\infty}{(-i\omega)\tau_w + 1} = 0. \tag{75}$$

Here, for convenience, we have subsumed δW_C, δW_A, and δW_Φ into δW_F.

Let us now also define $\delta W_\infty = \delta W_F + \delta W_V^\infty$, the sum of the plasma fluid and vacuum perturbed potential energies when the wall is placed at infinity (the no-wall condition), and $\delta W_b = \delta W_F + \delta W_V^b$, the sum of the plasma fluid and vacuum δWs when the wall is placed at a specific location b. These two contributions to the energy principle have been theoretically developed for years[49], and computer codes have been written to solve for them, such as `PEST`[43] and `DCON`[50]. Note that setting $\tau_w = 0$ leads to the internal kink mode dispersion relation[26], starting from $\delta I + \delta W_K + \delta W_\infty = 0$.

The above expression is valid in the range where $\delta W_b > 0$, that is where $\beta < \beta_b$, the "with-wall" or "ideal" limit. Also, in order for the problem to be considered ideal, not resistive, the wall time should be less than the ideal-wall tearing mode growth time (see appendix C).

Now, if we solve for $-i\omega = \gamma - i\omega_r$, the dispersion relation is

$$(\gamma - i\omega_r)\tau_w = -\frac{\delta W_\infty + \delta W_K + \delta I}{\delta W_b + \delta W_K + \delta I}. \tag{76}$$

We have already seen that the δW_K and δI terms also include γ and ω_r in their formulations, so the above expression is non-linear. In fact, in general, there are three possible roots of the RWM from this dispersion relation[5,12,16,51]. Further discussion of the multiple roots of the RWM can be found in appendix D. Putting that complication aside for the moment, we can simplify the above expression one step further by neglecting plasma inertia, which sets $\delta I \to 0$. Usually this approximation is not made for the internal kink mode, but it is often made for the RWM.

(Can this be justified by looking at a comparison of the δI and δW_K terms?)

$$\frac{\delta I}{\delta W_K} \approx \frac{\omega_\phi \boldsymbol{\xi}_\perp^* \cdot \mathbf{v}}{v^2 \boldsymbol{\xi}_\perp^* \cdot \boldsymbol{\kappa}} \approx \frac{\omega_\phi}{v\boldsymbol{\kappa}} \approx \frac{a}{R}. \tag{77}$$

With large aspect ratio, then, the dispersion relation can be written[9,10,12]:

$$(\gamma - i\omega_r)\tau_w = -\frac{\delta W_\infty + \delta W_K}{\delta W_b + \delta W_K}. \tag{78}$$

Finally, if the complex kinetic term, δW_K, is also neglected, the result is the fluid growth rate for resistive wall modes neglecting plasma inertia and kinetic effects, which is written[49]: $\gamma_F\tau_w = -\delta W_\infty/\delta W_b$.

The change in potential energy due to kinetic effects, δW_K, in general has both real and imaginary parts, while δI is real. The real part of ω is the mode rotation frequency, and is given by:

$$\omega_r\tau_w = \frac{Im(\delta W_K)(\delta W_b - \delta W_\infty)}{(\delta W_b + Re(\delta W_K) + \delta I)^2 + (Im(\delta W_K))^2}, \tag{79}$$

and solving for the imaginary part, we find the normalized growth rate:

$$\gamma\tau_w = -\frac{\delta W_\infty \delta W_b + (Im(\delta W_K))^2 + (Re(\delta W_K) + \delta I)(\delta W_\infty + \delta W_b + Re(\delta W_K) + \delta I)}{(\delta W_b + Re(\delta W_K) + \delta I)^2 + (Im(\delta W_K))^2}. \quad (80)$$

Since the denominator is always positive, a condition for stability is that the numerator is positive (therefore the growth rate is negative). The first term of the numerator is the fluid (MHD) instability drive. It is negative and therefore destabilizing when $\beta_\infty < \beta < \beta_b$ (the "wall-stabilized" regime of β) and it is positive when $\beta < \beta_\infty$. The second term shows that the imaginary part of δW_K is always stabilizing. The third term is stabilizing or destabilizing depending on the sign and the relative magnitudes of the real part of $\delta W_K + \delta I$, δW_∞, and δW_b.

Another way of writing the stability condition is[18]

$$\left(Re\left(\delta W_K\right) + \delta I - a\right)^2 + \left(Im\left(\delta W_K\right)\right)^2 = r^2, \quad (81)$$

where

$$a = \frac{1}{2}\left(\delta W_b + \delta W_\infty\right) + \frac{1}{2}\left(\delta W_b - \delta W_\infty\right)\frac{\gamma\tau_w}{1+\gamma\tau_w}, \quad (82)$$

and

$$r = \frac{1}{2}\left(\delta W_b - \delta W_\infty\right)\frac{1}{1+\gamma\tau_w}. \quad (83)$$

It is easy to see that on a plot of $Im(\delta W_K)$ vs. $Re(\delta W_K) + \delta I$, contours of constant $\gamma\tau_w$ form circles with offset a and radius r.

Therefore, for a given δW_∞ and δW_b, the plasma is stable if $Re(\delta W_K) + \delta I$ and $Im(\delta W_K)$ lie outside of a circle centered at $(-\frac{1}{2}(\delta W_b + \delta W_\infty), 0)$ with radius $\frac{1}{2}(\delta W_b - \delta W_\infty)$. Once the values of δW_∞ and δW_b are known, it can be predicted what values of $Im(\delta W_K)$ and $Re(\delta W_K) + \delta I$ will be necessary to provide stabilization. If $Re(\delta W_K) + \delta I > -\frac{1}{2}(\delta W_b + \delta W_\infty)$, which is usually the case, then increasing $Re(\delta W_K) + \delta I$ will decrease the growth rate. Wait, positive δI is stabilizing? Not intuitive - is it right? Increasing $|Im(\delta W_K)|$ always decreases the growth rate.

Therefore determining the stability of a plasma equilibrium to resistive wall modes involves the calculation of δW_∞, δW_b, δW_K, and δI, if it is not neglected. Let us now turn our attention to the solution for the δW, which we have already formulated up to the point of dependence on $\mathbb{P}$, and therefore, in turn, on $\tilde{f}_j$. First we will derive an expression for $\tilde{f}_j$ in terms of a general f_j, then use it in the various δW terms, and then finally consider some specific distributions f_j.

V. PERTURBED DISTRIBUTION FUNCTION

To find an expression for the perturbed distribution function, $\tilde{f}_j$, for particles j, we start with the drift kinetic equation,

$$\frac{d\tilde{f}_j}{dt} + \left[\frac{\tilde{\mathbf{F}}}{m_j}\right]\boldsymbol{\nabla}_v f_j = C(\tilde{f}_j), \quad (84)$$

where $\tilde{\mathbf{F}}$ is the perturbed force, and $C(\tilde{f}_j)$ is the collision operator. There are several possibilities of increasing complexity that can be used for the collisionality[20]. These are discussed in appendix G. For now we will consider the collisionless drift kinetic (Vlasov) equation:

$$\frac{d\tilde{f}_j}{dt} = -\left[\frac{\tilde{\mathbf{F}}}{m_j}\right]\boldsymbol{\nabla}_v f_j, \quad (85)$$

with collisionality to be added later in an ad-hoc manner (in appendix H). Using the method of characteristics on the above equation,

$$\tilde{f}_j = \int d\tilde{f}_j = -\int_{-\infty}^{t} \left[\frac{\tilde{\mathbf{F}}}{m_j}\right] \boldsymbol{\nabla}_v f_j dt'. \tag{86}$$

If $f_j = f_j(\varepsilon, P_\phi, \mu)$, in general, where the particle total energy $\varepsilon = m_j v^2/2 + Z_j e\Phi$, the toroidal canonical momentum $P_\phi = m_j R v_\phi + Z_j e\Psi$, and the particle magnetic moment $\mu = m_j v_\perp^2/(2B)$, then

$$\tilde{f}_j = -\int_{-\infty}^{t} \frac{Z_j e}{m_j}\left(\tilde{\mathbf{E}} + \mathbf{v}\times\tilde{\mathbf{B}}\right)\cdot\left[\hat{\mathbf{e}}_{\mathbf{v}}\frac{\partial f_j}{\partial v} + \hat{\mathbf{e}}_\phi \frac{\partial f_j}{\partial v_\phi} + \hat{\mathbf{e}}_\perp \frac{\partial f_j}{\partial v_\perp}\right] dt' \tag{87}$$

$$= -\int_{-\infty}^{t} Z_j e\left(\tilde{\mathbf{E}} + \mathbf{v}\times\tilde{\mathbf{B}}\right)\cdot\left[\mathbf{v}\frac{\partial f_j}{\partial \varepsilon} + \hat{\mathbf{e}}_\phi R\frac{\partial f_j}{\partial P_\phi} + \frac{\mathbf{v}_\perp}{B}\frac{\partial f_j}{\partial \mu}\right] dt' \tag{88}$$

$$= -\int_{-\infty}^{t} Z_j e\left[\tilde{\mathbf{E}}\cdot\mathbf{v}\frac{\partial f_j}{\partial \varepsilon} + \cancel{(\mathbf{v}\times\tilde{\mathbf{B}})\cdot\mathbf{v}\frac{\partial f_j}{\partial \varepsilon}} + R\tilde{\mathbf{E}}_\phi\frac{\partial f_j}{\partial P_\phi} + R\left(\mathbf{v}\times\tilde{\mathbf{B}}\right)\cdot\hat{\mathbf{e}}_\phi\frac{\partial f_j}{\partial P_\phi} + \left(\tilde{\mathbf{E}} + \mathbf{v}\times\tilde{\mathbf{B}}\right)\cdot\mathbf{v}_\perp\frac{1}{B}\frac{\partial f_j}{\partial \mu}\right] dt'. \tag{89}$$

The first term above includes $\delta\varepsilon = Z_j e\int_{-\infty}^{t}\tilde{\mathbf{E}}\cdot\mathbf{v}dt'$, which is the change in energy of a particle moving across time-varying fields and is also associated with kinetic compressibility[52]. Let us now examine the second to last term separately:

$$R\left(\mathbf{v}\times\tilde{\mathbf{B}}\right)\cdot\hat{\mathbf{e}}_\phi\frac{\partial f_j}{\partial P_\phi} = R\mathbf{v}\cdot\left(\tilde{\mathbf{B}}\times\hat{\mathbf{e}}_\phi\right)\frac{\partial f_j}{\partial P_\phi} \tag{90}$$

$$= R\mathbf{v}\cdot\left(\frac{1}{i\omega}\left(\boldsymbol{\nabla}\times\tilde{\mathbf{E}}\right)\times\hat{\mathbf{e}}_\phi\right)\frac{\partial f_j}{\partial P_\phi} \tag{91}$$

$$= \frac{1}{i\omega}\mathbf{v}\cdot\left(-\nabla\left(R\tilde{\mathbf{E}}_\phi\right) + \mathbf{e}_\phi\cdot\nabla\left(R\tilde{\mathbf{E}}\right)\right)\frac{\partial f_j}{\partial P_\phi} \tag{92}$$

$$= \left(-\frac{1}{i\omega}\mathbf{v}\cdot\nabla\left(R\tilde{\mathbf{E}}_\phi\right) - \frac{n}{\omega}\mathbf{v}\cdot\tilde{\mathbf{E}}\right)\frac{\partial f_j}{\partial P_\phi} \tag{93}$$

Here we have used the general perturbed forms $\mathbf{B} = \mathbf{B_0} + \tilde{\mathbf{B}}e^{-i\omega t - in\phi}$ and $\mathbf{E} = \mathbf{E_0} + \tilde{\mathbf{E}}e^{-i\omega t - in\phi}$ so that $\partial/\partial t \to -i\omega$ and $\nabla \to -in$. Now returning to Eq. 89:

$$\tilde{f}_j = -\int_{-\infty}^{t} Z_j e\left[\left(\tilde{\mathbf{E}}\cdot\mathbf{v}\right)\left(\frac{\partial f_j}{\partial\varepsilon} - \frac{n}{\omega}\frac{\partial f_j}{\partial P_\phi}\right) - \frac{1}{i\omega}\frac{\partial f_j}{\partial P_\phi}\left(-i\omega\left(R\tilde{\mathbf{E}}_\phi\right) + \mathbf{v}\cdot\nabla\left(R\tilde{\mathbf{E}}_\phi\right)\right) + \left(\tilde{\mathbf{E}} + \mathbf{v}\times\tilde{\mathbf{B}}\right)\cdot\mathbf{v}_\perp\frac{1}{B}\frac{\partial f_j}{\partial\mu}\right] dt' \tag{94}$$

$$= -\int_{-\infty}^{t} Z_j e\left[\left(\tilde{\mathbf{E}}\cdot\mathbf{v}\right)\left(\frac{\partial f_j}{\partial\varepsilon} - \frac{n}{\omega}\frac{\partial f_j}{\partial P_\phi}\right) - \frac{1}{i\omega}\frac{\partial f_j}{\partial P_\phi}\left(\frac{d\left(R\tilde{\mathbf{E}}_\phi\right)}{dt'}\right) + \left(\tilde{\mathbf{E}} + \mathbf{v}\times\tilde{\mathbf{B}}\right)\cdot\mathbf{v}_\perp\frac{1}{B}\frac{\partial f_j}{\partial\mu}\right] dt' \tag{95}$$

$$= -Z_j e\left(\frac{\partial f_j}{\partial\varepsilon} - \frac{n}{\omega}\frac{\partial f_j}{\partial P_\phi}\right)\int_{-\infty}^{t}\tilde{\mathbf{E}}\cdot\mathbf{v}dt' + \frac{Z_j e}{i\omega}\frac{\partial f_j}{\partial P_\phi}R\tilde{\mathbf{E}}_\phi - Z_j e\frac{1}{B}\frac{\partial f_j}{\partial\mu}\int_{-\infty}^{t}\left(\tilde{\mathbf{E}} + \mathbf{v}\times\tilde{\mathbf{B}}\right)\cdot\mathbf{v}_\perp dt'. \tag{96}$$

This is the same as Eq. (17) of Ref. [32], with the addition of the $\partial f_j/\partial\mu$ term. Let us now look at the $\tilde{\mathbf{E}}\cdot\mathbf{v}$ term, using $\tilde{\mathbf{E}}$ from Eq. 29.

$$\int_{-\infty}^{t} \tilde{\mathbf{E}} \cdot \mathbf{v} dt' = \int_{-\infty}^{t} \left(i\omega \left(\boldsymbol{\xi}_{\perp} \times \mathbf{B_0} \right) \cdot \mathbf{v} - \nabla \tilde{\boldsymbol{\Phi}} \cdot \mathbf{v} \right) dt' \tag{97}$$

$$= \int_{-\infty}^{t} \left(-i\omega \left(\mathbf{v} \times \mathbf{B_0} \right) \cdot \boldsymbol{\xi}_{\perp} - \mathbf{v} \cdot \nabla \tilde{\boldsymbol{\Phi}} \right) dt' \tag{98}$$

$$= \int_{-\infty}^{t} \left(-i\omega \left(\mathbf{v} \times \mathbf{B_0} + \mathbf{E_0} + \nabla \boldsymbol{\Phi_0} \right) \cdot \boldsymbol{\xi}_{\perp} - \left(\frac{d\tilde{\boldsymbol{\Phi}}}{dt} - \frac{\partial \tilde{\boldsymbol{\Phi}}}{\partial t} \right) \right) dt' \tag{99}$$

$$= \int_{-\infty}^{t} \left(-i\omega \left(\mathbf{E_0} + \mathbf{v} \times \mathbf{B_0} \right) \cdot \boldsymbol{\xi}_{\perp} - i\omega \boldsymbol{\xi}_{\perp} \cdot \boldsymbol{\nabla} \Phi_0 - \frac{d\tilde{\boldsymbol{\Phi}}}{dt} - i\omega \tilde{\boldsymbol{\Phi}} \right) dt' \tag{100}$$

$$= \int_{-\infty}^{t} \left(-i\omega \frac{m_j}{Z_j e} \frac{d\mathbf{v}}{dt'} \cdot \boldsymbol{\xi}_{\perp} - \frac{d\tilde{\boldsymbol{\Phi}}}{dt'} - i\omega \left(\tilde{\boldsymbol{\Phi}} + \boldsymbol{\xi}_{\perp} \cdot \boldsymbol{\nabla} \Phi_0 \right) \right) dt' \tag{101}$$

$$= \int_{-\infty}^{t} \left(-i\omega \frac{m_j}{Z_j e} \left(\frac{d \left(\mathbf{v} \cdot \boldsymbol{\xi}_{\perp} \right)}{dt'} - \mathbf{v} \cdot \frac{d\boldsymbol{\xi}_{\perp}}{dt'} \right) - \frac{d\tilde{\boldsymbol{\Phi}}}{dt'} - i\omega \left(\tilde{\boldsymbol{\Phi}} + \boldsymbol{\xi}_{\perp} \cdot \boldsymbol{\nabla} \Phi_0 \right) \right) dt' \tag{102}$$

$$= -i\omega \frac{m_j}{Z_j e} \left(\mathbf{v} \cdot \boldsymbol{\xi}_{\perp} \right) - \tilde{\boldsymbol{\Phi}} + i\omega \int_{-\infty}^{t} \left(\frac{m_j}{Z_j e} \mathbf{v} \cdot \frac{d\boldsymbol{\xi}_{\perp}}{dt'} - \left(\tilde{\boldsymbol{\Phi}} + \boldsymbol{\xi}_{\perp} \cdot \boldsymbol{\nabla} \Phi_0 \right) \right) dt' \tag{103}$$

Returning to Eq. 96,

$$\tilde{f}_j = i m_j \left(\omega \frac{\partial f_j}{\partial \varepsilon} - n \frac{\partial f_j}{\partial P_\phi} \right) \left[\mathbf{v} \cdot \boldsymbol{\xi}_{\perp} + \frac{Z_j e}{i\omega m_j} \tilde{\boldsymbol{\Phi}} - \int_{\infty}^{t} \left(\mathbf{v} \cdot \frac{d\boldsymbol{\xi}_{\perp}}{dt'} - \frac{Z_j e}{m_j} \left(\tilde{\boldsymbol{\Phi}} + \boldsymbol{\xi}_{\perp} \cdot \boldsymbol{\nabla} \Phi_0 \right) \right) dt' \right]$$
$$+ \frac{Z_j e}{i\omega} \frac{\partial f_j}{\partial P_\phi} \left(R \tilde{\mathbf{E}}_\phi \right) - \frac{Z_j e}{B} \frac{\partial f_j}{\partial \mu} \int_{-\infty}^{t} \left(\tilde{\mathbf{E}} + \mathbf{v} \times \tilde{\mathbf{B}} \right) \cdot \mathbf{v}_{\perp} dt'. \tag{104}$$

Using $\tilde{\mathbf{E}}_\phi = \tilde{\mathbf{E}} \cdot \hat{\mathbf{e}}_\phi$, with $\tilde{\mathbf{E}}$ from Eq. 29, and defining

$$\tilde{s}_j = \int_{-\infty}^{t} \left(\mathbf{v} \cdot \frac{d\boldsymbol{\xi}_{\perp}}{dt'} - \frac{Z_j e}{m_j} \left(\tilde{\boldsymbol{\Phi}} + \boldsymbol{\xi}_{\perp} \cdot \boldsymbol{\nabla} \Phi_0 \right) \right) dt', \tag{105}$$

we have

$$\tilde{f}_j = i m_j \left(\omega \frac{\partial f_j}{\partial \varepsilon} - n \frac{\partial f_j}{\partial P_\phi} \right) \left(\mathbf{v} \cdot \boldsymbol{\xi}_{\perp} - \tilde{s}_j \right) - \frac{Z_j e}{B} \frac{\partial f_j}{\partial \mu} \int_{-\infty}^{t} \left(\tilde{\mathbf{E}} + \mathbf{v} \times \tilde{\mathbf{B}} \right) \cdot \mathbf{v}_{\perp} dt'$$
$$+ \left(Z_j e \frac{\partial f_j}{\partial \varepsilon} - \frac{n}{\omega} Z_j e \frac{\partial f_j}{\partial P_\phi} \right) \tilde{\boldsymbol{\Phi}} + \frac{Z_j e}{i\omega} \frac{\partial f_j}{\partial P_\phi} R \left(i\omega \boldsymbol{\xi}_{\perp} \times \mathbf{B} - \nabla \tilde{\boldsymbol{\Phi}} \right) \cdot \hat{\mathbf{e}}_\phi. \tag{106}$$

Here $\tilde{s}_j$ is the integral along the unperturbed orbits. Now, let's look at the second line above. Rewritten, its

$$Z_j e \frac{\partial f_j}{\partial \varepsilon} \tilde{\boldsymbol{\Phi}} - Z_j e \frac{\partial f_j}{\partial P_\phi} \left(\frac{n}{\omega} \tilde{\boldsymbol{\Phi}} - R \left(\boldsymbol{\xi}_{\perp} \times \mathbf{B} \right) \cdot \hat{\mathbf{e}}_\phi + \frac{R}{i\omega} \nabla \tilde{\boldsymbol{\Phi}} \cdot \hat{\mathbf{e}}_\phi \right) \tag{107}$$

$$= Z_j e \frac{\partial f_j}{\partial \varepsilon} \tilde{\boldsymbol{\Phi}} + Z_j e \frac{\partial f_j}{\partial P_\phi} R \left(\boldsymbol{\xi}_{\perp} \cdot \mathbf{B} \right) \times \hat{\mathbf{e}}_\phi \tag{108}$$

$$= Z_j e \frac{\partial f_j}{\partial \varepsilon} \tilde{\boldsymbol{\Phi}} - \frac{\partial f_j}{\partial P_\phi} \left(\boldsymbol{\xi}_{\perp} \cdot \boldsymbol{\nabla} Z_j e \Psi \right), \tag{109}$$

since $R\mathbf{B}_\phi \times \hat{\mathbf{e}}_\phi = \boldsymbol{\nabla} \Psi$. Returning to Eq. 106, we have

$$\tilde{f}_j = i m_j \left(\omega \frac{\partial f_j}{\partial \varepsilon} - n \frac{\partial f_j}{\partial P_\phi} \right) \left(\mathbf{v} \cdot \boldsymbol{\xi}_{\perp} - \tilde{s}_j \right) - \frac{Z_j e}{B} \frac{\partial f_j}{\partial \mu} \int_{-\infty}^{t} \left(\tilde{\mathbf{E}} + \mathbf{v} \times \tilde{\mathbf{B}} \right) \cdot \mathbf{v}_{\perp} dt'$$
$$+ Z_j e \frac{\partial f_j}{\partial \varepsilon} \tilde{\boldsymbol{\Phi}} - \frac{\partial f_j}{\partial P_\phi} \left(\boldsymbol{\xi}_{\perp} \cdot \boldsymbol{\nabla} Z_j e \Psi \right), \tag{110}$$

which is the same as Eq. (25) of Ref. [53]. Simplifying the second line above further, we see that it is

$$= Z_j e \frac{\partial f_j}{\partial \varepsilon} \tilde{\mathbf{\Phi}} - \frac{\partial f_j}{\partial P_\phi} \left(\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} \left(P_\phi - m_j R v_\phi\right)\right) \tag{111}$$

$$= Z_j e \frac{\partial f_j}{\partial \varepsilon} \tilde{\mathbf{\Phi}} - \frac{\partial f_j}{\partial P_\phi} \left(\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} P_\phi\right) + \frac{\partial f_j}{\partial P_\phi} \left(\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} m_j R v_\phi\right) \tag{112}$$

$$= -\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} f_j + Z_j e \frac{\partial f_j}{\partial \varepsilon} \tilde{\mathbf{\Phi}} + \frac{\partial f_j}{\partial P_\phi} \left(\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} m_j R v_\phi\right). \tag{113}$$

Returning to Eq. 106:

$$\tilde{f}_j = -\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} f_j + i m_j \left(\omega \frac{\partial f_j}{\partial \varepsilon} - n \frac{\partial f_j}{\partial P_\phi}\right) \left(\mathbf{v} \cdot \boldsymbol{\xi}_\perp - \tilde{s}_j\right) - \frac{m_j}{B} \frac{\partial f_j}{\partial \mu} M(\boldsymbol{\xi}_\perp) + Z_j e \frac{\partial f_j}{\partial \varepsilon} \tilde{\mathbf{\Phi}} + \frac{\partial f_j}{\partial P_\phi} \left(\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} m_j R v_\phi\right). \tag{114}$$

Following Ref. [54], we have defined:

$$M(\boldsymbol{\xi}_\perp) = \frac{Z_j e}{m_j} \int_{-\infty}^{t} \left(\tilde{\mathbf{E}} + \mathbf{v} \times \tilde{\mathbf{B}}\right) \cdot \mathbf{v}_\perp dt' \tag{115}$$

$$= \frac{1}{\mathbf{B}} \int_{-\infty}^{t} \left[i\omega \left(\boldsymbol{\xi}_\perp \times \mathbf{B}\right) \cdot \mathbf{v}_\perp + \left(\left(\mathbf{v}_\perp + \mathbf{v}_\parallel\right) \times \tilde{\mathbf{B}}\right) \cdot \mathbf{v}_\perp \right] dt' \tag{116}$$

$$= \frac{1}{\mathbf{B}} \int_{-\infty}^{t} \left[-i\omega \left(\boldsymbol{\xi}_\perp \cdot \mathbf{v}_\perp\right) \times \mathbf{B} + \left(\mathbf{v}_\perp \times \mathbf{v}_\perp\right) \cdot \tilde{\mathbf{B}} + \left(\mathbf{v}_\parallel \times \mathbf{v}_\perp\right) \cdot \tilde{\mathbf{B}} \right] dt' \tag{117}$$

$$= \int_{-\infty}^{t} \left[-i\omega \boldsymbol{\xi}_\perp \cdot \mathbf{v}_\perp + \frac{v_\perp^2}{\mathbf{B}} \frac{\mathbf{B}}{B} \cdot \tilde{\mathbf{B}} + \frac{\mathbf{v}_\parallel}{\mathbf{B}} \frac{\mathbf{B}}{B} \mathbf{v}_\perp \cdot \tilde{\mathbf{B}} \right] dt' \tag{118}$$

$$= \int_{-\infty}^{t} \left[-i\omega \boldsymbol{\xi}_\perp \cdot \mathbf{v}_\perp + v_\perp^2 \tilde{\mathbf{B}}_\parallel + \frac{v_\parallel \mathbf{v}_\perp}{B} \cdot \tilde{\mathbf{B}} \right] dt' \tag{119}$$

$$= -i\omega \boldsymbol{\xi}_\perp \cdot \mathbf{v}_\perp + \left(\frac{\mu}{m_j} \frac{\mathbf{B}}{B} + \frac{v_\parallel \mathbf{v}_\perp}{B}\right) \cdot \tilde{\mathbf{B}}, \tag{120}$$

so that

$$\begin{aligned} \tilde{f}_j = & -\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} f_j + Z_j e \frac{\partial f_j}{\partial \varepsilon} \tilde{\mathbf{\Phi}} + \frac{\partial f_j}{\partial P_\phi} \left(\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} m_j R v_\phi\right) \\ & + i m_j \left(\omega \frac{\partial f_j}{\partial \varepsilon} - n \frac{\partial f_j}{\partial P_\phi}\right) \left(\mathbf{v} \cdot \boldsymbol{\xi}_\perp - \tilde{s}_j\right) - \frac{m_j}{B} \frac{\partial f_j}{\partial \mu} \left(-i\omega \boldsymbol{\xi}_\perp \cdot \mathbf{v}_\perp + \left(\frac{\mu}{m_j} \frac{\mathbf{B}}{B} + \frac{v_\parallel \mathbf{v}_\perp}{B}\right) \cdot \tilde{\mathbf{B}}\right). \end{aligned} \tag{121}$$

Note that $\tilde{\mathbf{B}}$ is given by Eq. 25, and in the middle term of $M(\boldsymbol{\xi}_\perp)$, $(\mu/m_j)\mathbf{B} \cdot \tilde{\mathbf{B}}/B = (\mu/m_j)\tilde{\mathbf{B}}_\parallel$, where we have introduced $\tilde{\mathbf{B}}_\parallel$. We will now spend a moment to derive an expression for $\tilde{\mathbf{B}}_\parallel$ which will become useful later.

$$\tilde{\mathbf{B}}_\parallel = \mathbf{B} \cdot \tilde{\mathbf{B}}/B \tag{122}$$

$$= \hat{\mathbf{b}} \cdot \left(\boldsymbol{\nabla} \times \left(\boldsymbol{\xi}_\perp \times \mathbf{B}\right)\right) \tag{123}$$

$$= \hat{\mathbf{b}} \cdot \left(\boldsymbol{\xi}_\perp \left(\cancel{\boldsymbol{\nabla} \cdot \mathbf{B}}\right) - \mathbf{B} \left(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp\right)\right) \tag{124}$$

$$= \hat{\mathbf{b}} \cdot \left(-\mathbf{B} \boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp + \boldsymbol{\xi}_\perp \cancel{\boldsymbol{\nabla} \cdot \mathbf{B}} + \mathbf{B} \cdot \boldsymbol{\nabla} \boldsymbol{\xi}_\perp - \boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} \mathbf{B}\right) \tag{125}$$

$$= -B \boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp + \mathbf{B} \cdot \boldsymbol{\nabla} \left(\cancel{\boldsymbol{\xi}_\perp \cdot \hat{\mathbf{b}}}\right) - \boldsymbol{\xi}_\perp \cdot \left(\mathbf{B} \cdot \boldsymbol{\nabla} \hat{\mathbf{b}}\right) - \hat{\mathbf{b}} \cdot \boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} \mathbf{B} \tag{126}$$

$$= -B \left(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp + \boldsymbol{\xi}_\perp \cdot \left(\hat{\mathbf{b}} \cdot \boldsymbol{\nabla} \hat{\mathbf{b}}\right)\right) - \hat{\mathbf{b}} \cdot \boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} \mathbf{B} \tag{127}$$

$$= -B \left(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp + \boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp\right) - \hat{\mathbf{b}} \cdot \boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} \mathbf{B}, \tag{128}$$

where we have defined the magnetic curvature $\boldsymbol{\kappa} = \hat{\mathbf{b}} \cdot \boldsymbol{\nabla} \hat{\mathbf{b}}$. For the last term we can now use

$$\hat{\mathbf{b}} \cdot \boldsymbol{\xi}_\perp \cdot \nabla \mathbf{B} = \boldsymbol{\xi}_\perp \cdot \nabla \left(\hat{\mathbf{b}} \cdot \mathbf{B} \right) - \mathbf{B} \cdot \left(\boldsymbol{\xi}_\perp \cdot \nabla \hat{\mathbf{b}} \right), \tag{129}$$

or,

$$2\mathbf{B} \cdot \boldsymbol{\xi}_\perp \cdot \nabla \mathbf{B} = \boldsymbol{\xi}_\perp \cdot \nabla \left(\mathbf{B} \cdot \mathbf{B} \right). \tag{130}$$

Returning to Eq. 128,

$$\tilde{\mathbf{B}}_\parallel = - B \left(\nabla \cdot \boldsymbol{\xi}_\perp + \boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp + \frac{1}{B^2} \boldsymbol{\xi}_\perp \cdot \nabla \left(\frac{\mathbf{B}^2}{2} \right) \right), \tag{131}$$

Finally, using the equilibrium relation[29] (see appendix E, Eq. E5)

$$\nabla_\perp \left(\mu_0 p_\perp + \frac{\mathbf{B}^2}{2} \right) = \boldsymbol{\kappa} \left(B^2 + \mu_0 \left(p_\perp - p_\parallel \right) \right), \tag{132}$$

we have

$$\tilde{\mathbf{B}}_\parallel = - B \left(\nabla \cdot \boldsymbol{\xi}_\perp + 2\boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp + \boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp \left(\frac{\mu_0 \left(p_\perp - p_\parallel \right)}{B^2} \right) - \boldsymbol{\xi}_\perp \cdot \nabla \left(\frac{\mu_0 p_\perp}{B^2} \right) \right). \tag{133}$$

Now we can use a definition of an anisotropy parameter[29,55–57]

$$\sigma = 1 + \frac{\mu_0 \left(p_\perp - p_\parallel \right)}{B^2}, \tag{134}$$

to write the alternative form:

$$\tilde{\mathbf{B}}_\parallel = - B \left(\nabla \cdot \boldsymbol{\xi}_\perp + (1 + \sigma) \boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp - \boldsymbol{\xi}_\perp \cdot \nabla \left(\frac{\mu_0 p_\perp}{B^2} \right) \right), \tag{135}$$

which is the final form of $\tilde{\mathbf{B}}_\parallel$.

So now writing $\tilde{f}_j$ in terms of $\tilde{\mathbf{B}}_\parallel$:

$$\begin{aligned} \tilde{f}_j = &- \boldsymbol{\xi}_\perp \cdot \nabla f_j + Z_j e \frac{\partial f_j}{\partial \varepsilon} \tilde{\boldsymbol{\Phi}} + \frac{\partial f_j}{\partial P_\phi} \left(\boldsymbol{\xi}_\perp \cdot \nabla m_j R v_\phi \right) \\ &+ i m_j \left(\omega \frac{\partial f_j}{\partial \varepsilon} - n \frac{\partial f_j}{\partial P_\phi} \right) \left(\mathbf{v} \cdot \boldsymbol{\xi}_\perp - \tilde{s}_j \right) - \frac{m_j}{B} \frac{\partial f_j}{\partial \mu} \left(-i\omega \boldsymbol{\xi}_\perp \cdot \mathbf{v}_\perp + \frac{\mu}{m_j} \tilde{\mathbf{B}}_\parallel + \frac{v_\parallel}{B} \mathbf{v}_\perp \cdot \tilde{\mathbf{B}} \right). \end{aligned} \tag{136}$$

(What happened to the $\nabla m_j R v_\phi$ term?)

$$\tilde{f}_j = - \boldsymbol{\xi}_\perp \cdot \nabla f_j + Z_j e \frac{\partial f_j}{\partial \varepsilon} \tilde{\boldsymbol{\Phi}} + i m_j \left(\omega \frac{\partial f_j}{\partial \varepsilon} - n \frac{\partial f_j}{\partial P_\phi} \right) \left(\mathbf{v} \cdot \boldsymbol{\xi}_\perp - \tilde{s}_j \right) - \frac{m_j}{B} \frac{\partial f_j}{\partial \mu} \left(-i\omega \boldsymbol{\xi}_\perp \cdot \mathbf{v}_\perp + \frac{\mu}{m_j} \tilde{\mathbf{B}}_\parallel + \frac{v_\parallel}{B} \mathbf{v}_\perp \cdot \tilde{\mathbf{B}} \right). \tag{137}$$

The quantity $L_j \left(\boldsymbol{\xi}_\perp \right)$ in Ref. [54] contains the finite Larmor radius terms that we have neglected in Eq. 137.

VI. SOLUTION FOR δW_F AND δW_A

Returning now to the derivation of Sec. III, we start with Eq. 72 for $\tilde{\mathbb{P}}$:

$$\tilde{\mathbb{P}} = \sum_j m_j \int \mathbf{v}\mathbf{v}\left(\tilde{f}_j + \frac{\partial f_j}{\partial B}\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\mathbf{B} + \frac{\partial f_j}{\partial \Phi}\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\boldsymbol{\Phi}\right)d^3\mathbf{v}, \tag{138}$$

and Eq. 137 for $\tilde{f}_j$:

$$\tilde{f}_j = -\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla} f_j + Z_j e\frac{\partial f_j}{\partial\varepsilon}\tilde{\boldsymbol{\Phi}} + im_j\left(\omega\frac{\partial f_j}{\partial\varepsilon} - n\frac{\partial f_j}{\partial P_\phi}\right)(\mathbf{v}\cdot\boldsymbol{\xi}_\perp - \tilde{s}_j) - \frac{m_j}{B}\frac{\partial f_j}{\partial\mu}\left(-i\omega\boldsymbol{\xi}_\perp\cdot\mathbf{v}_\perp + \frac{\mu}{m_j}\tilde{\mathbf{B}}_\parallel + \frac{v_\parallel}{B}\mathbf{v}_\perp\cdot\tilde{\mathbf{B}}\right). \tag{139}$$

Now, expanding $\mathbf{v}\mathbf{v}$:

$$\begin{aligned}\tilde{\mathbb{P}} = &\sum_j m_j \int \left(v_\parallel^2\hat{\mathbf{b}}\hat{\mathbf{b}} + v_\parallel\left(\hat{\mathbf{b}}\mathbf{v}_\perp + \mathbf{v}_\perp\hat{\mathbf{b}}\right) + \frac{1}{2}v_\perp^2\left(\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}\right)\right)\\ &\left[-\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla} f_j + \mathbf{v}\cdot\boldsymbol{\xi}_\perp i m_j\left(\omega\frac{\partial f_j}{\partial\varepsilon} - n\frac{\partial f_j}{\partial P_\phi} + \omega\frac{1}{B}\frac{\partial f_j}{\partial\mu}\right) + im_j\left(\omega\frac{\partial f_j}{\partial\varepsilon} - n\frac{\partial f_j}{\partial P_\phi}\right)\tilde{s}_j\right.\\ &\left.-\mu\frac{\partial f_j}{\partial\mu}\left(\frac{1}{B}\left(\tilde{\mathbf{B}}_\parallel + \boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\mathbf{B}\right)\right) - \frac{m_j v_\parallel}{B^2}\frac{\partial f_j}{\partial\mu}\mathbf{v}_\perp\cdot\tilde{\mathbf{B}} + Z_j e\left(\tilde{\boldsymbol{\Phi}} + \boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\Phi_0\right)\frac{\partial f_j}{\partial\varepsilon}\right]d^3\mathbf{v}.\end{aligned} \tag{140}$$

From Eq. 128,

$$\tilde{\mathbf{B}}_\parallel + \boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\mathbf{B} = -B\left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right), \tag{141}$$

and we will also now define

$$\tilde{\boldsymbol{\mathcal{Z}}} = Z_j e\left(\tilde{\boldsymbol{\Phi}} + \boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\Phi_0\right). \tag{142}$$

Then

$$\begin{aligned}\tilde{\mathbb{P}} = &\sum_j m_j \int \left(v_\parallel^2\hat{\mathbf{b}}\hat{\mathbf{b}} + \frac{1}{2}v_\perp^2\left(\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}\right)\right)\left[-\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla} f_j + im_j\left(\omega\frac{\partial f_j}{\partial\varepsilon} - n\frac{\partial f_j}{\partial P_\phi}\right)\tilde{s}_j + \left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right)\mu\frac{\partial f_j}{\partial\mu} + \tilde{\boldsymbol{\mathcal{Z}}}\frac{\partial f_j}{\partial\varepsilon}\right]d^3\mathbf{v}\\ &+ \sum_j m_j \int v_\parallel\left(\hat{\mathbf{b}}\mathbf{v}_\perp + \mathbf{v}_\perp\hat{\mathbf{b}}\right)(\mathbf{v}\cdot\boldsymbol{\xi}_\perp)\, im_j\left(\omega\frac{\partial f_j}{\partial\varepsilon} - n\frac{\partial f_j}{\partial P_\phi} + \omega\frac{1}{B}\frac{\partial f_j}{\partial\mu}\right)d^3\mathbf{v}\\ &- \sum_j m_j \int v_\parallel\left(\hat{\mathbf{b}}\mathbf{v}_\perp + \mathbf{v}_\perp\hat{\mathbf{b}}\right)\left(\mathbf{v}_\perp\cdot\tilde{\mathbf{B}}\right)\left(\frac{m_j v_\parallel}{B^2}\frac{\partial f_j}{\partial\mu}\right)d^3\mathbf{v},\end{aligned} \tag{143}$$

where all other terms gyro-average to zero. (Is that correct?) Let us now separate the perturbed pressure tensor into a fluid and anisotropy term and a kinetic and electrostatic term.

$$\begin{aligned}\tilde{\mathbb{P}}_{F+A} = &\sum_j m_j \int \left(v_\parallel^2\hat{\mathbf{b}}\hat{\mathbf{b}} + \frac{1}{2}v_\perp^2\left(\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}\right)\right)\left[-\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla} f_j + \left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right)\mu\frac{\partial f_j}{\partial\mu}\right]d^3\mathbf{v}\\ &+ \sum_j m_j \int \left(\hat{\mathbf{b}}\boldsymbol{\xi}_\perp + \boldsymbol{\xi}_\perp\hat{\mathbf{b}}\right)\left(v_\parallel\frac{1}{2}v_\perp^2\right) im_j\left(\omega\frac{\partial f_j}{\partial\varepsilon} - n\frac{\partial f_j}{\partial P_\phi} + \omega\frac{1}{B}\frac{\partial f_j}{\partial\mu}\right)d^3\mathbf{v}\\ &- \sum_j m_j \int \left(\hat{\mathbf{b}}\tilde{\mathbf{B}}_\perp + \tilde{\mathbf{B}}_\perp\hat{\mathbf{b}}\right)\left(v_\parallel\frac{1}{2}v_\perp^2\right)\left(\frac{m_j v_\parallel}{B^2}\frac{\partial f_j}{\partial\mu}\right)d^3\mathbf{v},\end{aligned} \tag{144}$$

and

$$\tilde{\mathbb{P}}_{K+\Phi} = \sum_j m_j \int \left(v_\parallel^2 \hat{\mathbf{b}}\hat{\mathbf{b}} + \frac{1}{2} v_\perp^2 \left(\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}} \right) \right) \left[i m_j \left(\omega \frac{\partial f_j}{\partial \varepsilon} - n \frac{\partial f_j}{\partial P_\phi} \right) \tilde{s}_j + \tilde{\boldsymbol{\mathcal{Z}}} \frac{\partial f_j}{\partial \varepsilon} \right] d^3\mathbf{v}. \tag{145}$$

First, in this section, we will continue with the fluid and anisotropy contributions to δW, and then in the next section we will return to the kinetic and electrostatic parts. Continuing with the fluid term,

$$\begin{aligned}
\tilde{\mathbb{P}}_{F+A} =& \hat{\mathbf{b}}\hat{\mathbf{b}} \left[-\sum_j m_j \int v_\parallel^2 \boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} f_j d^3\mathbf{v} + \left(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp + \boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp \right) \sum_j m_j \int v_\parallel^2 \frac{m_j v_\perp^2}{2B} \frac{\partial f_j}{\partial \mu} d^3\mathbf{v} \right] \\
&+ \left(\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}} \right) \left[-\sum_j m_j \int \frac{1}{2} v_\perp^2 \boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} f_j d^3\mathbf{v} + \left(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp + \boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp \right) \sum_j m_j \int \frac{1}{2} v_\perp^2 \frac{m_j v_\perp^2}{2B} \frac{\partial f_j}{\partial \mu} d^3\mathbf{v} \right] \\
&+ \left(\hat{\mathbf{b}} \boldsymbol{\xi}_\perp + \boldsymbol{\xi}_\perp \hat{\mathbf{b}} \right) \left[\sum_j m_j \int \left(v_\parallel \frac{1}{2} v_\perp^2 \right) i m_j \left(\omega \frac{\partial f_j}{\partial \varepsilon} - n \frac{\partial f_j}{\partial P_\phi} + \omega \frac{1}{B} \frac{\partial f_j}{\partial \mu} \right) d^3\mathbf{v} \right] \\
&- \left(\hat{\mathbf{b}} \tilde{\mathbf{B}}_\perp + \tilde{\mathbf{B}}_\perp \hat{\mathbf{b}} \right) \left[\sum_j m_j \int \left(v_\parallel^2 \frac{1}{2} v_\perp^2 \right) \left(\frac{m_j}{B^2} \frac{\partial f_j}{\partial \mu} \right) d^3\mathbf{v} \right].
\end{aligned} \tag{146}$$

Let us now define

$$I_1 = \sum_j m_j \int v_\parallel^2 \frac{1}{2} v_\perp^2 \frac{m_j}{B} \frac{\partial f_j}{\partial \mu} d^3\mathbf{v}, \tag{147}$$

$$I_2 = \sum_j m_j \int \frac{1}{4} v_\perp^4 \frac{m_j}{B} \frac{\partial f_j}{\partial \mu} d^3\mathbf{v}, \tag{148}$$

and

$$\Pi_{\parallel\perp} = \sum_j m_j \int \left(v_\parallel \frac{1}{2} v_\perp^2 \right) i m_j \left(\omega \frac{\partial f_j}{\partial \varepsilon} - n \frac{\partial f_j}{\partial P_\phi} + \omega \frac{1}{B} \frac{\partial f_j}{\partial \mu} \right) d^3\mathbf{v}. \tag{149}$$

Then

$$\begin{aligned}
\tilde{\mathbb{P}}_{F+A} =& \hat{\mathbf{b}}\hat{\mathbf{b}} \left[-\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} p_\parallel + \left(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp + \boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp \right) I_1 \right] + \left(\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}} \right) \left[-\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} p_\perp + \left(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp + \boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp \right) I_2 \right] \\
&+ \left(\hat{\mathbf{b}} \boldsymbol{\xi}_\perp + \boldsymbol{\xi}_\perp \hat{\mathbf{b}} \right) \Pi_{\parallel\perp} - \frac{1}{B} \left(\hat{\mathbf{b}} \tilde{\mathbf{B}}_\perp + \tilde{\mathbf{B}}_\perp \hat{\mathbf{b}} \right) I_1.
\end{aligned} \tag{150}$$

Evaluating I_2 is straightforward,

$$I_2 = \sum_j m_j \int \frac{1}{4} v_\perp^4 \frac{m_j}{B} \frac{\partial f_j}{\partial \mu} d^3\mathbf{v} \tag{151}$$

$$= \sum_j m_j \int \frac{1}{4} v_\perp^4 \frac{m_j}{B} \left(\frac{-2B^2}{m_j v_\perp^2} \right) \frac{\partial f_j}{\partial B} d^3\mathbf{v} \tag{152}$$

$$= -\sum_j m_j \int \frac{1}{2} v_\perp^2 B \frac{\partial f_j}{\partial B} d^3\mathbf{v} \tag{153}$$

$$= -B \frac{\partial p_\perp}{\partial B}. \tag{154}$$

Similarly, for I_1

$$I_1 = \sum_j m_j \int v_\parallel^2 \frac{1}{2} v_\perp^2 \frac{m_j}{B} \frac{\partial f_j}{\partial \mu} d^3\mathbf{v} \tag{155}$$

$$= \sum_j m_j \int v_\parallel^2 \frac{1}{2} v_\perp^2 \frac{m_j}{B} \left(\frac{-2B^2}{m_j v_\perp^2} \right) \frac{\partial f_j}{\partial B} d^3\mathbf{v} \tag{156}$$

$$= -\sum_j m_j \int v_\parallel^2 B \frac{\partial f_j}{\partial B} d^3\mathbf{v} \tag{157}$$

$$= -B \frac{\partial p_\parallel}{\partial B}. \tag{158}$$

However, I_1 can also be evaluated in a different way, leading to a more useful result. To do that we replace $\partial f_j / \partial \mu$ using:

$$\frac{\partial f_j}{\partial v_\perp^2} = \frac{\partial f_j}{\partial \varepsilon} \frac{\partial \varepsilon}{\partial v_\perp^2} + \frac{\partial f_j}{\partial \mu} \frac{\partial \mu}{\partial v_\perp^2}, \tag{159}$$

$$\frac{\partial f_j}{\partial v_\parallel^2} = \frac{\partial f_j}{\partial \varepsilon} \frac{\partial \varepsilon}{\partial v_\parallel^2} + \frac{\partial f_j}{\partial \mu} \cancel{\frac{\partial \mu}{\partial v_\parallel^2}}. \tag{160}$$

Then

$$\frac{\partial f_j}{\partial \mu} = \frac{\partial v_\perp^2}{\partial \mu} \left(\frac{\partial f_j}{\partial v_\perp^2} - \frac{\partial f_j}{\partial \varepsilon} \frac{\partial \varepsilon}{\partial v_\perp^2} \right) \tag{161}$$

$$= \frac{\partial v_\perp^2}{\partial \mu} \left(\frac{\partial f_j}{\partial v_\perp^2} - \frac{\partial f_j}{\partial v_\parallel^2} \frac{\partial v_\parallel^2}{\partial \varepsilon} \frac{\partial \varepsilon}{\partial v_\perp^2} \right) \tag{162}$$

$$= \frac{2B}{m_j} \left(\frac{\partial f_j}{\partial v_\perp^2} - \frac{\partial f_j}{\partial v_\parallel^2} \frac{2}{m_j} \frac{m_j}{2} \right) \tag{163}$$

$$= \frac{2B}{m_j} \left(\frac{\partial f_j}{\partial v_\perp^2} - \frac{\partial f_j}{\partial v_\parallel^2} \right), \tag{164}$$

so that

$$I_1 = \sum_j m_j \int v_\parallel^2 \frac{1}{2} v_\perp^2 \frac{m_j}{B} \frac{\partial f_j}{\partial \mu} d^3\mathbf{v} \tag{165}$$

$$= \sum_j m_j \int v_\parallel^2 v_\perp^2 \left(\frac{\partial f_j}{\partial v_\perp^2} - \frac{\partial f_j}{\partial v_\parallel^2} \right) d^3\mathbf{v} \tag{166}$$

$$= 2\pi \sum_j m_j \int v_\parallel^2 \int \left(v_\perp^2 \frac{\partial f_j}{\partial v_\perp^2} v_\perp dv_\perp \right) dv_\parallel - 2\pi \sum_j m_j \int v_\perp^2 \int \left(v_\parallel^2 \frac{\partial f_j}{\partial v_\parallel^2} dv_\parallel \right) v_\perp dv_\perp \tag{167}$$

$$= 2\pi \sum_j m_j \int v_\parallel^2 \int \left(\frac{1}{2} v_\perp^2 \right) \int df_j dv_\parallel - 2\pi \sum_j m_j \int v_\perp^2 \int \left(\frac{1}{2} v_\parallel \right) \int df_j v_\perp dv_\perp \tag{168}$$

$$= 2\pi \sum_j m_j \int v_\parallel^2 \int \left(\int v_\perp dv_\perp \right) f_j dv_\parallel - 2\pi \sum_j m_j \int v_\perp^2 \int \left(\frac{1}{2} \int v_\parallel dv_\parallel \right) f_j v_\perp dv_\perp \tag{169}$$

$$= \sum_j m_j \int f_j v_\parallel^2 d^3\mathbf{v} - \sum_j m_j \int f_j \frac{1}{2} v_\perp^2 d^3\mathbf{v} \tag{170}$$

$$= p_\perp - p_\parallel. \tag{171}$$

(Somehow I get the wrong sign between the last two lines above) where we have used

$$\int d^3\mathbf{v} = 2\pi \int\int v_\perp dv_\perp dv_\parallel. \tag{172}$$

Note also, from Eq. 158, that

$$p_\parallel - p_\perp = B\frac{\partial p_\parallel}{\partial B}. \tag{173}$$

Returning now to Eq. 150, and dropping the $\Pi_{\parallel\perp}$ term (Why?), it can be written

$$\begin{aligned}\tilde{\mathbb{P}}_{F+A} =& \hat{\mathbf{b}}\hat{\mathbf{b}}\left[-\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}p_\parallel - \left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right)\left(p_\parallel - p_\perp\right)\right] + \left(\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}\right)\left[-\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}p_\perp - \left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right)B\frac{\partial p_\perp}{\partial B}\right]\\ &+ \frac{1}{B}\left(\hat{\mathbf{b}}\tilde{\mathbf{B}}_\perp + \tilde{\mathbf{B}}_\perp\hat{\mathbf{b}}\right)\left(p_\parallel - p_\perp\right).\end{aligned} \tag{174}$$

Finally, the above expression represents a form of the perturbed pressure tensor that we can use to evaluate $\delta W_F + \delta W_A$, from Eq. 71:

$$\delta W_F + \delta W_A = \frac{1}{2}\int \boldsymbol{\xi}_\perp^*\cdot\left(\mathbf{j_0}\times\tilde{\mathbf{B}} + \tilde{\mathbf{j}}\times\mathbf{B_0} - \boldsymbol{\nabla}\cdot\tilde{\mathbb{P}}_{F+A}\right)d\mathbf{V}. \tag{175}$$

Using also Eq. 68 for $\mathbf{j_0}\times\tilde{\mathbf{B}} + \tilde{\mathbf{j}}\times\mathbf{B_0}$,

$$\begin{aligned}\delta W_F + \delta W_A =& \frac{1}{2}\int \boldsymbol{\xi}_\perp^*\cdot\left\{\frac{1}{\mu_0}\left[-\boldsymbol{\nabla}\left(B\tilde{\mathbf{B}}_\parallel\right) + \mathbf{B}\cdot\boldsymbol{\nabla}\left(\tilde{\mathbf{B}}_\parallel\hat{\mathbf{b}} + \hat{\mathbf{b}}\tilde{\mathbf{B}}_\parallel\right)\right] + \frac{1}{\mu_0}\left(1 - \frac{\mu_0\left(p_\parallel - p_\perp\right)}{B^2}\right)\left(\boldsymbol{\nabla}\cdot\left(\mathbf{B}\tilde{\mathbf{B}}_\perp + \tilde{\mathbf{B}}_\perp\mathbf{B}\right)\right)\right.\\ &-\boldsymbol{\nabla}\cdot\hat{\mathbf{b}}\hat{\mathbf{b}}\left[-\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}p_\parallel - \left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right)\left(p_\parallel - p_\perp\right)\right]\\ &\left.+\boldsymbol{\nabla}\cdot\left(\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}\right)\left(-\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}p_\perp - \left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right)B\frac{\partial p_\perp}{\partial B}\right)\right\}d\mathbf{V}\end{aligned} \tag{176}$$

Using the anisotropy correction factor σ from Eq. 134 and with

$$\boldsymbol{\nabla}\cdot\left(\hat{\mathbf{b}}\hat{\mathbf{b}}\right) = \hat{\mathbf{b}}\nabla_\parallel + \boldsymbol{\kappa} + \hat{\mathbf{b}}\left(\boldsymbol{\nabla}\cdot\hat{\mathbf{b}}\right), \tag{177}$$

$$\boldsymbol{\nabla}\cdot\left(\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}\right) = \boldsymbol{\nabla}_\perp - \boldsymbol{\kappa} - \hat{\mathbf{b}}\left(\boldsymbol{\nabla}\cdot\hat{\mathbf{b}}\right), \tag{178}$$

we have

$$
\begin{aligned}
\delta W_F + \delta W_A =& \frac{1}{2}\int \boldsymbol{\xi}_\perp^* \cdot \left\{ \frac{1}{\mu_0}\left[-\boldsymbol{\nabla}\left(B\tilde{\mathbf{B}}_\parallel\right) + \mathbf{B}\cdot\boldsymbol{\nabla}\left(\tilde{\mathbf{B}}_\parallel\hat{\mathbf{b}} + \hat{\mathbf{b}}\tilde{\mathbf{B}}_\parallel\right)\right] + \frac{1}{\mu_0}\sigma\left(\boldsymbol{\nabla}\cdot\left(\mathbf{B}\tilde{\mathbf{B}}_\perp + \tilde{\mathbf{B}}_\perp\mathbf{B}\right)\right)\right. \\
&+\hat{\mathbf{b}}\nabla_\parallel\left(-\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}p_\parallel - \left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right)\left(p_\parallel - p_\perp\right)\right) - \boldsymbol{\nabla}_\perp\left(-\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}p_\perp - \left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right)B\frac{\partial p_\perp}{\partial B}\right) \\
&\left.-\left(\boldsymbol{\kappa} + \hat{\mathbf{b}}\left(\boldsymbol{\nabla}\cdot\hat{\mathbf{b}}\right)\right)\left(-\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}p_\parallel - \left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right)\left(p_\parallel - p_\perp - B\frac{\partial p_\perp}{\partial B}\right) + \boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}p_\perp\right)\right\} d\mathbf{V} \qquad (179)\\
=& \frac{1}{2}\int \left\{-\boldsymbol{\xi}_\perp^*\cdot\boldsymbol{\nabla}\left(B\tilde{\mathbf{B}}_\parallel/\mu_0\right) + 2\left(B\tilde{\mathbf{B}}_\parallel/\mu_0\right)\left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right) + \sigma\boldsymbol{\xi}_\perp^*\cdot\boldsymbol{\nabla}\cdot\left(\mathbf{B}\tilde{\mathbf{B}}_\perp + \tilde{\mathbf{B}}_\perp\mathbf{B}\right)/\mu_0\right. \\
&+\cancel{\boldsymbol{\xi}_\perp^*\cdot\hat{\mathbf{b}}}\nabla_\parallel\left(-\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}p_\parallel - \left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right)\left(p_\parallel - p_\perp\right)\right) - \boldsymbol{\xi}_\perp^*\cdot\boldsymbol{\nabla}\left(-\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}p_\perp - \left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right)B\frac{\partial p_\perp}{\partial B}\right) \\
&\left.-\left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^* + \cancel{\boldsymbol{\xi}_\perp^*\cdot\hat{\mathbf{b}}}\left(\boldsymbol{\nabla}\cdot\hat{\mathbf{b}}\right)\right)\left(-\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}p_\parallel - \left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right)\left(p_\parallel - p_\perp - B\frac{\partial p_\perp}{\partial B}\right) + \boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}p_\perp\right)\right\} d\mathbf{V} \\
&\qquad (180)\\
=& \frac{1}{2}\int \left\{\left(B\tilde{\mathbf{B}}_\parallel/\mu_0\right)\left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp^* + 2\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right) + \sigma\boldsymbol{\xi}_\perp^*\cdot\boldsymbol{\nabla}\cdot\left(\mathbf{B}\tilde{\mathbf{B}}_\perp + \tilde{\mathbf{B}}_\perp\mathbf{B}\right)/\mu_0\right. \\
&+\left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp^*\right)\left(-\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}p_\perp - \left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right)B\frac{\partial p_\perp}{\partial B}\right) \\
&\left.+\left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right)\left(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\left(p_\parallel - p_\perp\right) + \left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right)\left(p_\parallel - p_\perp - B\frac{\partial p_\perp}{\partial B}\right)\right)\right\} d\mathbf{V}. \qquad (181)
\end{aligned}
$$

Now using an expression for $\tilde{\mathbf{B}}_\parallel$ from Eq. 141:

$$
\begin{aligned}
\delta W_F + \delta W_A =& \frac{1}{2}\int \left\{\sigma\boldsymbol{\xi}_\perp^*\cdot\boldsymbol{\nabla}\cdot\left(\mathbf{B}\tilde{\mathbf{B}}_\perp + \tilde{\mathbf{B}}_\perp\mathbf{B}\right)/\mu_0\right. \\
&+\left(-B^2/\mu_0\right)\left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp^* + 2\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right)\left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + (1+\sigma)\,\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp - \boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\left(\frac{\mu_0 p_\perp}{B^2}\right)\right) \\
&+\left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp^*\right)\left(-\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}p_\perp\right) - \left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp^*\right)\left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right)B\frac{\partial p_\perp}{\partial B} \\
&\left.+\left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right)\left(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\left(p_\parallel - p_\perp\right)\right) + \left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right)\left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right)\left(p_\parallel - p_\perp - B\frac{\partial p_\perp}{\partial B}\right)\right\} d\mathbf{V} \qquad (182)\\
=& \frac{1}{2}\int \left\{\sigma\boldsymbol{\xi}_\perp^*\cdot\boldsymbol{\nabla}\cdot\left(\mathbf{B}\tilde{\mathbf{B}}_\perp + \tilde{\mathbf{B}}_\perp\mathbf{B}\right)/\mu_0 - B^2/\mu_0\left|\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp^* + 2\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right|^2\right. \\
&+\left(p_\parallel - p_\perp\right)\left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp^* + 2\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right)\left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right) + \left(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}p_\perp\right)\left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp^* + 2\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right) \\
&-\left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp^*\right)\left(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}p_\perp\right) + \left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right)\left(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\left(p_\parallel - p_\perp\right)\right) \\
&\left.+\left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^* + \boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp^*\right)\left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right)\left(-B\frac{\partial p_\perp}{\partial B}\right) + \left(p_\parallel - p_\perp\right)\left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right)\left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right)\right\} d\mathbf{V} \\
&\qquad (183)\\
=& \frac{1}{2}\int \left\{\sigma\boldsymbol{\xi}_\perp^*\cdot\boldsymbol{\nabla}\cdot\left(\mathbf{B}\tilde{\mathbf{B}}_\perp + \tilde{\mathbf{B}}_\perp\mathbf{B}\right)/\mu_0 - B^2/\mu_0\left|\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp^* + 2\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right|^2\right. \\
&+\left(p_\parallel - p_\perp\right)\left(2\left|\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right|^2 + \left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp^*\right)\left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right) + \left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right)\left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp\right) + \left|\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right|^2\right) \\
&+\left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right)\left(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}p_\parallel\right) + \left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp^* - \boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp^* + 2\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^* - \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right)\left(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}p_\perp\right) \\
&\left.+\left|\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right|^2\left(-B\frac{\partial p_\perp}{\partial B}\right)\right\} d\mathbf{V} \qquad (184)\\
=& \frac{1}{2}\int \left\{\sigma\boldsymbol{\xi}_\perp^*\cdot\boldsymbol{\nabla}\cdot\left(\mathbf{B}\tilde{\mathbf{B}}_\perp + \tilde{\mathbf{B}}_\perp\mathbf{B}\right)/\mu_0 - B^2/\mu_0\left|\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp^* + 2\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right|^2 + \left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right)\left(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\left(p_\parallel + p_\perp\right)\right)\right. \\
&-\left(p_\parallel - p_\perp\right)\left(-3\left|\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right|^2 - \left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp^*\right)\left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right) - \left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right)\left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp\right)\right) \\
&\left.+\left|\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right|^2\left(-B\frac{\partial p_\perp}{\partial B}\right)\right\} d\mathbf{V}. \qquad (185)
\end{aligned}
$$

Manipulating the expression in the second line leads to

$$\delta W_F + \delta W_A = \frac{1}{2}\int \Big\{ \sigma \boldsymbol{\xi}_\perp^* \cdot \boldsymbol{\nabla} \cdot \left(\mathbf{B}\tilde{\mathbf{B}}_\perp + \tilde{\mathbf{B}}_\perp \mathbf{B} \right)/\mu_0 - B^2/\mu_0 \left| \boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp^* + 2\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^* \right|^2 + \left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right)\left(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\left(p_\parallel + p_\perp\right)\right)$$
$$- \left(p_\parallel - p_\perp\right)\left(\left|\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right|^2 + \left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp^*\right)\left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right) + \left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right)\left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp\right) + \left|\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp\right|^2 \right)$$
$$- \left(p_\parallel - p_\perp\right)\left(-4\left|\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right|^2 - 2\left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp^*\right)\left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right) - 2\left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right)\left(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp\right) - \left|\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp\right|^2 \right)$$
$$+ \left|\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right|^2 \left(-B\frac{\partial p_\perp}{\partial B} \right) \Big\} d\mathbf{V} \tag{186}$$

$$= \frac{1}{2}\int \Big\{ \sigma \boldsymbol{\xi}_\perp^* \cdot \boldsymbol{\nabla} \cdot \left(\mathbf{B}\tilde{\mathbf{B}}_\perp + \tilde{\mathbf{B}}_\perp \mathbf{B} \right)/\mu_0 + \left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right)\left(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\left(p_\parallel + p_\perp\right)\right)$$
$$- B^2/\mu_0 \left(1 - \frac{\mu_0\left(p_\parallel - p_\perp\right)}{B^2} \right) \left| \boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp^* + 2\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^* \right|^2$$
$$+ \left|\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right|^2 \left(p_\perp - p_\parallel - B\frac{\partial p_\perp}{\partial B} \right) \Big\} d\mathbf{V} \tag{187}$$

$$= \frac{1}{2}\int \Big\{ \sigma \left(\frac{1}{\mu_0}\boldsymbol{\xi}_\perp^* \cdot \boldsymbol{\nabla} \cdot \left(\mathbf{B}\tilde{\mathbf{B}}_\perp + \tilde{\mathbf{B}}_\perp \mathbf{B} \right) - \frac{B^2}{\mu_0} \left| \boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp^* + 2\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^* \right|^2 \right) + \left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right)\left(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\left(p_\parallel + p_\perp\right)\right)$$
$$- B\left|\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right|^2 \left(\frac{\partial}{\partial B}\left(p_\parallel + p_\perp\right) \right) \Big\} d\mathbf{V}, \tag{188}$$

where we have used Eq. 173 for $p_\parallel - p_\perp$. Recalling the discussion from subsection III A, the first term can be rewritten in a different way.

$$\frac{1}{\mu_0}\boldsymbol{\xi}_\perp^* \cdot \boldsymbol{\nabla} \cdot \left(\mathbf{B}\tilde{\mathbf{B}}_\perp + \tilde{\mathbf{B}}_\perp \mathbf{B} \right) = ? \tag{189}$$
$$= -\frac{|\tilde{\boldsymbol{B}}_\perp|^2}{\mu_0} + j_\parallel \left(\boldsymbol{\xi}_\perp^* \times \hat{\boldsymbol{b}} \right) \cdot \tilde{\boldsymbol{B}}_\perp \tag{190}$$

Finally, we have

$$\delta W_{F+A} = \frac{1}{2}\int \Big\{ \sigma \left(-\frac{|\tilde{\boldsymbol{B}}_\perp|^2}{\mu_0} - \frac{B^2}{\mu_0}|\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}|^2 + j_\parallel \left(\boldsymbol{\xi}_\perp^* \times \hat{\boldsymbol{b}} \right) \cdot \tilde{\boldsymbol{B}}_\perp \right) + \left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right)\left(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\left(p_\parallel + p_\perp\right)\right)$$
$$- B\left|\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right|^2 \left(\frac{\partial}{\partial B}\left(p_\parallel + p_\perp\right) \right) \Big\} d\mathbf{V}. \tag{191}$$

It is now useful to split δW_{F+A} into isotropic and anisotropic parts. This is easily done by defining

$$p_{\text{avg}} = \frac{1}{2}\left(p_\parallel + p_\perp\right). \tag{192}$$

Then

$$\delta W_F = \frac{1}{2}\int \Big\{ \left(-\frac{|\tilde{\boldsymbol{B}}_\perp|^2}{\mu_0} - \frac{B^2}{\mu_0}|\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}|^2 + j_\parallel \left(\boldsymbol{\xi}_\perp^* \times \hat{\boldsymbol{b}} \right) \cdot \tilde{\boldsymbol{B}}_\perp \right) + 2\left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right)\left(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}p_{\text{avg}}\right) \Big\} d\mathbf{V}, \tag{193}$$

and

$$\delta W_A = \frac{1}{2}\int \Big\{ (1-\sigma) \left(-\frac{|\tilde{\boldsymbol{B}}_\perp|^2}{\mu_0} - \frac{B^2}{\mu_0}|\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}|^2 + j_\parallel \left(\boldsymbol{\xi}_\perp^* \times \hat{\boldsymbol{b}} \right) \cdot \tilde{\boldsymbol{B}}_\perp \right) - 2B\left|\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right|^2 \frac{\partial p_{\text{avg}}}{\partial B} \Big\} d\mathbf{V}. \tag{194}$$

One can easily see that if the equilibrium pressure is isotropic δW_A is zero.

The fluid δW should be self-adjoint and therefore strictly real ? . In particular, the last term of δW_A in Eq. 194 is obviously real, as are the first two terms (the X and the X terms) of of δW_A in Eq. 194 and δW_F in Eq. 193. When the equilibrium pressure is isotropic ($\sigma = 1$ and $p_{\rm avg} = p$), one can show that the imaginary parts of the last two terms of δW_F cancel ? . With anisotropy that property is no longer obvious, but a lengthly calculation can be used to show that indeed $\delta W_F + \delta W_A$ is still self-adjoint (see appendix L).

VII. SOLUTION FOR δW_Φ AND δW_K

Let us now return to Eq. 145 for $\tilde{\mathbb{P}}_{K+\Phi}$, and solve for the kinetic and electrostatic parts of δW.

$$\delta W_K + \delta W_\Phi = -\frac{1}{2}\int \boldsymbol{\xi}_\perp^* \cdot \left(\boldsymbol{\nabla}\cdot\tilde{\mathbb{P}}_{K+\Phi}\right) d\mathbf{V} \tag{195}$$

$$= -\frac{1}{2}\sum_j m_j \int\int \boldsymbol{\xi}_\perp^* \cdot \boldsymbol{\nabla}\cdot \left(v_\parallel^2 \hat{\mathbf{b}}\hat{\mathbf{b}} + \frac{1}{2}v_\perp^2\left(\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}\right)\right)\left[im_j\left(\omega\frac{\partial f_j}{\partial\varepsilon} - n\frac{\partial f_j}{\partial P_\phi}\right)\tilde{s}_j + \tilde{\boldsymbol{\mathcal{Z}}}\frac{\partial f_j}{\partial \varepsilon}\right] d^3\mathbf{v} d\mathbf{V}. \tag{196}$$

From Eqs. 177 and 178, we can see that

$$\boldsymbol{\xi}_\perp^* \cdot \boldsymbol{\nabla}\cdot\left(\hat{\mathbf{b}}\hat{\mathbf{b}}\right) = \boldsymbol{\xi}_\perp^* \cdot \boldsymbol{\kappa} = \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*, \tag{197}$$

$$\boldsymbol{\xi}_\perp^* \cdot \boldsymbol{\nabla}\cdot\left(\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}\right) = \boldsymbol{\xi}_\perp^* \cdot \boldsymbol{\nabla}_\perp - \boldsymbol{\xi}_\perp^*\cdot\boldsymbol{\kappa} = -\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp^* - \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*. \tag{198}$$

Now

$$\delta W_K + \delta W_\Phi = \frac{1}{2}\sum_j \int\int \frac{1}{2}m_j v^2 \left(\frac{v_\perp^2}{v^2}\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp^* + \left(\frac{v_\perp^2}{v^2} - 2\frac{v_\parallel^2}{v^2}\right)\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right)\left[im_j\left(\omega\frac{\partial f_j}{\partial\varepsilon} - n\frac{\partial f_j}{\partial P_\phi}\right)\tilde{s}_j + \tilde{\boldsymbol{\mathcal{Z}}}\frac{\partial f_j}{\partial \varepsilon}\right] d^3\mathbf{v} d\mathbf{V}. \tag{199}$$

Then we introduce the pitch angle, $\chi = v_\parallel/v$. Note that $\Lambda = \mu B_0/\varepsilon$ doesn't distinguish between positive and negative parallel velocity; using χ is therefore preferable in a general sense. Also using $\varepsilon = \frac{1}{2}m_j v^2 + Z_j e\Phi$, we have

$$\delta W_K + \delta W_\Phi = \frac{1}{2}\sum_j \int\int \left(\varepsilon - Z_j e\Phi\right)\left(\left(1-\chi^2\right)\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp^* + \left(1-3\chi^2\right)\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp^*\right)\left[im_j\left(\omega\frac{\partial f_j}{\partial\varepsilon} - n\frac{\partial f_j}{\partial P_\phi}\right)\tilde{s}_j + \tilde{\boldsymbol{\mathcal{Z}}}\frac{\partial f_j}{\partial \varepsilon}\right] d^3\mathbf{v} d\mathbf{V}. \tag{200}$$

Let's return to $\tilde{s}_j$ from Eq. 105. Then

$$\tilde{s}_j = \int_{-\infty}^{t}\left(\mathbf{v}\cdot\frac{d\boldsymbol{\xi}_\perp}{dt'} - \frac{\tilde{\boldsymbol{\mathcal{Z}}}}{m_j}\right)dt'. \tag{201}$$

Now,

Derive this first step (look in Ref. 32, Eqs. 25 and 26):

$$\mathbf{v}\cdot\frac{d\boldsymbol{\xi}_\perp}{dt} = \boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\left(\mathbf{v}\mathbf{v}\right) \tag{202}$$

$$\approx \frac{v_\perp^2}{2}\boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \left(\frac{v_\perp^2}{2} - v_\parallel^2\right)\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp, \tag{203}$$

where terms that gyro-average to nearly zero have been neglected[32].

Now we note that we have seen the quantity in Eq. 203 once already, in Eq. 199. Let us introduce:

$$\langle H \rangle = \frac{\varepsilon - Z_j e \Phi}{T_j} \left(\left(1-\chi^2\right) \boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp^* + \left(1 - 3\chi^2\right) \boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp^* \right) + \frac{\tilde{\boldsymbol{\mathcal{Z}}}}{T_j} \tag{204}$$

so that

$$\langle \tilde{s}_j \rangle = -\int_{-\infty}^{t} \frac{\langle HT_j \rangle}{m_j} dt', \tag{205}$$

where $\langle \cdot \rangle$ indicates a gyro-averaged quantity. The evaluation of the integral over the unperturbed orbits is performed in appendix H, and results in:

$$\langle \tilde{s}_j \rangle = \sum_{l=-\infty}^{\infty} \frac{-\langle HT_j \rangle}{i m_j \left(n \langle \omega_D^j \rangle + l \omega_b^j - i \nu_{\text{eff}}^j + n\omega_E - \omega \right)}. \tag{206}$$

The bounce, ω_b^j, magnetic precession drift, ω_D^j, and $E \times B$, ω_E frequencies are defined in appendices I, J, and K. The summation is over l, the bounce harmonic. (Make general for trapped or circulating particles.) The effective collision frequency, ν_{eff}^j is discussed further in appendix G.

Returning now to Eq. 200,

$$\delta W_K + \delta W_\Phi = -\frac{1}{2} \sum_j \int \int \left(\langle HT_j \rangle - \tilde{\boldsymbol{\mathcal{Z}}} \right) \left[\frac{\left(\omega \frac{\partial f_j}{\partial \varepsilon} - n \frac{\partial f_j}{\partial P_\phi} \right) \langle HT_j \rangle}{n \langle \omega_D^j \rangle + l \omega_b^j - i \nu_{\text{eff}}^j + n\omega_E - \omega} - \tilde{\boldsymbol{\mathcal{Z}}} \frac{\partial f_j}{\partial \varepsilon} \right] d^3\mathbf{v} d\mathbf{V}, \tag{207}$$

and separating the two terms:

Somehow the sign is wrong here?

$$\delta W_\Phi = -\frac{1}{2} \sum_j \int \int |\tilde{\boldsymbol{\mathcal{Z}}}|^2 \frac{\partial f_j}{\partial \varepsilon} d^3\mathbf{v} d\mathbf{V}, \tag{208}$$

and

$$\delta W_K = -\frac{1}{2} \sum_j \int \int |\langle HT_j \rangle|^2 \frac{\omega \frac{\partial f_j}{\partial \varepsilon} - n \frac{\partial f_j}{\partial P_\phi}}{n \langle \omega_D^j \rangle + l \omega_b^j - i \nu_{\text{eff}}^j + n\omega_E - \omega} d^3\mathbf{v} d\mathbf{V}. \tag{209}$$

What happened to the cross-terms?

VIII. RECONSTITUTION OF EQUATIONS IN TERMS OF ε, χ, AND Ψ

For the purposes of the `MISK` code[10,18], which performs integration over ε, $\chi = v_\parallel / v$, and Ψ, we will now replace the integration over $d^3\mathbf{v}$ and $d\mathbf{V}$ with $d\varepsilon$, $d\chi$, and $d\Psi$. First, from Eq. 172,

$$\int d^3\mathbf{v} = 2\pi \int \int v_\perp dv_\perp dv_\parallel, \tag{210}$$

and then using

$$v_\perp = \sqrt{v^2 - v_\parallel^2} = \sqrt{\frac{2\varepsilon}{m_j} \left(1 - \chi^2\right)}, \tag{211}$$

$$\frac{d\varepsilon}{dv_\perp} = \frac{d}{dv_\perp}\left(\frac{1}{2}m_j\left(v_\parallel^2 + v_\perp^2\right)\right) = m_j v_\perp = m_j\sqrt{\frac{2\varepsilon}{m_j}(1-\chi^2)}, \tag{212}$$

$$\frac{d\chi}{dv_\parallel} = \frac{d}{dv_\parallel}\left(\frac{v_\parallel}{v}\right) = \sqrt{\frac{m_j}{2\varepsilon}}, \tag{213}$$

we find

$$\int d^3\mathbf{v} = 2\pi\int\int\left(\sqrt{\frac{2\varepsilon}{m_j}(1-\chi^2)}\right)\left(\frac{1}{\sqrt{2m_j\varepsilon}\sqrt{1-\chi^2}}d\varepsilon\right)\left(\sqrt{\frac{2\varepsilon}{m_j}}d\chi\right) = \frac{2\sqrt{2}\pi}{m_j^{\frac{3}{2}}}\int\int\varepsilon^{\frac{1}{2}}d\chi d\varepsilon. \tag{214}$$

Next,

$$\int d\mathbf{V} = \int\int\int \mathbf{r}d\mathbf{r}d\boldsymbol{\phi}d\boldsymbol{\theta} = 2\pi\int\int \mathbf{r}d\mathbf{r}d\boldsymbol{\theta}, \tag{215}$$

assuming axisymmetry. Now we will replace the $d\mathbf{r}$ and $d\boldsymbol{\theta}$ integration with integration over $d\boldsymbol{\Psi}$ and $d\boldsymbol{\ell}$, the particle trajectory. First, $d\boldsymbol{\ell}$ is defined as

$$d\boldsymbol{\ell} = \frac{d\boldsymbol{\theta}}{\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\theta} = \frac{Bd\boldsymbol{\theta}}{\boldsymbol{B_\theta}\cdot\boldsymbol{\nabla}\theta}. \tag{216}$$

Then with

$$\boldsymbol{B_\theta} = \frac{1}{R_0}\left(\boldsymbol{\nabla}\Psi \times \hat{\boldsymbol{e}}_{\boldsymbol{\phi}}\right), \tag{217}$$

$$d\boldsymbol{\ell} = \frac{BR_0 d\boldsymbol{\theta}}{\left(\frac{d\Psi}{d\mathbf{r}} \times \hat{\boldsymbol{e}}_{\boldsymbol{\phi}}\right)\cdot\frac{\hat{\boldsymbol{e}}_{\boldsymbol{\theta}}}{r}}, \tag{218}$$

so,

$$\int d\mathbf{V} = 2\pi\int\int \mathbf{r}d\mathbf{r}d\boldsymbol{\theta} = 2\pi\int\int BR_0 d\Psi d\boldsymbol{\ell}. \tag{219}$$

Finally, if we define

$$\hat{\tau} = \int\frac{d\boldsymbol{\ell}}{|\chi|}, \tag{220}$$

(Where does $\hat{\tau}/2$ come from for trapped particles and not for circulating in Ref. [26]?) then

$$\int d\mathbf{V} = 2\pi\hat{\tau}\int\frac{|\chi|}{B}d\Psi. \tag{221}$$

Therefore a three dimensional volume integral has been reduced to one dimension in Ψ by the assumptions of axisymmetry and that f_j (and therefore n_j and T_j) are flux functions, i.e. they are functions of Ψ and not r and θ). Finally,

$$\int\int d^3\mathbf{v}d\mathbf{V} = \int\int\int 4\sqrt{2}\pi^2\frac{\hat{\tau}}{m_j^{\frac{3}{2}}B}|\chi|\varepsilon^{\frac{1}{2}}d\varepsilon d\chi d\Psi. \tag{222}$$

Secondly, we must replace the P_ϕ derivative in Eq. 209. Since $P_\phi = m_j R v_\phi + Z_j e\Psi$,

$$\frac{\partial P_\phi}{\partial \Psi} = Z_j e + m_j v_\phi \frac{\partial R}{\partial \Psi} + m_j R \frac{\partial v_\phi}{\partial \Psi}. \tag{223}$$

Then

$$\frac{\partial f_j}{\partial \Psi} = \frac{\partial f_j}{\partial P_\phi}\frac{\partial P_\phi}{\partial \Psi} \tag{224}$$

$$= \frac{\partial f_j}{\partial P_\phi}\left(Z_j e + m_j v_\phi \frac{\partial R}{\partial \Psi}\right) + m_j R \frac{\partial v_\phi}{\partial \Psi}\frac{\partial f_j}{\partial P_\phi} \tag{225}$$

$$= \frac{\partial f_j}{\partial P_\phi}\left(Z_j e + m_j v_\phi \frac{\partial R}{\partial \Psi}\right) + Z_j e \frac{\partial f_j}{\partial \varepsilon}\frac{d\Phi}{d\Psi}. \tag{226}$$

Here we have used $m_j R = \partial P_\phi / \partial v_\phi$ and $Z_j e = \partial \varepsilon / \partial \Phi$. Therefore,

$$\frac{\partial f_j}{\partial P_\phi} = \frac{\frac{\partial f_j}{\partial \Psi} - Z_j e \frac{\partial f_j}{\partial \varepsilon}\frac{d\Phi}{d\Psi}}{Z_j e + m_j v_\phi \frac{\partial R}{\partial \Psi}}. \tag{227}$$

If we take $\partial R/\partial \Psi \approx 0$ (Why?) and use $\omega_E = -d\Phi/d\Psi$ (see Eq. K4), then

$$\frac{\partial f_j}{\partial P_\phi} = \frac{1}{Z_j e}\frac{\partial f_j}{\partial \Psi} + \omega_E \frac{\partial f_j}{\partial \varepsilon}. \tag{228}$$

Finally, the four δW terms are:

$$\delta W_K = \sum_j \sum_{l=-\infty}^{\infty} 2\sqrt{2}\pi^2 \int\int\int \left[\left|\langle HT_j \rangle\right|^2 \lambda_{j,l} \frac{f_j}{T_j}\right] \frac{\hat{\tau}}{m_j^{\frac{3}{2}} B} |\chi| \varepsilon^{\frac{1}{2}} d\varepsilon d\chi d\Psi, \tag{229}$$

$$\delta W_\Phi = \sum_j 2\sqrt{2}\pi^2 \int\int\int \left[\left|\tilde{\boldsymbol{\mathcal{Z}}}\right|^2 \frac{\partial f_j}{\partial \varepsilon}\right] \frac{\hat{\tau}}{m_j^{\frac{3}{2}} B} |\chi| \varepsilon^{\frac{1}{2}} d\varepsilon d\chi d\Psi, \tag{230}$$

$$\delta W_F =, \tag{231}$$

$$\delta W_A =, \tag{232}$$

where we have defined the frequency resonance fraction, $\lambda_{j,l}$ as:

$$\lambda_{j,l} = \frac{\frac{T_j}{f_j}\left((\omega - n\omega_E)\frac{\partial f_j}{\partial \varepsilon} - \frac{n}{Z_j e}\frac{\partial f_j}{\partial \Psi}\right)}{n\langle \omega_D^j \rangle + l\omega_b^j - i\nu_{\text{eff}}^j + n\omega_E - \omega}. \tag{233}$$

Note that in appendix F we use the quasineutrality condition to solve for $\tilde{\boldsymbol{\mathcal{Z}}}$ in terms of known quantities. We will now turn our attention in the next section to specific distribution functions for use in Eqs. 229-232. (Correct?)

IX. SPECIFIC DISTRIBUTION FUNCTIONS

Equations 229-232 are expressions for δW_K, δW_Φ, δW_F, and δW_A that have so far not assumed a specific form of the distribution function. Once f_j is specified, the derivatives $\partial f_j/\partial\varepsilon$, $\partial f_j/\partial\Psi$, and $\partial f_j/\partial\chi$ are used to evaluate the δW terms. In principle one can use any form of f_j, including numerical forms, so long as they provide smooth derivatives. Here we will utilize four different analytical forms for f_j: Maxwellian, bi-Maxwellian, isotropic slowing-down, and anisotropic slowing-down. Other distributions for future consideration might include an energetic particle distribution modified by high harmonic fast wave (HHFW) heating[58] or Maxwellian electrons modified by electron cyclotron current drive (ECCD) or electron cyclotron resonance heating (ECRH)[59].

A. Maxwellian Distribution Function

The Maxwellian distribution function, f_j^M, is

$$f_j^M(\varepsilon,\Psi) = n_j\left(\frac{m_j}{2\pi T_j}\right)^{\frac{3}{2}} e^{-\varepsilon/T_j}. \tag{234}$$

It is isotropic with respect to pitch angle (independent of χ), so $\partial f_j/\partial\chi = 0$. Also,

$$\frac{\partial f_j}{\partial\varepsilon} = -\frac{f_j^M}{T_j}, \tag{235}$$

and

$$\frac{\partial f_j}{\partial\Psi} = \left(\frac{1}{T_j^{\frac{3}{2}}}\frac{dn_j}{d\Psi} - \frac{3}{2}\frac{n_j}{T_j^{\frac{5}{2}}}\frac{dT_j}{d\Psi} + \left(\frac{\varepsilon}{T_j}\right)\frac{n_j}{T_j^{\frac{5}{2}}}\frac{dT_j}{d\Psi}\right)\left(\frac{m_j}{2\pi}\right)^{\frac{3}{2}} e^{-\varepsilon/T_j}. \tag{236}$$

Defining $\hat{\varepsilon} = \varepsilon/T_j$,

$$\frac{\partial f_j}{\partial\Psi} = -\frac{f_j^M}{T_j}\left(-\frac{T_j}{n_j}\frac{dn_j}{d\Psi} - \left(\hat{\varepsilon} - \frac{3}{2}\right)\frac{dT_j}{d\Psi}\right), \tag{237}$$

or,

$$\frac{\partial f_j}{\partial\Psi} = -\frac{f_j^M}{T_j}Z_j e\left(\omega_{*N}^j + \left(\hat{\varepsilon} - \frac{3}{2}\right)\omega_{*T}^j\right), \tag{238}$$

where we have defined

$$\omega_{*N}^j = -\frac{T_j}{Z_j e n_j}\frac{dn_j}{d\Psi}, \tag{239}$$

$$\omega_{*T}^j = -\frac{1}{Z_j e}\frac{dT_j}{d\Psi}. \tag{240}$$

So, then:

$$(\omega - n\omega_E)\frac{\partial f_j}{\partial\varepsilon} - \frac{n}{eZ_j}\frac{\partial f_j}{\partial\Psi} = \frac{f_j^M}{T_j}\left(n\left(\omega_{*N}^j + \left(\hat{\varepsilon} - \frac{3}{2}\right)\omega_{*T}^j + \omega_E\right) - \omega\right) \tag{241}$$

Then from Eqs. 233 and 241, for a Maxwellian,

$$\lambda_{j,l}^{M} = \frac{n\left(\omega_{*N}^{j} + \left(\hat{\varepsilon} - \frac{3}{2}\right)\omega_{*T}^{j} + \omega_E\right) - \omega}{n\langle\omega_D^j\rangle + l\omega_b^j - i\nu_{\text{eff}}^j + n\omega_E - \omega}. \tag{242}$$

Equation 229 can be written as:

$$\delta W_K^M = \sum_j \sum_{l=-\infty}^{\infty} 2\sqrt{2}\pi^2 \int\int\int \left[|\langle HT_j\rangle|^2 \; \frac{n\left(\omega_{*N}^{j} + \left(\hat{\varepsilon} - \frac{3}{2}\right)\omega_{*T}^{j} + \omega_E\right) - \omega}{n\langle\omega_D^j\rangle + l\omega_b^j - i\nu_{\text{eff}}^j + n\omega_E - \omega} \right] \frac{f_j^M}{T_j} \frac{\hat{\tau}}{m_j^{\frac{3}{2}} B} |\chi| \varepsilon^{\frac{1}{2}} d\varepsilon d\chi d\Psi \tag{243}$$

$$= \sum_j \sum_{l=-\infty}^{\infty} 2\sqrt{2}\pi^2 \int\int\int \left[|\langle H/\hat{\varepsilon}\rangle|^2 \; \frac{n\left(\omega_{*N}^{j} + \left(\hat{\varepsilon} - \frac{3}{2}\right)\omega_{*T}^{j} + \omega_E\right) - \omega}{n\langle\omega_D^j\rangle + l\omega_b^j - i\nu_{\text{eff}}^j + n\omega_E - \omega} \right] T_j^{\frac{5}{2}} n_j \left(\frac{m_j}{2\pi T_j}\right)^{\frac{3}{2}} e^{-\hat{\varepsilon}} \frac{\hat{\tau}}{m_j^{\frac{3}{2}} B} |\chi| \hat{\varepsilon}^{\frac{5}{2}} d\hat{\varepsilon} d\chi d\Psi \tag{244}$$

$$= \sum_j \sum_{l=-\infty}^{\infty} \sqrt{\pi} \int\int\int \left[|\langle H/\hat{\varepsilon}\rangle|^2 \; \frac{n\left(\omega_{*N}^{j} + \left(\hat{\varepsilon} - \frac{3}{2}\right)\omega_{*T}^{j} + \omega_E\right) - \omega}{n\langle\omega_D^j\rangle + l\omega_b^j - i\nu_{\text{eff}}^j + n\omega_E - \omega} \right] n_j T_j \frac{\hat{\tau}}{B} |\chi| \hat{\varepsilon}^{\frac{5}{2}} e^{-\hat{\varepsilon}} d\hat{\varepsilon} d\chi d\Psi. \tag{245}$$

For the electrostatic term we return to the form of Eq. 208, for convenience, and write:

$$\delta W_\Phi^M = -\frac{1}{2}\int e^2 \left|\tilde{\mathbf{\Phi}} + \boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla}\Phi_0\right|^2 \sum_j Z_j^2 \frac{1}{T_j} \int f_j^M d^3\mathbf{v} d\mathbf{V} \tag{246}$$

$$= -\frac{1}{2}\int e^2 \left|\tilde{\mathbf{\Phi}} + \boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla}\Phi_0\right|^2 \sum_j Z_j^2 \frac{n_j}{T_j} d\mathbf{V}. \tag{247}$$

Equation 247 is the same as Eq. (3) of Ref. [10]. Alternatively, using Eq. **??**,

$$\delta W_\Phi^M = -\sum_j 2\sqrt{2}\pi^2 \int\int\int \left[(Z_j e)^2 \left|\tilde{\mathbf{\Phi}} + \boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla}\Phi_0\right|^2 \right] \frac{f_j^M}{T_j} \frac{\hat{\tau}}{m_j^{\frac{3}{2}} B} |\chi| \varepsilon^{\frac{1}{2}} d\varepsilon d\chi d\Psi \tag{248}$$

$$= -\sum_j \sqrt{\pi} \int\int \left[\left(\frac{Z_j e}{T_j}\right)^2 \left|\tilde{\mathbf{\Phi}} + \boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla}\Phi_0\right|^2 \right] \left[\int \hat{\varepsilon}^{\frac{1}{2}} e^{-\hat{\varepsilon}} d\hat{\varepsilon} \right] n_j T_j \frac{\hat{\tau}}{B} |\chi| d\chi d\Psi \tag{249}$$

$$= -\sum_j \frac{\pi}{2} \int\int \left[\left(\frac{Z_j e}{T_j}\right)^2 \left|\tilde{\mathbf{\Phi}} + \boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla}\Phi_0\right|^2 \right] n_j T_j \frac{\hat{\tau}}{B} |\chi| d\chi d\Psi, \tag{250}$$

which is equivalent.

With an isotropic Maxwellian distribution, $\sigma = 1$, $p_{\text{avg}} = p$ in δW_F (Eq. **??**), and $\delta W_A = 0$.

B. Bi-Maxwellian Distribution Function

In order to be consistent with the assumption of different pressures in the parallel and perpendicular directions, one should use a bi-Maxwellian distribution. The Maxwellian distribution is really just a special case of the more general bi-Maxwellian, with $T_{j\parallel} = T_{j\perp}$, so the bi-Maxwellian forms could be used to be more general. The bi-Maxwellian distribution is given by:

$$f_j^{bM}(\varepsilon, \Psi, \mu) = n_j \left(\frac{m_j}{2\pi}\right)^{\frac{3}{2}} \frac{1}{T_{j\perp} T_{j\parallel}^{\frac{1}{2}}} e^{-(\varepsilon - \mu B)/T_{j\parallel}} e^{-\mu B/T_{j\perp}}, \tag{251}$$

or, in terms of χ,

$$f_j^{bM}(\varepsilon,\Psi,\chi) = n_j\left(\frac{m_j}{2\pi}\right)^{\frac{3}{2}}\frac{1}{T_{j\perp}T_{j\|}^{\frac{1}{2}}}e^{-\varepsilon\chi^2/T_{j\|}}e^{-\varepsilon(1-\chi^2)/T_{j\perp}}, \tag{252}$$

or:

$$f_j^{bM}(\hat{\varepsilon},\Psi,\chi) = n_j\left(\frac{m_j}{2\pi}\right)^{\frac{3}{2}}\frac{1}{T_{j\perp}T_{j\|}^{\frac{1}{2}}}e^{-\hat{\varepsilon}\chi^2(T_j/T_{j\|})}e^{-\hat{\varepsilon}(1-\chi^2)(T_j/T_{j\perp})}. \tag{253}$$

We can see from Eq. 251 that

$$\frac{\partial f_j}{\partial\varepsilon} = -\frac{f_j^{bM}}{T_j}\left(\frac{T_j}{T_{j\|}}\right). \tag{254}$$

Then $\partial f_j/\partial\Psi$ takes the form:

$$\frac{\partial f_j}{\partial\Psi} = -\frac{f_j^{bM}}{T_j}\left(-\frac{T_j}{n_j}\frac{dn_j}{d\Psi} - \left(\varepsilon\chi^2\frac{T_j}{T_{j\|}^2} - \frac{1}{2}\frac{T_j}{T_{j\|}}\right)\frac{dT_{j\|}}{d\Psi} - \left(\varepsilon(1-\chi^2)\frac{T_j}{T_{j\perp}^2} - \frac{T_j}{T_{j\perp}}\right)\frac{dT_{j\perp}}{d\Psi}\right), \tag{255}$$

which is analogous to Eq. 237 (and, of course, reduces to Eq. 237 when $T_{j\|} = T_{j\perp}$). Defining

$$\omega^j_{*T_\|} = -\frac{1}{Z_je}\frac{dT_{j\|}}{d\Psi}, \tag{256}$$

and

$$\omega^j_{*T_\perp} = -\frac{1}{Z_je}\frac{dT_{j\perp}}{d\Psi}, \tag{257}$$

then

$$\frac{\partial f_j}{\partial\Psi} = -\frac{f_j^{bM}}{T_j}Z_je\left(\omega^j_{*N} + \left(\frac{\varepsilon\chi^2T_j}{T_{j\|}^2} - \frac{1}{2}\frac{T_j}{T_{j\|}}\right)\omega^j_{*T_\|} + \left(\frac{\varepsilon(1-\chi^2)T_j}{T_{j\perp}^2} - \frac{T_j}{T_{j\perp}}\right)\omega^j_{*T_\perp}\right), \tag{258}$$

or

$$\frac{\partial f_j}{\partial\Psi} = -\frac{f_j^{bM}}{T_j}Z_je\left[\omega^j_{*N} + \left(\hat{\varepsilon}\chi^2\left(\frac{T_j}{T_{j\|}}\right) - \frac{1}{2}\right)\left(\frac{T_j}{T_{j\|}}\right)\omega^j_{*T_\|} + \left(\hat{\varepsilon}(1-\chi^2)\left(\frac{T_j}{T_{j\perp}}\right) - 1\right)\left(\frac{T_j}{T_{j\perp}}\right)\omega^j_{*T_\perp}\right]. \tag{259}$$

So that

$$(\omega - n\omega_E)\frac{\partial f}{\partial\varepsilon} - \frac{n}{Z_je}\frac{\partial f}{\partial\Psi} =$$
$$\frac{f_j^{bM}}{T_j}\left[n\left(\omega^j_{*N} + \left(\hat{\varepsilon}\chi^2\left(\frac{T_j}{T_{j\|}}\right) - \frac{1}{2}\right)\left(\frac{T_j}{T_{j\|}}\right)\omega^j_{*T_\|} + \left(\hat{\varepsilon}(1-\chi^2)\left(\frac{T_j}{T_{j\perp}}\right) - 1\right)\left(\frac{T_j}{T_{j\perp}}\right)\omega^j_{*T_\perp} + \left(\frac{T_j}{T_{j\|}}\right)\omega_E\right) - \left(\frac{T_j}{T_{j\|}}\right)\omega\right]. \tag{260}$$

Finally, in this case $\partial f_j/\partial\chi \neq 0$, but rather,

$$\frac{\partial f_j}{\partial \chi} = - f_j^{bM} \left(2\varepsilon|\chi|\right) \left(\frac{1}{T_{j\parallel}} - \frac{1}{T_{j\perp}}\right) \tag{261}$$

$$= - f_j^{bM} \left(2\hat{\varepsilon}|\chi|\right) \left(\frac{T_j}{T_{j\parallel}} - \frac{T_j}{T_{j\perp}}\right). \tag{262}$$

Now examining the terms of δW for the bi-Maxwellian distribution, from Eqs. 229-232, we find:

$$\delta W_K^{bM} = \sum_j \sum_{l=-\infty}^{\infty} 2\sqrt{2}\pi^2 \int\int\int \frac{f_j^{bM}}{T_j} \frac{\hat{\tau}}{m_j^{\frac{3}{2}} B} |\chi| \varepsilon^{\frac{1}{2}} d\varepsilon d\chi d\Psi \left[|\langle HT_j \rangle|^2 \frac{n\left(\omega_{*N}^j + \left(\hat{\varepsilon}\chi^2 \left(\frac{T_j}{T_{j\parallel}}\right) - \frac{1}{2}\right)\left(\frac{T_j}{T_{j\parallel}}\right)\omega_{*T_\parallel}^j + \left(\hat{\varepsilon}(1-\chi^2)\left(\frac{T_j}{T_{j\perp}}\right) - 1\right)\left(\frac{T_j}{T_{j\perp}}\right)\omega_{*T_\perp}^j + \left(\frac{T_j}{T_{j\parallel}}\right)\omega_E\right) - \left(\frac{T_j}{T_{j\parallel}}\right)\omega}{n\langle\omega_D^j\rangle + l\omega_b^j - i\nu_{\text{eff}}^j + n\omega_E - \omega} \right] \tag{263}$$

$$= \sum_j \sum_{l=-\infty}^{\infty} 2\sqrt{2}\pi^2 \int\int\int \frac{1}{T_j} n_j \left(\frac{m_j}{2\pi}\right)^{\frac{3}{2}} \frac{1}{T_{j\perp} T_{j\parallel}^{\frac{1}{2}}} e^{-\varepsilon\chi^2/T_{j\parallel}} e^{-\varepsilon(1-\chi)^2/T_{j\perp}} \frac{\hat{\tau}}{m_j^{\frac{3}{2}} B} |\chi| \varepsilon^{\frac{1}{2}} d\varepsilon d\chi d\Psi \left[|\langle HT_j \rangle|^2 \frac{n\left(\omega_{*N}^j + \left(\hat{\varepsilon}\chi^2 \left(\frac{T_j}{T_{j\parallel}}\right) - \frac{1}{2}\right)\left(\frac{T_j}{T_{j\parallel}}\right)\omega_{*T_\parallel}^j + \left(\hat{\varepsilon}(1-\chi^2)\left(\frac{T_j}{T_{j\perp}}\right) - 1\right)\left(\frac{T_j}{T_{j\perp}}\right)\omega_{*T_\perp}^j + \left(\frac{T_j}{T_{j\parallel}}\right)\omega_E\right) - \left(\frac{T_j}{T_{j\parallel}}\right)\omega}{n\langle\omega_D^j\rangle + l\omega_b^j - i\nu_{\text{eff}}^j + n\omega_E - \omega} \right] \tag{264}$$

$$= \sum_j \sum_{l=-\infty}^{\infty} \sqrt{\pi} \int\int\int n_j T_j \left(\frac{1}{T_j^2}\right) \left(\frac{1}{T_{j\perp} T_{j\parallel}^{\frac{1}{2}}}\right) \frac{\hat{\tau}}{B} |\chi| \varepsilon^{\frac{5}{2}} e^{-\varepsilon\chi^2/T_{j\parallel}} e^{-\varepsilon(1-\chi)^2/T_{j\perp}} d\varepsilon d\chi d\Psi \left[|\langle HT_j/\varepsilon \rangle|^2 \frac{n\left(\omega_{*N}^j + \left(\hat{\varepsilon}\chi^2 \left(\frac{T_j}{T_{j\parallel}}\right) - \frac{1}{2}\right)\left(\frac{T_j}{T_{j\parallel}}\right)\omega_{*T_\parallel}^j + \left(\hat{\varepsilon}(1-\chi^2)\left(\frac{T_j}{T_{j\perp}}\right) - 1\right)\left(\frac{T_j}{T_{j\perp}}\right)\omega_{*T_\perp}^j + \left(\frac{T_j}{T_{j\parallel}}\right)\omega_E\right) - \left(\frac{T_j}{T_{j\parallel}}\right)\omega}{n\langle\omega_D^j\rangle + l\omega_b^j - i\nu_{\text{eff}}^j + n\omega_E - \omega} \right] \tag{265}$$

$$= \sum_j \sum_{l=-\infty}^{\infty} \sqrt{\pi} \int\int\int n_j T_j \left(\frac{T_j^{\frac{3}{2}}}{T_{j\perp} T_{j\parallel}^{\frac{1}{2}}}\right) \frac{\hat{\tau}}{B} |\chi| \hat{\varepsilon}^{\frac{5}{2}} e^{-\hat{\varepsilon}\chi^2 (T_j/T_{j\parallel})} e^{-\hat{\varepsilon}(1-\chi)^2 (T_j/T_{j\perp})} d\hat{\varepsilon} d\chi d\Psi \left[|\langle H/\hat{\varepsilon} \rangle|^2 \frac{n\left(\omega_{*N}^j + \left(\hat{\varepsilon}\chi^2 \frac{T_j}{T_{j\parallel}} - \frac{1}{2}\right)\left(\frac{T_j}{T_{j\parallel}}\right)\omega_{*T_\parallel}^j + \left(\hat{\varepsilon}(1-\chi^2)\frac{T_j}{T_{j\perp}} - 1\right)\left(\frac{T_j}{T_{j\perp}}\right)\omega_{*T_\perp}^j + \left(\frac{T_j}{T_{j\parallel}}\right)\omega_E\right) - \left(\frac{T_j}{T_{j\parallel}}\right)\omega}{n\langle\omega_D^j\rangle + l\omega_b^j - i\nu_{\text{eff}}^j + n\omega_E - \omega} \right]. \tag{266}$$

One can see that when $T_{j\parallel} = T_{j\perp} = T_j$, this equation reduces to the form of Eq. 245 for Maxwellian particles, since the exponential terms together become $e^{-\hat{\varepsilon}}$, and the ω_{*T} terms become $(\hat{\varepsilon} - \frac{3}{2})\omega_{*T}$.

For the electrostatic term, from Eq. 230:

$$\delta W_{\Phi}^{bM} = -\sum_j 2\sqrt{2}\pi^2 \int\int\int \left[(Z_j e)^2 \left| \tilde{\mathbf{\Phi}} + \boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla}\Phi_0 \right|^2 \right] \frac{f_j^{bM}}{T_j} \left(\frac{T_j}{T_{j\parallel}} \right) \frac{\hat{\tau}}{m_j^{\frac{3}{2}} B} |\chi| \varepsilon^{\frac{1}{2}} d\varepsilon d\chi d\Psi \tag{267}$$

$$= -\sum_j \sqrt{\pi} \int\int\int \left[(Z_j e)^2 \left| \tilde{\mathbf{\Phi}} + \boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla}\Phi_0 \right|^2 \right] \frac{n_j T_j}{T_j^2} \left(\frac{T_j}{T_{j\parallel}} \right) \left(\frac{1}{T_{j\perp} T_{j\parallel}^{\frac{1}{2}}} \right) \frac{\hat{\tau}}{B} |\chi| \varepsilon^{\frac{1}{2}} e^{-\varepsilon\chi^2/T_{j\parallel}} e^{-\varepsilon(1-\chi)^2/T_{j\perp}} d\varepsilon d\chi d\Psi \tag{268}$$

$$= -\sum_j \sqrt{\pi} \int\int \left[\left(\frac{Z_j e}{T_j} \right)^2 \left| \tilde{\mathbf{\Phi}} + \boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla}\Phi_0 \right|^2 \right] n_j T_j \left(\frac{T_j}{T_{j\parallel}} \right) \left(\frac{T_j^{\frac{3}{2}}}{T_{j\perp} T_{j\parallel}^{\frac{1}{2}}} \right) \frac{\hat{\tau}}{B} |\chi| d\chi d\Psi \int_0^\infty \hat{\varepsilon}^{\frac{1}{2}} e^{-\hat{\varepsilon}\chi^2 \left(T_j/T_{j\parallel}\right)} e^{-\hat{\varepsilon}(1-\chi)^2 (T_j/T_{j\perp})} d\hat{\varepsilon} \tag{269}$$

$$= -\sum_j \frac{\pi}{2} \int\int \left[\left(\frac{Z_j e}{T_j} \right)^2 \left| \tilde{\mathbf{\Phi}} + \boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla}\Phi_0 \right|^2 \right] n_j T_j \left(\frac{T_j}{T_{j\parallel}} \right) \left(\frac{T_j^{\frac{3}{2}}}{T_{j\perp} T_{j\parallel}^{\frac{1}{2}}} \right) \frac{\hat{\tau}}{B} |\chi| \left[\chi^2 \left(\frac{T_j}{T_{j\parallel}} \right) + (1-\chi^2) \left(\frac{T_j}{T_{j\perp}} \right) \right]^{-\frac{3}{2}} d\chi d\Psi. \tag{270}$$

Here we have used

$$\int_0^\infty x^{\frac{1}{2}} e^{-ax} dx = \frac{\sqrt{\pi}}{2a^{\frac{3}{2}}}, \quad \text{for } a > 0. \tag{271}$$

The δW_A term is no longer zero, but rather, from Eq. 232:
(Redo all this)

$$\delta W_A^{bM} = -\sum_j 2\sqrt{2}\pi^2 \int\int\int \left[\langle HT_j \rangle^* \frac{\tilde{\mathbf{B}}_\parallel}{B} \frac{\chi^2 - 1}{2|\chi|} \right] f_j^{bM} (2\varepsilon|\chi|) \left(\frac{1}{T_{j\parallel}} - \frac{1}{T_{j\perp}} \right) \frac{\hat{\tau}}{m_j^{\frac{3}{2}} B} |\chi| \varepsilon^{\frac{1}{2}} d\varepsilon d\chi d\mathbf{\Psi} \tag{272}$$

$$= \sum_j \sqrt{\pi} \int\int\int \left[\langle HT_j \rangle^* \frac{\tilde{\mathbf{B}}_\parallel}{B} (\chi^2 - 1) \right] n_j \frac{1}{T_{j\perp} T_{j\parallel}^{\frac{1}{2}}} e^{-\varepsilon\chi^2/T_{j\parallel}} e^{-\varepsilon(1-\chi^2)/T_{j\perp}} \left(\frac{1}{T_{j\perp}} - \frac{1}{T_{j\parallel}} \right) \frac{\hat{\tau}}{B} |\chi| \varepsilon^{\frac{3}{2}} d\varepsilon d\chi d\mathbf{\Psi} \tag{273}$$

$$= \sum_j \sqrt{\pi} \int\int\int n_j T_j \left(\frac{1}{T_j} \right) \left(\frac{1}{T_{j\perp} T_{j\parallel}^{\frac{1}{2}}} \right) \left(\frac{1}{T_{j\perp}} - \frac{1}{T_{j\parallel}} \right) \frac{\hat{\tau}}{B} |\chi| \varepsilon^{\frac{5}{2}} T_{j\parallel}^{-\frac{7}{2}} e^{-\varepsilon\chi^2/T_{j\parallel}} e^{-\varepsilon(1-\chi^2)/T_{j\perp}} d\varepsilon d\chi d\mathbf{\Psi}$$
$$\left[\langle HT_j/\varepsilon \rangle^* \frac{\tilde{\mathbf{B}}_\parallel}{B} (\chi^2 - 1) \right] \tag{274}$$

$$= \sum_j \sqrt{\pi} \int\int\int n_j T_j \left(\frac{T_j^{\frac{3}{2}}}{T_{j\perp} T_{j\parallel}^{\frac{1}{2}}} \right) \left(\frac{T_j}{T_{j\perp}} - \frac{T_j}{T_{j\parallel}} \right) \frac{\hat{\tau}}{B} |\chi| \hat{\varepsilon}^{\frac{5}{2}} e^{-\hat{\varepsilon}\chi^2 \left(T_j/T_{j\parallel}\right)} e^{-\hat{\varepsilon}\left(1-\chi^2\right)(T_j/T_{j\perp})} d\hat{\varepsilon} d\chi d\mathbf{\Psi}$$
$$\left[\langle H/\hat{\varepsilon} \rangle^* \frac{\tilde{\mathbf{B}}_\parallel}{B} (\chi^2 - 1) \right]. \tag{275}$$

Like δW_Φ, δW_A does not involve a frequency resonance fraction with various energy dependent terms in the same way that δW_K does. Therefore we can simply perform the energy integration, using[60]:

$$\int_0^\infty x^{\frac{5}{2}} e^{-ax} dx = \frac{15\sqrt{\pi}}{8a^{\frac{7}{2}}}, \quad \text{for } a > 0. \tag{276}$$

Then we have:

$$\delta W_A^{bM} = \sum_j \frac{15\pi}{8} \int\int n_j T_j \left(\frac{T_j^{\frac{3}{2}}}{T_{j\perp} T_{j\parallel}^{\frac{1}{2}}}\right) \left(\frac{T_j}{T_{j\perp}} - \frac{T_j}{T_{j\parallel}}\right) \frac{\hat{\tau}}{B} \left[\langle H/\hat{\varepsilon}\rangle^* \frac{\tilde{\mathbf{B}}_\parallel}{B}\right]$$
$$|\chi| \left(\chi^2 - 1\right) \left[\chi^2 \left(\frac{T_j}{T_{j\parallel}}\right) + \left(1 - \chi^2\right) \left(\frac{T_j}{T_{j\perp}}\right)\right]^{-\frac{7}{2}} d\chi d\mathbf{\Psi}$$

C. Isotropic Slowing-down Distribution Function

If instead of thermal particles we consider the stabilizing kinetic effects of energetic particles[21], such as beam ions[19,61,62] or alpha particles[19,26,63], then we must make use of a different distribution function. An isotropic slowing-down distribution function that is a good model for alpha particles is:

$$f_j^\alpha(\varepsilon, \Psi) = n_j A_\alpha \left(\frac{m_j}{\varepsilon_\alpha}\right)^{\frac{3}{2}} \frac{1}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}}, \quad 0 \le \varepsilon \le \varepsilon_\alpha, \tag{277}$$

where ε_α (which takes the place of T_j as a defining energy of the distribution) is the birth energy (3.52 MeV for alpha particles), $\hat{\varepsilon} = \varepsilon/\varepsilon_\alpha$, $\hat{\varepsilon}_c = \varepsilon_c/\varepsilon_\alpha$, and the critical energy between slowing down on electrons ($\varepsilon > \varepsilon_c$) vs. slowing down on ions ($\varepsilon < \varepsilon_c$) is[63,64]:

$$\varepsilon_c = \left(\frac{3\sqrt{\pi}}{4}\right)^{\frac{2}{3}} \left(\frac{m_j}{m_e}\right) \left(\frac{m_e}{n_e} \sum_i \left(\frac{n_i Z_i^2}{m_i}\right)\right)^{\frac{2}{3}} T_e. \tag{278}$$

Note that in a deuterium plasma with $n_i = n_e$, $\varepsilon_c/T_e = 18.65(m_j/m_i)$ so that for alpha particles $\hat{\varepsilon}_c \approx 0.01\ T_e$ [keV]. For alpha particles, then, $\hat{\varepsilon}_c \ll 1$ for plasmas with $T_e \ll 100$ keV, so $\hat{\varepsilon}_c$ is quite small in Eq. 277 compared to the range of $0 \le \hat{\varepsilon} \le 1$. For deuterium beam ions $\hat{\varepsilon}_c \approx 0.2\ T_e$ [keV], so $\hat{\varepsilon}_c$ and $\hat{\varepsilon}$ are comparable for plasmas with $T_e \approx 1 - 10$ keV. Note that in a plasma with a significant fraction of tritium, one must use the full form of Eq. 278, as it can make a difference in the calculation for alpha particles[65].

The constant A_α is defined by the first moment:

$$n_j = \int f_j^\alpha d^3\mathbf{v} = \int_{-1}^{1} \int_0^{\varepsilon_\alpha} \frac{2\sqrt{2}\pi}{m_j^{\frac{3}{2}}} \varepsilon^{\frac{1}{2}} f_j^\alpha d\varepsilon d\chi = \int_0^1 \frac{4\sqrt{2}\pi}{m_j^{\frac{3}{2}}} \varepsilon_\alpha^{\frac{3}{2}} \hat{\varepsilon}^{\frac{1}{2}} f_j^\alpha d\hat{\varepsilon}. \tag{279}$$

Using equations 277 and 279, we find:

$$A_\alpha = \left(4\sqrt{2}\pi \int_0^1 \frac{\hat{\varepsilon}^{\frac{1}{2}}}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}} d\hat{\varepsilon}\right)^{-1} \tag{280}$$

which is the same as Eq. 20 in Ref. [26]. This integral is analytically solvable, and results in[39]:

$$A_\alpha = \frac{3}{8\sqrt{2}\pi} \left(\ln\left(1 + \hat{\varepsilon}_c^{-\frac{3}{2}}\right)\right)^{-1}, \tag{281}$$

so that

$$f_j^\alpha(\varepsilon, \Psi) = n_j \frac{3}{8\sqrt{2}\pi} \left(\ln\left(1 + \hat{\varepsilon}_c^{-\frac{3}{2}}\right)\right)^{-1} \left(\frac{m_j}{\varepsilon_\alpha}\right)^{\frac{3}{2}} \frac{1}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}}, \quad 0 \le \varepsilon \le \varepsilon_\alpha. \tag{282}$$

Note that this distribution function is isotropic in pitch angle (independent of χ), so that $\partial f_j/\partial\chi = 0$.

Now, performing the partial derivatives from Eq. **??**, first:

$$\frac{\partial f_j}{\partial \varepsilon} = -\frac{f_j^\alpha}{\varepsilon_\alpha}\frac{3}{2}\frac{\hat{\varepsilon}^{\frac{1}{2}}}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}}. \tag{283}$$

And, second:

$$\frac{\partial f_j}{\partial \Psi} = f_j^\alpha \left(\frac{1}{n_j}\frac{dn_j}{d\Psi} + \frac{1}{A_\alpha}\frac{dA_\alpha}{d\Psi} + \frac{1}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}}\left(-\frac{d\hat{\varepsilon}_c^{\frac{3}{2}}}{d\Psi}\right)\right). \tag{284}$$

Now, analogously to Eq. 241,

$$(\omega - n\omega_E)\frac{\partial f_j}{\partial \varepsilon} - \frac{n}{Z_j e}\frac{\partial f_j}{\partial \Psi} = \frac{f_j^\alpha}{\varepsilon_\alpha}\left(\frac{\varepsilon_\alpha n}{Z_j e}\left(\frac{1}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}}\frac{d\hat{\varepsilon}_c^{\frac{3}{2}}}{d\Psi} - \frac{1}{A_\alpha}\frac{dA_\alpha}{d\Psi} - \frac{1}{n_j}\frac{dn_j}{d\Psi}\right) + \frac{3}{2}\frac{\hat{\varepsilon}^{\frac{1}{2}}}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}}(n\omega_E - \omega)\right). \tag{285}$$

From Eq. 233, the frequency resonance fraction is:

$$\lambda_{j,l}^\alpha = \frac{\frac{\varepsilon_\alpha n}{Z_j e}\left(\frac{1}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}}\frac{d\hat{\varepsilon}_c^{\frac{3}{2}}}{d\Psi} - \frac{1}{A_\alpha}\frac{dA_\alpha}{d\Psi} - \frac{1}{n_j}\frac{dn_j}{d\Psi}\right) + \frac{3}{2}\frac{\hat{\varepsilon}^{\frac{1}{2}}}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}}(n\omega_E - \omega)}{n\langle\omega_D^j\rangle + l\omega_b^j - i\nu_{\text{eff}}^j + n\omega_E - \omega}. \tag{286}$$

And, finally, substituting into Eq. 229,

$$\delta W_K^\alpha = \sum_j \sum_{l=-\infty}^{\infty} 2\sqrt{2}\pi^2 \int\int\int \frac{\hat{\tau}}{m_j^{\frac{3}{2}} B}|\chi|\varepsilon^{\frac{1}{2}} d\varepsilon d\chi d\Psi \left[|\langle HT_j\rangle|^2 \frac{\frac{\varepsilon_\alpha n}{Z_j e}\left(\frac{1}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}}\frac{d\hat{\varepsilon}_c^{\frac{3}{2}}}{d\Psi} - \frac{1}{A_\alpha}\frac{dA_\alpha}{d\Psi} - \frac{1}{n_j}\frac{dn_j}{d\Psi}\right) + \frac{3}{2}\frac{\hat{\varepsilon}^{\frac{1}{2}}}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}}(n\omega_E - \omega)}{n\langle\omega_D^j\rangle + l\omega_b^j - i\nu_{\text{eff}}^j + n\omega_E - \omega}\frac{f_j^\alpha}{\varepsilon_\alpha}\right] \tag{287}$$

$$= \sum_j \sum_{l=-\infty}^{\infty} 2\sqrt{2}\pi^2 \int\int\int n_j \varepsilon_\alpha A_\alpha \frac{\hat{\tau}}{B}|\chi| \frac{\hat{\varepsilon}^{\frac{5}{2}}}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}} d\hat{\varepsilon} d\chi d\Psi \left[|\langle H/\hat{\varepsilon}\rangle|^2 \frac{\frac{\varepsilon_\alpha n}{Z_j e}\left(\frac{1}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}}\frac{d\hat{\varepsilon}_c^{\frac{3}{2}}}{d\Psi} - \frac{1}{A_\alpha}\frac{dA_\alpha}{d\Psi} - \frac{1}{n_j}\frac{dn_j}{d\Psi}\right) + \frac{3}{2}\frac{\hat{\varepsilon}^{\frac{1}{2}}}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}}(n\omega_E - \omega)}{n\langle\omega_D^j\rangle + l\omega_b^j - i\nu_{\text{eff}}^j + n\omega_E - \omega}\right], \tag{288}$$

which is the result found in Ref. [26], Eq. (19). For the electrostatic term:

$$\delta W_\Phi^\alpha = \sum_j 2\sqrt{2}\pi^2 \int\int\int \left[-(Z_j e)^2 \left|\tilde{\mathbf{\Phi}} + \boldsymbol{\xi}_\perp \cdot \nabla\Phi_0\right|^2 \frac{f_j^\alpha}{\varepsilon_\alpha}\frac{3}{2}\frac{\hat{\varepsilon}^{\frac{1}{2}}}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}}\right] \frac{\hat{\tau}}{m_j^{\frac{3}{2}} B}|\chi|\varepsilon^{\frac{1}{2}} d\varepsilon d\chi d\Psi \tag{289}$$

$$= -\sum_j 3\sqrt{2}\pi^2 \int\int\int (Z_j e)^2 \left|\tilde{\mathbf{\Phi}} + \boldsymbol{\xi}_\perp \cdot \nabla\Phi_0\right|^2 \frac{\hat{\tau}}{B}|\chi| d\chi d\Psi \frac{n_j A_\alpha}{\varepsilon_\alpha}\left(\frac{m_j}{\varepsilon_\alpha}\right)^{\frac{3}{2}} \frac{1}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}}\frac{\hat{\varepsilon}^{\frac{1}{2}}}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}}\left(\frac{\varepsilon_\alpha}{m_j}\right)^{\frac{3}{2}} \hat{\varepsilon}^{\frac{1}{2}} d\hat{\varepsilon} \tag{290}$$

$$= -\sum_j 3\sqrt{2}\pi^2 \int\int\int \left(\frac{Z_j e}{\varepsilon_\alpha}\right)^2 \left|\tilde{\mathbf{\Phi}} + \boldsymbol{\xi}_\perp \cdot \nabla\Phi_0\right|^2 n_j \varepsilon_\alpha A_\alpha \frac{\hat{\tau}}{B}|\chi| \left(\frac{\hat{\varepsilon}^{\frac{1}{2}}}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}}\right)^2 d\hat{\varepsilon} d\chi d\Psi. \tag{291}$$

For the isotropic slowing-down distribution, $\sigma = 1$, $p_{\text{avg}} = p$ in δW_F (Eq. **??**), and $\delta W_A = 0$.

D. Anisotropic Slowing-down Distribution Function

For beam ions, a logical extension of the previous slowing-down distribution function is to add pitch angle dependence. The anisotropy of the energetic particles' distribution function may have an influence on stability[19,24,66–68]. A standard solution is a Gaussian distribution of particles in χ,[69] so that, modifying Eq. 282, we have:

$$f_j^b(\varepsilon,\Psi,\chi) = \sum_s\sum_k\sum_p f_{s,k,p}(\varepsilon,\Psi,\chi) \tag{292}$$

$$= \sum_s\sum_k\sum_p n_{s,k,p}(\Psi)A_{s,k,p}(\Psi)\left(\frac{m_j}{\varepsilon_{s,k}}\right)^{\frac{3}{2}}\frac{1}{\hat{\varepsilon}_{s,k}^{\frac{3}{2}}+\hat{\varepsilon}_c^{\frac{3}{2}}(\Psi)}\frac{1}{\delta\chi_{s,k,p}(\hat{\varepsilon}_{s,k},\Psi)}$$
$$\times\left[e^{-(\chi-\chi_{0\ s,p}(\Psi))^2/\delta\chi^2_{s,k,p}(\hat{\varepsilon}_{s,k},\Psi)}+e^{-(\chi+2+\chi_{0\ s,p}(\Psi))^2/\delta\chi^2_{s,k,p}(\hat{\varepsilon}_{s,k},\Psi)}+e^{-(\chi-2+\chi_{0\ s,p}(\Psi))^2/\delta\chi^2_{s,k,p}(\hat{\varepsilon}_{s,k},\Psi)}\right],$$
$$0\le\hat{\varepsilon}_{s,k}\le 1, -1\le\chi\le 1. \tag{293}$$

We have made the above expression general by making it a sum of many different particle types. We will allow there to be s number of sources, k number of energy components, and p surfaces of deposition, so that the total energetic particle distribution is fully described by the linear superposition of $s\times k\times p$ separate distributions of the above form. Here $\varepsilon_{s,k}$ is the maximum beam energy for source s and energy component k. So, if the source $s=A$ is set to 90 keV, for example, then one might notate the full energy component maximum energy as $\varepsilon_{s,k}=\varepsilon_{A,\mathrm{full}}=90$ keV, while the half energy component maximum energy for source B set to 70 keV would be: $\varepsilon_{B,\mathrm{half}}=35$ keV. In addition, each beam source can deposit on a particular Ψ surface more than once, for example $p=$ out when it enters the surface on the outboard side, and then $p=$ in when it exits on the inboard side.

The additional two exponential terms are included here in order to satisfy the boundary conditions of no diffusive flux[69] at $\chi=-1$ or 1. Here, however, we do not require any conditions on the trapped/circulating boundaries, or symmetry about $\chi=0$.

The general form above gives each surface of deposition a central pitch angle $\chi_{0\ s,p}$, and width $\delta\chi_{s,k,p}$. The center of the Gaussian is determined geometrically, by the intersection of the beam line with the magnetic field lines of the particular surfaces, and is therefore, ostensibly, a known quantity. Note that for the general case of non-perpendicular injection, $\chi_{s,p}\neq 0$ and the distribution is not symmetric and therefore the recasting of the formulation of the problem in terms of χ rather than Λ in subsection VIII was indeed necessary.

The spread of the Gaussian depends on the energy of the particles, because the broadening is determined by Coulomb scattering[69]. The form of the Gaussian width is given by[69–71]:

$$\delta\chi_{s,k,p}(\hat{\varepsilon}_{s,k},\Psi) = \sqrt{\delta\chi^2_{0\ s,p}(\Psi)-\frac{1}{3}\ln\left[1+\hat{\varepsilon}_c^{\frac{3}{2}}(\Psi)\right]-\frac{1}{3}\ln\left[\frac{\hat{\varepsilon}_{s,k}^{\frac{3}{2}}}{\hat{\varepsilon}_{s,k}^{\frac{3}{2}}+\hat{\varepsilon}_c^{\frac{3}{2}}(\Psi)}\right]}. \tag{294}$$

Here it is implicitly assumed that the energetic particles have the same mass as the thermal ions they are slowing down on (I think).

An alternative form for $\delta\chi$ is given in Ref. [72] (where the parameter α is used, such that $\delta\chi=\sqrt{4\alpha}$):

$$\delta\chi = \sqrt{\frac{4m_iZ_{\mathrm{eff}}}{6m_b}(1-\chi_0^2)\ln\left[\frac{1+\hat{\varepsilon}_c^{\frac{3}{2}}/\hat{\varepsilon}^{\frac{3}{2}}}{1+\hat{\varepsilon}_c^{\frac{3}{2}}}\right]}. \tag{295}$$

With $m_b=m_i$ and $Z_{\mathrm{eff}}=1$, this can be written,

$$\delta\chi_{s,k,p}(\hat{\varepsilon}_{s,k},\Psi) = \sqrt{2\left(\chi^2_{0\ s,p}(\Psi)-1\right)\left(\frac{1}{3}\ln\left[1+\hat{\varepsilon}_c^{\frac{3}{2}}(\Psi)\right]+\frac{1}{3}\ln\left[\frac{\hat{\varepsilon}_{s,k}^{\frac{3}{2}}}{\hat{\varepsilon}_{s,k}^{\frac{3}{2}}+\hat{\varepsilon}_c^{\frac{3}{2}}(\Psi)}\right]\right)}, \tag{296}$$

which is similar, but not the same, as Eq. 294. Why?

If we now solve for A_b in terms of the other quantities, then the unknowns that must be specified to fully describe the energetic particle distribution function are $s\times p$ number of χ_0 terms, $s\times p$ number of $\delta\chi_0$ terms, s number of

$\varepsilon_{s,\text{full}}$ (the other energy components being fractions of the full energy), and $s \times k \times p$ number of $n_{s,k,p}$ terms. The $\varepsilon_{s,\text{full}}$ energies are known, and the χ_0 terms can be determined by geometry. The $\delta\chi_0$ terms will most likely need to be modeled. Finally, the breakdown of the total energetic particle density profile into its constituent profiles is not easily determined. It is probably best modeled as well, in which case it is not important to know each $n_{s,k,p}$ specifically, but one could rather use a weighting quantity $C_{s,k,p}(\Psi) = n_{s,k,p}(\Psi)A_{s,k,p}(\Psi)\left(m_j/\varepsilon_{s,k}\right)^{\frac{3}{2}}$ in Eq. 293 instead. For completeness, however, we will now demonstrate that $A_{s,k,p}(\Psi)$ can be solved for in terms of the other quantities, as in Eq. 279, by the definition of the density profile in terms of the distribution function. The difference is that now each individual particle type must have its own density profile defined, so that:

$$n_{s,k,p}(\Psi) = \int f_{s,k,p} d^3\mathbf{v} = \int_{-1}^{1}\int_{0}^{1} 2\sqrt{2}\pi\left(\frac{\varepsilon_{s,k}}{m_j}\right)^{\frac{3}{2}} \hat{\varepsilon}^{\frac{1}{2}} f_{s,k,p} d\hat{\varepsilon} d\chi, \tag{297}$$

where, of course,

$$n_j(\Psi) = \sum_s \sum_k \sum_p n_{s,k,p}(\Psi), \tag{298}$$

(which is a further useful constraint on the system not available if C is used instead).

Let us consider a single beam source, s, at a single injection energy, $\varepsilon_{s,k} = \varepsilon_b$, with a single deposition surface, $\chi_{0\ s,p} = \chi_0$. Since every other particle type follows the exact same form, the following derivations will hold for each. Substituting Eq. 294 into Eq. 293, we have:

$$f_j^b(\varepsilon,\Psi,\chi) = n_j A_b \left(\frac{m_j}{\varepsilon_b}\right)^{\frac{3}{2}} \frac{1}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}} \frac{1}{\delta\chi}\left(\exp\left[\frac{-\left(\chi-\chi_0\right)^2}{\delta\chi^2}\right] + \exp\left[\frac{-\left(\chi+2+\chi_0\right)^2}{\delta\chi^2}\right] + \exp\left[\frac{-\left(\chi-2+\chi_0\right)^2}{\delta\chi^2}\right]\right). \tag{299}$$

In order to solve for A_b, we substitute Eq. 299 into Eq. 297:

$$n_j = 2\sqrt{2}\pi n_j A_b \int_{-1}^{1}\int_{0}^{1} \frac{\hat{\varepsilon}^{\frac{1}{2}}}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}} \frac{1}{\delta\chi}\left(\exp\left[\frac{-\left(\chi-\chi_0\right)^2}{\delta\chi^2}\right] + \exp\left[\frac{-\left(\chi+2+\chi_0\right)^2}{\delta\chi^2}\right] + \exp\left[\frac{-\left(\chi-2+\chi_0\right)^2}{\delta\chi^2}\right]\right) d\hat{\varepsilon} d\chi. \tag{300}$$

Then we perform the χ integration, using :

$$\int_{-1}^{1} \frac{e^{\frac{-(\chi-c)^2}{\delta\chi^2}}}{\delta\chi} d\chi = \frac{\sqrt{\pi}}{2}\left[\text{erf}\left(\frac{c+1}{\delta\chi}\right) - \text{erf}\left(\frac{c-1}{\delta\chi}\right)\right], \tag{301}$$

so that

$$\begin{aligned} n_j &= \sqrt{2}\pi^{\frac{3}{2}} n_j A_b \int_0^1 \frac{\hat{\varepsilon}^{\frac{1}{2}}}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}} d\hat{\varepsilon} \\ &\quad \left[\text{erf}\left(\frac{1-\chi_0}{\delta\chi}\right) - \text{erf}\left(\frac{-1-\chi_0}{\delta\chi}\right) + \text{erf}\left(\frac{3+\chi_0}{\delta\chi}\right) - \text{erf}\left(\frac{1+\chi_0}{\delta\chi}\right) + \text{erf}\left(\frac{-1+\chi_0}{\delta\chi}\right) - \text{erf}\left(\frac{-3+\chi_0}{\delta\chi}\right)\right] && (302) \\ &= \sqrt{2}\pi^{\frac{3}{2}} n_j A_b \int_0^1 \frac{\hat{\varepsilon}^{\frac{1}{2}}}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}} d\hat{\varepsilon}\left[\text{erf}\left(\frac{\chi_0+3}{\delta\chi}\right) - \text{erf}\left(\frac{\chi_0-3}{\delta\chi}\right)\right]. && (303) \end{aligned}$$

Then

$$A_b = \left[4\sqrt{2}\pi \int_0^1 \frac{\hat{\varepsilon}^{\frac{1}{2}}}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}} \frac{\sqrt{\pi}}{4}\left[\text{erf}\left(\frac{\chi_0+3}{\delta\chi}\right) - \text{erf}\left(\frac{\chi_0-3}{\delta\chi}\right)\right] d\hat{\varepsilon}\right]^{-1} \tag{304}$$

which is, unfortunately, not as easily integrated as A_α was. Now, from Eq. 299

$$f_j^b(\varepsilon,\Psi,\chi) = n_j\left(\frac{m_j}{\varepsilon_b}\right)^{\frac{3}{2}}\frac{1}{\hat{\varepsilon}^{\frac{3}{2}}+\hat{\varepsilon}_c^{\frac{3}{2}}}\left[\delta\chi_0^2-\frac{1}{3}\ln\left[\frac{\hat{\varepsilon}^{\frac{3}{2}}(1+\hat{\varepsilon}_c^{\frac{3}{2}})}{\hat{\varepsilon}^{\frac{3}{2}}+\hat{\varepsilon}_c^{\frac{3}{2}}}\right]\right]^{-\frac{1}{2}}$$
$$\times\left(\exp\left[\frac{-(\chi-\chi_0)^2}{\delta\chi_0^2-\frac{1}{3}\ln\left[\frac{\hat{\varepsilon}^{\frac{3}{2}}(1+\hat{\varepsilon}_c^{\frac{3}{2}})}{\hat{\varepsilon}^{\frac{3}{2}}+\hat{\varepsilon}_c^{\frac{3}{2}}}\right]}\right]+\exp\left[\frac{-(\chi+2+\chi_0)^2}{\delta\chi_0^2-\frac{1}{3}\ln\left[\frac{\hat{\varepsilon}^{\frac{3}{2}}(1+\hat{\varepsilon}_c^{\frac{3}{2}})}{\hat{\varepsilon}^{\frac{3}{2}}+\hat{\varepsilon}_c^{\frac{3}{2}}}\right]}\right]+\exp\left[\frac{-(\chi-2+\chi_0)^2}{\delta\chi_0^2-\frac{1}{3}\ln\left[\frac{\hat{\varepsilon}^{\frac{3}{2}}(1+\hat{\varepsilon}_c^{\frac{3}{2}})}{\hat{\varepsilon}^{\frac{3}{2}}+\hat{\varepsilon}_c^{\frac{3}{2}}}\right]}\right]\right)\times$$
$$\left[4\sqrt{2}\pi\int_0^1\frac{\hat{\varepsilon}^{\frac{1}{2}}}{\hat{\varepsilon}^{\frac{3}{2}}+\hat{\varepsilon}_c^{\frac{3}{2}}}\frac{\sqrt{\pi}}{4}\left[\operatorname{erf}\left(\frac{\chi_0+3}{\left[\delta\chi_0^2-\frac{1}{3}\ln\left[\frac{\hat{\varepsilon}^{\frac{3}{2}}(1+\hat{\varepsilon}_c^{\frac{3}{2}})}{\hat{\varepsilon}^{\frac{3}{2}}+\hat{\varepsilon}_c^{\frac{3}{2}}}\right]\right]^{\frac{1}{2}}}\right)-\operatorname{erf}\left(\frac{\chi_0-3}{\left[\delta\chi_0^2-\frac{1}{3}\ln\left[\frac{\hat{\varepsilon}^{\frac{3}{2}}(1+\hat{\varepsilon}_c^{\frac{3}{2}})}{\hat{\varepsilon}^{\frac{3}{2}}+\hat{\varepsilon}_c^{\frac{3}{2}}}\right]\right]^{\frac{1}{2}}}\right)\right)d\hat{\varepsilon}\right]^{-1}. \tag{305}$$

Let us now consider the necessary partial derivatives. First, $\partial f_j/\partial\varepsilon$ begins in the same manner as the expression in the previous subsection (Eq. 283), but now includes additional terms because of the dependence of $\delta\chi$ on ε. If we write the exponential terms of Eq. 299 as

$$F_1+F_2+F_3=\frac{e^{-(a_1/\delta\chi)^2}}{\delta\chi}+\frac{e^{-(a_2/\delta\chi)^2}}{\delta\chi}+\frac{e^{-(a_3/\delta\chi)^2}}{\delta\chi}, \tag{306}$$

then

$$\frac{\partial f_j}{\partial\varepsilon}=\frac{1}{\varepsilon_b}\left[-\frac{3}{2}\frac{\hat{\varepsilon}^{\frac{1}{2}}}{\left(\hat{\varepsilon}^{\frac{3}{2}}+\hat{\varepsilon}_c^{\frac{3}{2}}\right)^2}n_jA_b\left(\frac{m_j}{\varepsilon_b}\right)^{\frac{3}{2}}(F_1+F_2+F_3)+\frac{1}{\left(\hat{\varepsilon}^{\frac{3}{2}}+\hat{\varepsilon}_c^{\frac{3}{2}}\right)^2}n_jA_b\left(\frac{m_j}{\varepsilon_b}\right)^{\frac{3}{2}}\left(\frac{\partial F_1}{\partial\hat{\varepsilon}}+\frac{\partial F_2}{\partial\hat{\varepsilon}}+\frac{\partial F_3}{\partial\hat{\varepsilon}}\right)\right] \tag{307}$$
$$=f_j^b\frac{1}{\varepsilon_b}\left[-\frac{3}{2}\frac{\hat{\varepsilon}^{\frac{1}{2}}}{\hat{\varepsilon}^{\frac{3}{2}}+\hat{\varepsilon}_c^{\frac{3}{2}}}+\frac{1}{F_1+F_2+F_3}\frac{1}{\delta\chi}\frac{d\delta\chi}{d\hat{\varepsilon}}\left[F_1\left(2\frac{a_1^2}{\delta\chi^2}-1\right)+F_2\left(2\frac{a_2^2}{\delta\chi^2}-1\right)+F_3\left(2\frac{a_3^2}{\delta\chi^2}-1\right)\right]\right] \tag{308}$$
$$=f_j^b\frac{1}{\varepsilon_b}\left[-\frac{3}{2}\frac{\hat{\varepsilon}^{\frac{1}{2}}}{\hat{\varepsilon}^{\frac{3}{2}}+\hat{\varepsilon}_c^{\frac{3}{2}}}+\frac{1}{\delta\chi}\frac{d\delta\chi}{d\hat{\varepsilon}}\left[\frac{2}{\delta\chi^2}\frac{F_1a_1^2+F_2a_2^2+F_3a_3^2}{F_1+F_2+F_3}-1\right]\right]. \tag{309}$$

Now, using

$$\frac{d\delta\chi}{d\hat{\varepsilon}}=-\frac{1}{4}\frac{1}{\delta\chi}\frac{1}{\hat{\varepsilon}}\left(\frac{\hat{\varepsilon}_c^{\frac{3}{2}}}{\hat{\varepsilon}^{\frac{3}{2}}+\hat{\varepsilon}_c^{\frac{3}{2}}}\right), \tag{310}$$

we have

$$\frac{\partial f_j}{\partial\varepsilon}=f_j^b\frac{1}{\varepsilon_b}\left[-\frac{3}{2}\frac{\hat{\varepsilon}^{\frac{1}{2}}}{\hat{\varepsilon}^{\frac{3}{2}}+\hat{\varepsilon}_c^{\frac{3}{2}}}+\frac{1}{4}\frac{1}{\delta\chi^2}\frac{1}{\hat{\varepsilon}}\left(\frac{\hat{\varepsilon}_c^{\frac{3}{2}}}{\hat{\varepsilon}^{\frac{3}{2}}+\hat{\varepsilon}_c^{\frac{3}{2}}}\right)\right.$$
$$\left.\left(1-\frac{2}{\delta\chi^2}\frac{(\chi-\chi_0)^2e^{-(\chi-\chi_0)^2/\delta\chi^2}+(\chi+2+\chi_0)^2e^{-(\chi+2+\chi_0)^2/\delta\chi^2}+(\chi-2+\chi_0)^2e^{-(\chi-2+\chi_0)^2/\delta\chi^2}}{e^{-(\chi-\chi_0)^2/\delta\chi^2}+e^{-(\chi+2+\chi_0)^2/\delta\chi^2}+e^{-(\chi-2+\chi_0)^2/\delta\chi^2}}\right)\right]. \tag{311}$$

Similarly, due to the additional dependence of χ_0 and $\delta\chi_0$ on Ψ (and of $\hat{\varepsilon}_c$ on Ψ in the $\delta\chi$ term), $\partial f_j/\partial\Psi$ is not the same as Eq. 284, but is instead given by the much more complex expression:

$$\frac{\partial f_j}{\partial \Psi} = f_j^b \left[\frac{1}{n_j}\frac{dn_j}{d\Psi} + \frac{1}{A_b}\frac{dA_b}{d\Psi} - \frac{1}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}}\frac{d\hat{\varepsilon}_c^{\frac{3}{2}}}{d\Psi} + \frac{\frac{\partial F_1}{\partial \Psi} + \frac{\partial F_2}{\partial \Psi} + \frac{\partial F_3}{\partial \Psi}}{F_1 + F_2 + F_3} \right] \tag{312}$$

$$\begin{aligned} &= f_j^b \left[\frac{1}{n_j}\frac{dn_j}{d\Psi} + \frac{1}{A_b}\frac{dA_b}{d\Psi} - \frac{1}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}}\frac{d\hat{\varepsilon}_c^{\frac{3}{2}}}{d\Psi} + \frac{1}{F_1 + F_2 + F_3} \right. \\ &\left(F_1 \left(\frac{1}{\delta\chi}\frac{\partial\delta\chi}{\partial\Psi}\left(\frac{2\left(\chi-\chi_0\right)^2}{\delta\chi^2} - 1 \right) + \frac{2}{\delta\chi^2}\left(\chi-\chi_0\right)\frac{d\chi_0}{d\Psi} \right) + \right. \\ &F_2 \left(\frac{1}{\delta\chi}\frac{\partial\delta\chi}{\partial\Psi}\left(\frac{2\left(\chi+2+\chi_0\right)^2}{\delta\chi^2} - 1 \right) - \frac{2}{\delta\chi^2}\left(\chi+2+\chi_0\right)\frac{d\chi_0}{d\Psi} \right) + \\ &\left.\left. F_3 \left(\frac{1}{\delta\chi}\frac{\partial\delta\chi}{\partial\Psi}\left(\frac{2\left(\chi-2+\chi_0\right)^2}{\delta\chi^2} - 1 \right) - \frac{2}{\delta\chi^2}\left(\chi-2+\chi_0\right)\frac{d\chi_0}{d\Psi} \right)\right)\right]. \end{aligned} \tag{313}$$

Continuing, with

$$\frac{\partial\delta\chi}{\partial\Psi} = \frac{1}{\delta\chi}\left[\delta\chi_0\frac{d\delta\chi_0}{d\Psi} + \frac{1}{6}\frac{d\hat{\varepsilon}_c^{\frac{3}{2}}}{d\Psi}\left(\frac{1}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}} - \frac{1}{1+\hat{\varepsilon}_c^{\frac{3}{2}}} \right) \right], \tag{314}$$

we have:

$$\begin{aligned} \frac{\partial f_j}{\partial \Psi} =& f_j^b \left[\frac{1}{n_j}\frac{dn_j}{d\Psi} + \frac{1}{A_b}\frac{dA_b}{d\Psi} - \frac{1}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}}\frac{d\hat{\varepsilon}_c^{\frac{3}{2}}}{d\Psi} + \frac{1}{e^{-(\chi-\chi_0)^2/\delta\chi^2} + e^{-(\chi+2+\chi_0)^2/\delta\chi^2} + e^{-(\chi-2+\chi_0)^2/\delta\chi^2}}\frac{1}{\delta\chi^2} \right. \\ &\left(e^{-(\chi-\chi_0)^2/\delta\chi^2}\left(\left[\delta\chi_0\frac{d\delta\chi_0}{d\Psi} + \frac{1}{6}\frac{d\hat{\varepsilon}_c^{\frac{3}{2}}}{d\Psi}\left(\frac{1}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}} - \frac{1}{1+\hat{\varepsilon}_c^{\frac{3}{2}}} \right) \right]\left(\frac{2\left(\chi-\chi_0\right)^2}{\delta\chi^2} - 1 \right) + 2\left(\chi-\chi_0\right)\frac{d\chi_0}{d\Psi} \right) + \right. \\ &e^{-(\chi+2+\chi_0)^2/\delta\chi^2}\left(\left[\delta\chi_0\frac{d\delta\chi_0}{d\Psi} + \frac{1}{6}\frac{d\hat{\varepsilon}_c^{\frac{3}{2}}}{d\Psi}\left(\frac{1}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}} - \frac{1}{1+\hat{\varepsilon}_c^{\frac{3}{2}}} \right) \right]\left(\frac{2\left(\chi+2+\chi_0\right)^2}{\delta\chi^2} - 1 \right) - 2\left(\chi+2+\chi_0\right)\frac{d\chi_0}{d\Psi} \right) + \\ &\left.\left. e^{-(\chi-2+\chi_0)^2/\delta\chi^2}\left(\left[\delta\chi_0\frac{d\delta\chi_0}{d\Psi} + \frac{1}{6}\frac{d\hat{\varepsilon}_c^{\frac{3}{2}}}{d\Psi}\left(\frac{1}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}} - \frac{1}{1+\hat{\varepsilon}_c^{\frac{3}{2}}} \right) \right]\left(\frac{2\left(\chi-2+\chi_0\right)^2}{\delta\chi^2} - 1 \right) - 2\left(\chi-2+\chi_0\right)\frac{d\chi_0}{d\Psi} \right)\right)\right]. \end{aligned} \tag{315}$$

Note that when $d\chi_0/d\Psi = 0$ and $d\delta\chi/d\Psi = 0$, the above equation reduces to only the first three terms, which is the same form as Eq. 284 for the isotropic slowing down distribution.

Finally, for this anisotropic case, $\partial f_j/\partial\chi \neq 0$, but rather:

$$\frac{\partial f_j}{\partial \chi} = f_j^b\left(-\frac{2}{\delta\chi^2} \right)\left[\frac{\left(\chi-\chi_0\right)e^{-(\chi-\chi_0)^2/\delta\chi^2} + \left(\chi+2+\chi_0\right)e^{-(\chi+2+\chi_0)^2/\delta\chi^2} + \left(\chi-2+\chi_0\right)e^{-(\chi-2+\chi_0)^2/\delta\chi^2}}{e^{-(\chi-\chi_0)^2/\delta\chi^2} + e^{-(\chi+2+\chi_0)^2/\delta\chi^2} + e^{-(\chi-2+\chi_0)^2/\delta\chi^2}} \right] \tag{316}$$

$$\begin{aligned} =& n_j A_b \left(\frac{m_j}{\varepsilon_b} \right)^{\frac{3}{2}} \frac{1}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}}\frac{1}{\delta\chi}\left(-\frac{2}{\delta\chi^2} \right) \times \\ &\left[\left(\chi-\chi_0\right)e^{-(\chi-\chi_0)^2/\delta\chi^2} + \left(\chi+2+\chi_0\right)e^{-(\chi+2+\chi_0)^2/\delta\chi^2} + \left(\chi-2+\chi_0\right)e^{-(\chi-2+\chi_0)^2/\delta\chi^2} \right]. \end{aligned} \tag{317}$$

Now, before proceeding with the full expression for the δW^b terms based upon these partial derivatives, let us first examine a special case which is greatly simplified.

1. The simplified case of a Gaussian with no radial or energy dependence

We note that the Gaussian term in Eq. 293 has energy dependence only in the $\delta\chi$ term. If, in Eq. 294, $\hat{\varepsilon}_c \ll \hat{\varepsilon}$ (ie. $\hat{\varepsilon}_c \to 0$), then the energy dependence disappears ($\delta\chi(\varepsilon,\Psi) = \delta\chi_0(\Psi)$). Such a case might be applicable, for example, if only very high energy particles (with $\hat{\varepsilon} \gg \hat{\varepsilon}_c$) are considered. Secondly, let us take χ_0 and $\delta\chi_0$ to be constants, independent of Ψ. This is a much simplified case which can not truly represent the geometry of neutral beam injection except, possibly, in the instance of perpendicular beam injection, in which case $\chi_0 = 0$ on all surfaces. Now the energy and Ψ dependencies of f_j^b are the same as f_j^α in Eq. 277, and therefore $\partial f_j^b/\partial\varepsilon = \partial f_j^\alpha/\partial\varepsilon$, $\partial f_j^b/\partial\Psi = \partial f_j^\alpha/\partial\Psi$, and the integration over $d\varepsilon$ in Eqs. 297 and 304 is the same as in Eq. 279. Then

$$A_b = \frac{4}{\sqrt{\pi}}\left[\mathrm{erf}\left(\frac{\chi_0+3}{\delta\chi_0}\right) - \mathrm{erf}\left(\frac{\chi_0-3}{\delta\chi_0}\right)\right]^{-1} A_\alpha \tag{318}$$

and

$$f_j^b(\varepsilon,\Psi,\chi) = n_j\frac{3}{8\sqrt{2}\pi}\left(\ln\left(1+\hat{\varepsilon}_c^{-\frac{3}{2}}\right)\right)^{-1}\left(\frac{m_j}{\varepsilon_b}\right)^{\frac{3}{2}}\frac{1}{\hat{\varepsilon}^{\frac{3}{2}}+\hat{\varepsilon}_c^{\frac{3}{2}}}4\left[\mathrm{erf}\left(\frac{\chi_0+3}{\delta\chi_0}\right) - \mathrm{erf}\left(\frac{\chi_0-3}{\delta\chi_0}\right)\right]^{-1}$$
$$\times\frac{1}{\sqrt{\pi}\delta\chi_0}\left[e^{-(\chi-\chi_0)^2/\delta\chi_0^2} + e^{-(\chi+2+\chi_0)^2/\delta\chi_0^2} + e^{-(\chi-2+\chi_0)^2/\delta\chi_0^2}\right],\quad 0\le\varepsilon\le\varepsilon_b, -1\le\chi\le 1. \tag{319}$$

Note that if $\chi_0 = 0$, and we take the limit of $\delta\chi_0 \to \infty$,

$$\lim_{\delta\chi_0\to\infty}\left(\frac{4}{\sqrt{\pi}}\left[2\mathrm{erf}\left(\frac{3}{\delta\chi_0}\right)\right]^{-1}\frac{1}{\delta\chi_0}\left[e^{-\chi^2/\delta\chi_0^2} + e^{-(\chi+2)^2/\delta\chi_0^2} + e^{-(\chi-2)^2/\delta\chi_0^2}\right]\right) = \lim_{\delta\chi_0\to\infty}\left(\frac{2}{\sqrt{\pi}\delta\chi_0\mathrm{erf}\left(3/\delta\chi_0\right)}\right) = 1. \tag{320}$$

So that with $\chi_0 = 0$, and $\delta\chi_0 \to \infty$, we recover $f_j^b = f_j^\alpha$ (actually if $\delta\chi_0 \to \infty$ it is not really necessary for χ_0 to be zero to represent an isotropic distribution).

Returning to Eq. 319, Eq. 229 for δW_K can now be written as

$$\delta W_K^b = \sum_j\sum_{l=-\infty}^{\infty}2\sqrt{2}\pi^2\int\int\int\frac{\hat{\tau}}{m_j^{\frac{3}{2}}B}|\chi|\varepsilon^{\frac{1}{2}}d\varepsilon d\chi d\Psi$$
$$\left[|\langle HT_j\rangle|^2\frac{\frac{\varepsilon_b n}{Z_j e}\left(\frac{1}{\hat{\varepsilon}^{\frac{3}{2}}+\hat{\varepsilon}_c^{\frac{3}{2}}}\frac{d\hat{\varepsilon}_c^{\frac{3}{2}}}{d\Psi} - \frac{1}{A_b}\frac{dA_b}{d\Psi} - \frac{1}{n_j}\frac{dn_j}{d\Psi}\right) + \frac{3}{2}\frac{\hat{\varepsilon}^{\frac{1}{2}}}{\hat{\varepsilon}^{\frac{3}{2}}+\hat{\varepsilon}_c^{\frac{3}{2}}}(n\omega_E-\omega)}{n\langle\omega_D^j\rangle + l\omega_b^j - i\nu_{\mathrm{eff}}^j + n\omega_E - \omega}\frac{f_j^b}{\varepsilon_b}\right] \tag{321}$$

$$= \sum_j\sum_{l=-\infty}^{\infty}2\sqrt{2}\pi^2\int\int\int n_j\varepsilon_b A_b\frac{\hat{\tau}}{B}|\chi|\frac{1}{\delta\chi_0}\left[e^{-(\chi-\chi_0)^2/\delta\chi_0^2} + e^{-(\chi+2+\chi_0)^2/\delta\chi_0^2} + e^{-(\chi-2+\chi_0)^2/\delta\chi_0^2}\right]\frac{\hat{\varepsilon}^{\frac{5}{2}}}{\hat{\varepsilon}^{\frac{3}{2}}+\hat{\varepsilon}_c^{\frac{3}{2}}}d\hat{\varepsilon}d\chi d\Psi$$
$$\left[|\langle H/\hat{\varepsilon}\rangle|^2\frac{\frac{\varepsilon_b n}{Z_j e}\left(\frac{1}{\hat{\varepsilon}^{\frac{3}{2}}+\hat{\varepsilon}_c^{\frac{3}{2}}}\frac{d\hat{\varepsilon}_c^{\frac{3}{2}}}{d\Psi} - \frac{1}{A_b}\frac{dA_b}{d\Psi} - \frac{1}{n_j}\frac{dn_j}{d\Psi}\right) + \frac{3}{2}\frac{\hat{\varepsilon}^{\frac{1}{2}}}{\hat{\varepsilon}^{\frac{3}{2}}+\hat{\varepsilon}_c^{\frac{3}{2}}}(n\omega_E-\omega)}{n\langle\omega_D^j\rangle + l\omega_b^j - i\nu_{\mathrm{eff}}^j + n\omega_E - \omega}\right]. \tag{322}$$

For the electrostatic term:

$$\delta W_\Phi^b = \sum_j 2\sqrt{2}\pi^2\int\int\int\left[-(Z_j e)^2\left|\tilde{\mathbf{\Phi}} + \boldsymbol{\xi}_\perp\cdot\nabla\Phi_0\right|^2\frac{f_j^b}{\varepsilon_b}\frac{3}{2}\frac{\hat{\varepsilon}^{\frac{1}{2}}}{\hat{\varepsilon}^{\frac{3}{2}}+\hat{\varepsilon}_c^{\frac{3}{2}}}\right]\frac{\hat{\tau}}{m_j^{\frac{3}{2}}B}|\chi|\varepsilon^{\frac{1}{2}}d\varepsilon d\chi d\Psi \tag{323}$$

$$= -\sum_j 3\sqrt{2}\pi^2\int\int\int\left(\frac{Z_j e}{\varepsilon_b}\right)^2\left|\tilde{\mathbf{\Phi}} + \boldsymbol{\xi}_\perp\cdot\nabla\Phi_0\right|^2 n_j\varepsilon_b A_b\frac{\hat{\tau}}{B}|\chi|$$
$$\frac{1}{\delta\chi_0}\left[e^{-(\chi-\chi_0)^2/\delta\chi_0^2} + e^{-(\chi+2+\chi_0)^2/\delta\chi_0^2} + e^{-(\chi-2+\chi_0)^2/\delta\chi_0^2}\right]\left(\frac{\hat{\varepsilon}^{\frac{1}{2}}}{\hat{\varepsilon}^{\frac{3}{2}}+\hat{\varepsilon}_c^{\frac{3}{2}}}\right)^2 d\hat{\varepsilon}d\chi d\Psi. \tag{324}$$

And the anisotropy term:
(Redo)

$$\delta W^b_{\tilde{B}_\parallel} = \sum_j 2\sqrt{2}\pi^2 \int\int\int \frac{\hat{\tau}}{m_j^{\frac{3}{2}} B} |\chi| \varepsilon^{\frac{1}{2}} \langle HT_j \rangle^* \frac{\tilde{\mathbf{B}}_\parallel}{B} \frac{\chi^2 - 1}{2|\chi|} \frac{\partial f_j^b}{\partial \chi} d\varepsilon d\chi d\mathbf{\Psi} \tag{325}$$

$$= \sum_j 2\sqrt{2}\pi^2 \int\int\int n_j A_b \left(\frac{m_j}{\varepsilon_b}\right)^{\frac{3}{2}} \frac{\hat{\tau}}{m_j^{\frac{3}{2}} B} \varepsilon^{\frac{1}{2}} \frac{1}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}} d\varepsilon d\chi d\mathbf{\Psi} \frac{1}{\delta\chi} \left(-\frac{2}{\delta\chi^2}\right) \langle HT_j \rangle^* \frac{\tilde{\mathbf{B}}_\parallel}{B} \frac{\chi^2 - 1}{2}$$
$$\left[(\chi - \chi_0)\, e^{-(\chi-\chi_0)^2/\delta\chi^2} + (\chi + 2 + \chi_0)\, e^{-(\chi+2+\chi_0)^2/\delta\chi^2} + (\chi - 2 + \chi_0)\, e^{-(\chi-2+\chi_0)^2/\delta\chi^2}\right] \tag{326}$$

$$= -\sum_j 2\sqrt{2}\pi^2 \int\int\int n_j \varepsilon_b A_b \frac{\hat{\tau}}{B} \frac{\hat{\varepsilon}^{\frac{3}{2}}}{\hat{\varepsilon}^{\frac{3}{2}} + \hat{\varepsilon}_c^{\frac{3}{2}}} d\hat{\varepsilon} d\chi d\mathbf{\Psi} \langle H/\hat{\varepsilon} \rangle^* \frac{\tilde{\mathbf{B}}_\parallel}{B} \frac{\chi^2 - 1}{\delta\chi^3}$$
$$\left[(\chi - \chi_0)\, e^{-(\chi-\chi_0)^2/\delta\chi^2} + (\chi + 2 + \chi_0)\, e^{-(\chi+2+\chi_0)^2/\delta\chi^2} + (\chi - 2 + \chi_0)\, e^{-(\chi-2+\chi_0)^2/\delta\chi^2}\right] \tag{327}$$

Note that when $\chi_0 = 0$, $\delta W^b_{\tilde{B}_\parallel} = 0$ because the $\langle H/\hat{\varepsilon} \rangle^*$ term and the $\chi^2 - 1$ term are even in χ, while the $\partial f/\partial \chi$ term is odd (the derivative of a Gaussian is positive on one side and negative on the other), making the integrand odd, and the integral from $\chi = -1$ to 1 equal to zero.

2. The general case

Let us now return to the general case, in which the Gaussian spread of beam ions in pitch angle has an energy dependence. Equations 229 and **??** must now be written using the more complicated versions of $\partial f_j/\partial \varepsilon$ and $\partial f_j \partial \Psi$ from Eqs. 311 and 315. The equation for $\partial f_j/\partial \chi$ is the same, from Eq. 317, so that $\delta W_{\tilde{B}_\parallel}$ is the same as in Eq. 327, but it now has the added complication in performing the integrations that χ_0 is a function of Ψ and $\delta\chi$ is a function of ε and Ψ.
(More)

X. DISCUSSION AND CONCLUSION

Theory shows that the stability of the resistive wall mode in tokamak fusion devices depends upon kinetic effects. Calculation of the complex mode frequency, ω, can either be performed by the solution of a set of self consistent equations, as was outlined in Sec. II, or through the perturbative approach of calculating changes in potential energy (δW) due to various effects and inputting these into a dispersion relation for ω. Here we have derived general forms for the δW terms that are independent of the particle distribution function that is being considered.

We find that δW can be neatly divided into four parts: δW_K, δW_Φ, δW_F, and δW_A.

In reality, only δW_K is a kinetic term that depends upon the frequency resonance fraction, while δW_Φ, δW_F, and δW_A are fluid terms, which are strictly real.

The fact that the electrostatic term is strictly real is evident in Eq. 230 through the $\left|\tilde{\mathbf{\Phi}} + \boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla}\Phi_0\right|^2$ term and the lack of any poles in the integration. Also, from Eqs. 247, 270, 291, and 324, the electrostatic term is negative definite for the four specific distribution functions considered here. Therefore it is generally destabilizing (see the discussion at the end of Sec. IV).

As we have seen, δW_A is zero for distribution functions that are isotropic with respect to pitch angle. Only for the bi-Maxwellian distribution for thermal particles and the anisotropic slowing down distribution for energetic particles is this term non-zero. One can also see that it is strictly real through the

Discuss the relative magnitudes of the four terms and δI.

Discuss the magnitudes in the context of the different distribution functions. For example, rotation dependence in the H term for Maxwellian, but not really for energetic species, because the frequencies are much higher.

Conclusion: Basically a summary of what we've done here.

ACKNOWLEDGMENTS

The authors would like to thank Bo Hu for the generous use of his notes, Holger Remierdes and Ming Chu for their suggestion to consider anisotropic energetic particles, Mario Podesta, Nikolai Gorelenkov and Brian Grierson for useful discussions on the anisotropic slowing down distribution function for beam ions, Greg Hammett for suggesting the full Krook operator in subsection G 1 and Luca Guazzotto, Janardhan Manickam, Jong-Kyu Park, Grigory Kagan, Zhirui Wang, and Yueqiang Liu for many useful discussions over the years.

This research was supported by the U.S. Department of Energy under grants DE-FG02-99ER54524 (Columbia University) and DE-FG02-93ER54215 (University of Rochester). And?

Appendix A: Radial Force Balance

A radial force balance can be obtained from the generalized Ohm's law, with no radial current or $\mathbf{j} \times \mathbf{B}$ force[8,73,74]:

$$E_r = \frac{\nabla p}{Z_j e n_j} + v_\phi B_\theta - v_\theta B_\phi. \tag{A1}$$

Which leads to the following equation after dividing through by RB_θ:

$$\omega_E = \Omega_j - \omega_{*j} - \frac{v_\theta^D}{R}\frac{B_\phi}{B_\theta}, \tag{A2}$$

where each term is in units of [rad/s], and:

$$\omega_{*j} = -\frac{1}{Z_j e n_j}\frac{d\left(n_j T_j\right)}{d\Psi}. \tag{A3}$$

Note that ω_E is the same, independent of the species, since all particles feel the same electric and magnetic fields. Therefore the balance between the toroidal rotation, diamagnetic frequency, and poloidal rotation terms must be the same for all particles in radial force balance. Poloidal rotation is often neglected, as it is generally small[74]. For thermal particles each term is on the same order, and $\Omega_j = \omega_\phi$, the plasma toroidal rotation. Look at Ref. [75] In contrast, one can see that for energetic particles it is easy to achieve $\omega_{*j} \gg \omega_E$ because of the high energy and low density of these particles (even if the pressure gradient is similar to thermal particles, the density is typically much less, so that ω_{*j} is large). So, for energetic particles $\Omega_j \approx \omega_{*j} \gg \omega_E,\ \omega_\phi$ is often true.

Appendix B: The Wall Time

The wall time, τ_w, used in Sec. IV for the RWM is given by[49]:

$$\tau_w = \frac{\mu_0 d_w}{\eta_w}\bar{b}, \tag{B1}$$

where η_w is the wall resistivity, d_w is the wall thickness and $\bar{b}$ is

$$\bar{b} = \frac{\int_{S_b}\left|\hat{\mathbf{n}} \times \tilde{\mathbf{A}}_\infty\right|^2 dS}{\int_{S_b}\left(\hat{\mathbf{n}} \times \tilde{\mathbf{A}}_\infty\right)\cdot \hat{\mathbf{n}} \times \left(\hat{\mathbf{n}} \times \boldsymbol{\nabla} \times \tilde{\mathbf{A}}_\infty\right) dS}. \tag{B2}$$

Here S_b is the surface of the wall, $\hat{\mathbf{n}}$ is the vector normal to the surface, $\tilde{\mathbf{A}}_\infty = \left(\boldsymbol{\xi}_\perp \times \mathbf{B}\right)_\infty$ is evaluated as if there were no wall (the wall is placed at ∞), and $\tilde{\mathbf{A}}_\mathbf{b} = \left(\boldsymbol{\xi}_\perp \times \mathbf{B}\right)_b$ is evaluated with the wall at location b.

Note that this definition of the wall time can give a different result from other determinations[63].

Appendix C: Ideal vs. Resistive

In order for the problem to be considered ideal, not resistive, the wall time should be less than the ideal-wall tearing mode growth time[76] $\tau_\eta = \tau_A^{\frac{2}{5}}\tau_R^{\frac{3}{5}}$, where the Alfvén time is $\tau_A = 2\pi R\sqrt{\mu_0 n_i m_i}/B_0$, and the resistive time is $\tau_R = a^2\mu_0/\eta$, with a the local minor radius and resistivity

$$\eta = \frac{\sqrt{2} m_e^{\frac{1}{2}} Z_{\text{eff}} e^2 \ln(\Lambda)}{12\pi^{\frac{3}{2}}\epsilon_0^2 T_e^{\frac{3}{2}}}. \tag{C1}$$

Appendix D: The Number of Roots

The RWM dispersion relation has three roots[5,12,16,51]. To see this, Eq. 76 can be rewritten:

$$(\gamma - i\omega_r)\,\tau_w = -\frac{\delta W_\infty + \delta W_K + \delta I}{\delta W_b + \delta W_K + \delta I} = \frac{1 - C}{\hat{\gamma}_f^{-1} + C}, \tag{D1}$$

or

$$D = (\hat{\gamma} - i\hat{\omega}_r)\left(\hat{\gamma}_f^{-1} + C\right) - 1 + C, \tag{D2}$$

where $\hat{\gamma} = \gamma\tau_w$ and $\hat{\omega}_r = \omega_r\tau_w$. Using a simplified model for δW_K from Ref. [51], where all frequencies are constant with respect to magnetic flux coordinate and pitch angle,

$$C = -\frac{\delta I}{\delta W_\infty} + c\frac{8}{15\sqrt{\pi}}\int_0^\infty \left[\frac{\hat{\omega}_{*N} + \left(\hat{\varepsilon} - \frac{3}{2}\right)\hat{\omega}_{*T} + \hat{\omega}_E - \hat{\omega}_r - i\hat{\gamma}}{\overline{\omega_D}\hat{\varepsilon}^{a_1} - i\overline{\nu}\hat{\varepsilon}^{a_2} + \hat{\omega}_E - \hat{\omega}_r - i\hat{\gamma}} + \frac{-\hat{\omega}_{*N} - \left(\hat{\varepsilon} - \frac{3}{2}\right)\hat{\omega}_{*T} + \hat{\omega}_E - \hat{\omega}_r - i\hat{\gamma}}{-\overline{\omega_D}\hat{\varepsilon}^{a_1} - i\sqrt{2m_i/m_e}\overline{\nu}\hat{\varepsilon}^{a_2} + \hat{\omega}_E - \hat{\omega}_r - i\hat{\gamma}}\right]\hat{\varepsilon}^{\frac{5}{2}}e^{-\hat{\varepsilon}}d\hat{\varepsilon}, \tag{D3}$$

one can see from Eqs. D2 and D3 that in general there are three roots of the RWM kinetic dispersion relation for $(\hat{\omega}_r, \hat{\gamma})$. This is because performing complex division on the ion and electron terms in Eq. D3 results in a quadratic in $(\hat{\omega}_r, \hat{\gamma})$. The nature of these roots was explored in detail in Ref. [51]. Here we examine various terms and circumstances which may or may not change the number of roots.

a. The Inertial Term

For an internal kink mode, with no wall present, the dispersion relation $\delta I = -\delta W_\infty - \delta W_K$ is recovered from Eq. D1 by setting $\tau_w = 0$. In that case, with δW_K quadratic in $(\hat{\omega}_r, \hat{\gamma})$, and with δI also quadratic in $(\hat{\omega}_r, \hat{\gamma})$ (see Eq. 62), there are two roots (the MHD and fishbone "branches") of the internal kink[26].

The plasma inertial term δI is usually neglected for the RWM. It was included in Eqs. D1 and D3 to demonstrate that it does not change the number of RWM roots. This is because the quadratic nature of C is already established by δW_K, so including δI may alter the RWM roots slightly, but it won't add any new roots.

b. Energetic Particles

If a situation is considered in which the energetic particle effects dominate over thermal ions and electrons, and $\delta W_{EP} \sim (\omega/\omega_D)\ln(1 - \omega_D/\omega)$, then the resulting dispersion relation has only two roots[63]. This is a scenario directly related to the fishbone mode. If, however, energetic particles are considered to have a kinetic effect with a similar magnitude and form to the thermal ion and electron terms in Eq. D3 (Refs. 19 and 26), then the number of roots remains three by extension of the arguments above.

c. Collisionless, Small Precession Drift Case

With thermal particles only, the number of roots can be reduced only under a special circumstance[11]: if $|\omega_D|$, $|\nu| \ll |\omega_E|$. Then not only are the denominators in Eq. D3 the same, but the ω_{*N}, ω_{*T}, and $\hat{\varepsilon}$ dependence in the numerator cancels, so that C becomes a real constant ($C \to 2c - \delta I/\delta W_\infty$) and the three roots become one. The rotation frequency of this single root is zero, while the growth rate is determined by Eq. D1. Whether it is stable or unstable would depend only on the magnitude of C compared to unity and $\hat{\gamma}_f$. One could imagine this scenario ($\omega_D, \nu \to 0$) might approximately represent a collisionless plasma with large $E \times B$ frequency from either $\omega_\phi \gg \omega_*$ or $\omega_* \gg \omega_\phi$ (ie large rotation and small density and temperature gradients, or vice versa).

Appendix E: Equilibrium

The plasma equilibrium relation comes from the force balance of Eq. 8:

$$\rho \frac{d\mathbf{v}}{dt} = \mathbf{j} \times \mathbf{B} - \boldsymbol{\nabla} \cdot \mathbb{P}. \tag{E1}$$

In equilibrium the plasma velocity is considered constant, so that

$$\mathbf{j} \times \mathbf{B} = \boldsymbol{\nabla} \cdot \mathbb{P}. \tag{E2}$$

Then using Ampere's Law and the anisotropic pressure tensor from Eq. 35, we have:

$$\frac{1}{\mu_0} \left(\boldsymbol{\nabla} \times \mathbf{B}\right) \times \mathbf{B} = \boldsymbol{\nabla} \cdot \left(p_\parallel \hat{\mathbf{b}}\hat{\mathbf{b}} + p_\perp \left(\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}\right)\right). \tag{E3}$$

Now using Eqs. 177 and 178 for $\boldsymbol{\nabla} \cdot \left(\hat{\mathbf{b}}\hat{\mathbf{b}}\right)$ and $\boldsymbol{\nabla} \cdot \left(\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}\right)$, we have

$$-\boldsymbol{\nabla} \frac{\mathbf{B}^2}{2\mu_0} + \left(\boldsymbol{\kappa} + \hat{\mathbf{b}}\left(\boldsymbol{\nabla} \cdot \hat{\mathbf{b}}\right)\right) \frac{B^2}{\mu_0} = \boldsymbol{\nabla}_\perp p_\perp + \hat{\mathbf{b}} \nabla_\parallel p_\parallel + \left(\boldsymbol{\kappa} + \hat{\mathbf{b}}\left(\boldsymbol{\nabla} \cdot \hat{\mathbf{b}}\right)\right)\left(p_\parallel - p_\perp\right). \tag{E4}$$

How did we get the left hand side, exactly?

In the perpendicular direction, the equilibrium is[29]

$$-\boldsymbol{\nabla}_\perp \frac{\mathbf{B}^2}{2\mu_0} + \boldsymbol{\kappa}\frac{B^2}{\mu_0} = \boldsymbol{\nabla}_\perp p_\perp + \boldsymbol{\kappa}\left(p_\parallel - p_\perp\right), \tag{E5}$$

$$\boldsymbol{\nabla}_\perp \left(\frac{\mathbf{B}^2}{2\mu_0} + p_\perp\right) = \boldsymbol{\kappa}\frac{B^2}{\mu_0}\left(1 - \frac{\mu_0\left(p_\parallel - p_\perp\right)}{B^2}\right) \tag{E6}$$

$$\boldsymbol{\nabla}_\perp \left(\frac{\mathbf{B}^2}{2\mu_0} + \frac{1}{2}\left(p_\parallel + p_\perp\right) - \frac{1}{2}\left(p_\parallel - p_\perp\right)\right) = \boldsymbol{\kappa}\frac{B^2}{\mu_0}\sigma \tag{E7}$$

$$\boldsymbol{\nabla}_\perp \left(\frac{\mathbf{B}^2}{2\mu_0}\sigma + \frac{1}{2}\left(p_\parallel + p_\perp\right)\right) = \boldsymbol{\kappa}\frac{B^2}{\mu_0}\sigma \tag{E8}$$

$$\boldsymbol{\nabla}_\perp \left(\frac{\mathbf{B}^2}{2\mu_0} + p_\sigma\right) = \boldsymbol{\kappa}\frac{B^2}{\mu_0}, \tag{E9}$$

where

$$p_\sigma = \frac{p_\parallel + p_\perp}{2\sigma}, \tag{E10}$$

with the anisotropy parameter σ defined in Eq. 134.

In the isotropic case, the equilibrium relation reduces to

$$\boldsymbol{\nabla}_\perp \left(\frac{\mathbf{B}^2}{2\mu_0} + p\right) = \boldsymbol{\kappa}\frac{B^2}{\mu_0}. \tag{E11}$$

The equilibrium relation in the anisotropic equilibrium pressure case can be written in this same form, if we define a new quantity which is like a corrected magnetic field

$$\mathbf{D} = \mathbf{B}\sqrt{\sigma}, \tag{E12}$$

so that,

$$\boldsymbol{\nabla}_\perp \left(\frac{\mathbf{D}^2}{2\mu_0} + \frac{1}{2}\left(p_\parallel + p_\perp\right)\right) = \boldsymbol{\kappa}\frac{D^2}{\mu_0}. \tag{E13}$$

This is useful because it means that the plasma equilibrium can be considered to first order the isotropic equilibrium, and then having an anisotropic correction of the second order.

Appendix F: The Quasineutrality Condition

In order to find the quantity $\tilde{\mathcal{Z}} = Z_j e\left(\tilde{\mathbf{\Phi}} + \boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla}\Phi_0\right)$ (from Eq. 142) to use in calculating the electrostatic terms δW_Φ, we must assume a quasineutrality condition. Specifically, if $n = n_0 + \tilde{n}e^{-i\omega t - in\phi}$, then

$$\sum_j Z_j n_j = 0 \rightarrow \sum_j Z_j \tilde{n}_j = 0. \tag{F1}$$

The solution for $\sum_j Z_j \tilde{n}_j$ is similar to that for $\tilde{\mathbb{P}}$ in Eq. 138,[29] with $\tilde{f}_j$ from Eq. 139, but with Z_j instead of $m_j\mathbf{vv}$. Therefore, from Eq. 140,

$$\sum_j Z_j \int \left[-\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} f_j + \mathbf{v}\cdot\boldsymbol{\xi}_\perp i m_j \left(\omega\frac{\partial f_j}{\partial \varepsilon} - n\frac{\partial f_j}{\partial P_\phi} + \omega\frac{1}{B}\frac{\partial f_j}{\partial \mu}\right) + i m_j\left(\omega\frac{\partial f_j}{\partial \varepsilon} - n\frac{\partial f_j}{\partial P_\phi}\right)\tilde{s}_j \right.$$
$$\left. -\mu\frac{\partial f_j}{\partial \mu}\left(\frac{1}{B}\left(\tilde{\mathbf{B}}_\parallel + \boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\mathbf{B}\right)\right) - \frac{m_j v_\parallel}{B^2}\frac{\partial f_j}{\partial \mu}\mathbf{v}_\perp\cdot\tilde{\mathbf{B}} + Z_j e\left(\tilde{\mathbf{\Phi}} + \boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\Phi_0\right)\frac{\partial f_j}{\partial \varepsilon}\right] d^3\mathbf{v} = 0. \tag{F2}$$

Now, noting that:

$$\sum_j Z_j \int \boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} f_j d^3\mathbf{v} = (\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla})\cancel{\sum_j Z_j n_j} = 0, \tag{F3}$$

we find

(Redo these equations.)

$$e\left(\tilde{\mathbf{\Phi}} + \boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\Phi_0\right)\sum_j \int Z_j^2 \frac{\partial f_j}{\partial \varepsilon} d^3\mathbf{v} = -\sum_j \sum_{l=-\infty}^{\infty} \int \frac{Z_j H T_j \left(\omega\frac{\partial f_j}{\partial \varepsilon} - n\frac{\partial f_j}{\partial P_\phi}\right)}{n\langle\omega_D^j\rangle + l\omega_b^j - i\nu_{\text{eff}}^j + n\omega_E - \omega} d^3\mathbf{v} + \sum_j \int Z_j \mu \frac{\tilde{B}_\parallel}{B}\frac{\partial f_j}{\partial \mu} d^3\mathbf{v}, \tag{F4}$$

$$e\left(\tilde{\mathbf{\Phi}} + \boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\Phi_0\right) = \left(\sum_j \int Z_j^2 \frac{\partial f_j}{\partial \varepsilon} d^3\mathbf{v}\right)^{-1}\left[-\sum_j \sum_{l=-\infty}^{\infty} \int \frac{Z_j H T_j \left(\omega\frac{\partial f_j}{\partial \varepsilon} - n\frac{\partial f_j}{\partial P_\phi}\right)}{n\langle\omega_D^j\rangle + l\omega_b^j - i\nu_{\text{eff}}^j + n\omega_E - \omega} d^3\mathbf{v} + \sum_j \int Z_j \mu \frac{\tilde{B}_\parallel}{B}\frac{\partial f_j}{\partial \mu} d^3\mathbf{v}\right]. \tag{F5}$$

Note that the quantity we wish to solve for, $e\left(\tilde{\mathbf{\Phi}} + \boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\Phi_0\right)$, appears on both sides of the equation, since it appears in H (see Eq. 204). The above equation is exact, but for simplification we will now make the assumption that the quasineutrality holds between isotropic thermal electrons and ions (i.e. that energetic ions are not important and $\partial f_j/\partial\mu = 0$). Now the quasineutrality condition is really $\tilde{n}_i = \tilde{n}_e$, or $\sum_{i,e} Z_j \int \tilde{f}_j d^3\mathbf{v} = 0$, and

$$e\left(\tilde{\mathbf{\Phi}} + \boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\Phi_0\right) = \left(\sum_{i,e}\int -Z_j^2\frac{f_j}{T_j}d^3\mathbf{v}\right)^{-1}\left[-\sum_{i,e}\sum_{l=-\infty}^{\infty}\int \frac{Z_j H T_j \left(\omega\frac{\partial f_j}{\partial \varepsilon} - n\frac{\partial f_j}{\partial P_\phi}\right)}{n\langle\omega_D^j\rangle + l\omega_b^j - i\nu_{\text{eff}}^j + n\omega_E - \omega}\left(\frac{2\sqrt{2}\pi}{m_j^{\frac{3}{2}}}\right)\varepsilon^{\frac{1}{2}}d\chi d\varepsilon\right] \tag{F6}$$

$$= \left(\frac{n_e}{T_e} + \frac{n_i}{T_i}\right)^{-1}\sum_{i,e}\sum_{l=-\infty}^{\infty}\frac{2\sqrt{2}\pi}{m_j^{\frac{3}{2}}}\int\int \frac{\left(Z_j H' T_j + Z_j^2 e\left(\tilde{\mathbf{\Phi}} + \boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\Phi_0\right)\right)\left(\omega\frac{\partial f_j}{\partial \varepsilon} - n\frac{\partial f_j}{\partial P_\phi}\right)}{n\langle\omega_D^j\rangle + l\omega_b^j - i\nu_{\text{eff}}^j + n\omega_E - \omega}\varepsilon^{\frac{1}{2}}d\chi d\varepsilon. \tag{F7}$$

Note that we have used:

$$\langle H'\rangle = \langle H\rangle - \frac{Z_j e}{T_j}\left(\tilde{\mathbf{\Phi}} + \boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\Phi_0\right), \tag{F8}$$

for the non-electrostatic terms of H, to more explicitly show the nonlinearity of the equation. This results from Eq. 204. Now, using the Maxwellian on the right hand side, and for $n = 1$, we have:

$$e\left(\tilde{\mathbf{\Phi}} + \boldsymbol{\xi}_\perp \cdot \nabla\Phi_0\right) = \left(\frac{n_i}{T_i} + \frac{n_e}{T_e}\right)^{-1} \sum_{i,e} \sum_{l=-\infty}^{\infty} 2\sqrt{2}\pi \int\int \left(Z_j H' T_j + e\left(\tilde{\mathbf{\Phi}} + \boldsymbol{\xi}_\perp \cdot \nabla\Phi_0\right)\right) \lambda_{j,l}^M \frac{f_j}{T_j} m_j^{-\frac{3}{2}} \varepsilon^{\frac{1}{2}} d\chi d\varepsilon \tag{F9}$$

$$= \left(\frac{n_i}{T_i} + \frac{n_e}{T_e}\right)^{-1} \sum_{i,e} \sum_{l=-\infty}^{\infty} \frac{1}{\sqrt{\pi}} \frac{n_j}{T_j} \int\int \left(Z_j H' T_j + e\left(\tilde{\mathbf{\Phi}} + \boldsymbol{\xi}_\perp \cdot \nabla\Phi_0\right)\right) \lambda_{j,l}^M d\chi \hat{\varepsilon}^{\frac{1}{2}} e^{-\hat{\varepsilon}} d\hat{\varepsilon} \tag{F10}$$

$$= \left(\frac{n_i}{T_i} + \frac{n_e}{T_e}\right)^{-1} \sum_{i,e} \sum_{l=-\infty}^{\infty} \frac{1}{\sqrt{\pi}} n_j \int\int \left(Z_j \langle H'/\hat{\varepsilon}\rangle \hat{\varepsilon}^{\frac{3}{2}} + e\left(\tilde{\mathbf{\Phi}} + \boldsymbol{\xi}_\perp \cdot \nabla\Phi_0\right) T_j^{-1} \hat{\varepsilon}^{\frac{1}{2}}\right) \lambda_{j,l}^M d\chi e^{-\hat{\varepsilon}} d\hat{\varepsilon} \tag{F11}$$

$$= \left(\frac{n_i}{T_i} + \frac{n_e}{T_e}\right)^{-1} \sum_{i,e} \sum_{l=-\infty}^{\infty} \frac{1}{\sqrt{\pi}} n_j \int \left(Z_j \langle H'/\hat{\varepsilon}\rangle \int \lambda_{j,l}^M \hat{\varepsilon}^{\frac{3}{2}} e^{-\hat{\varepsilon}} d\hat{\varepsilon} + e\left(\tilde{\mathbf{\Phi}} + \boldsymbol{\xi}_\perp \cdot \nabla\Phi_0\right) T_j^{-1} \int \lambda_{j,l}^M \hat{\varepsilon}^{\frac{1}{2}} e^{-\hat{\varepsilon}} d\hat{\varepsilon}\right) d\chi \tag{F12}$$

This is the same expression as in Ref. [10], Eq. (12), but with Λ integration replaced by χ.

Appendix G: Collisionality

There are several possibilities of increasing complexity that can be used for the collision operator $C(\tilde{f}_j)$ in the drift kinetic equation (Eq. 84)[20]. The simplest form is a Krook operator, where we define an effective collision frequency ν_{eff} so that $C(\tilde{f}_j) = -\nu_{\text{eff}} \tilde{f}_j$. Since the important collisions to consider in Eq. 84 are momentum-transferring collisions, we will consider ν_{ji}, where j is the test particle (electrons, thermal ions, fast ions, or alpha particles), and i denotes the bulk thermal ions. Three simple expressions are: collisionless:

$$\nu_0 = 0, \tag{G1}$$

no energy dependence (SI units)[45]:

$$\nu_1(\Psi) = \frac{\sqrt{2} n_j Z^4 e^4 \ln\Lambda}{12\pi^{\frac{3}{2}} \epsilon_0^2 m_{ji}^{\frac{1}{2}} T_j^{\frac{3}{2}}} \epsilon_r^{-1}, \tag{G2}$$

and simple energy dependence:

$$\nu_2(\varepsilon, \Psi) = \nu_1 \hat{\varepsilon}^{-\frac{3}{2}}. \tag{G3}$$

Here, $m_{ji} = m_j m_i/(m_j + m_i)$, and $\hat{\varepsilon}$ is the particle energy normalized by the representative distribution function energy. Also, the inclusion of the inverse aspect ratio, ϵ_r, makes ν_{eff} the frequency of collisions causing a scattering step on the order of the banana width[77].

As long as we use Eqs. G1-G3, ν_{eff} can be considered constant and the bounce-averaged collision operator can be written, simply:

$$C(\tilde{f}_j) = -\nu_{\text{eff}} \tilde{f}_j. \tag{G4}$$

However, if the collision operator is dependent on the pitch angle χ, a bounce-average must be taken. This case will be dealt with below, in subsection G 2.

Finally, recall that in Sec. V we considered the Vlasov equation for a solution to the collisionless perturbed distribution function, $\tilde{f}_j$. If we now assume a collisional distribution function g_j with the form $g_j = f_j e^{\nu_{\text{eff}} t}$, and substitute for f_j in the drift kinetic equation, Eq. 84, then we find

$$e^{-\nu_{\text{eff}}t}\frac{d\tilde{g}_j}{dt} - \nu_{\text{eff}}e^{-\nu_{\text{eff}}t}\tilde{g}_j + \left[\frac{\tilde{\mathbf{F}}}{m_j}\right]\boldsymbol{\nabla}_v g_j e^{-\nu_{\text{eff}}t} = \nu_{\text{eff}} g_j e^{-\nu_{\text{eff}}t}, \tag{G5}$$

$$\frac{d\tilde{g}_j}{dt} + \left[\frac{\tilde{\mathbf{F}}}{m_j}\right]\boldsymbol{\nabla}_v g_j = 0. \tag{G6}$$

In other words, using the solution for the collisionless $\tilde{f}$ is fine, so long as we add collisionality back into the formulation, in the form of $\tilde{f}e^{\nu_{\text{eff}}t}$, at the appropriate time. This is performed in appendix H.

1. Particle, Momentum, and Energy Conserving Krook Operator

The Krook operator discussed above, $C(\tilde{f}_j) = -\nu_{\text{eff}}\tilde{f}_j$, is simple and easily implemented, but it does not conserve momentum. A more complete form, which results from linearization of the Krook operator, that conserves particles, momentum, and energy could be used. Note that this form would be applicable for like particle collisions (such as ion-ion collisions) which conserve momentum, not for ion-electron collisions, which can transfer momentum between the species. In Refs. [78] and [79] such a form is written, but it assumes a non-energy dependent collision frequency and a Maxwellian distribution of particles:

$$C(\tilde{f}_j) = -\nu_{\text{eff}}\tilde{f}_j + \nu_{\text{eff}} f_j\left[\frac{\tilde{n}_j}{n_j} + \frac{m_j u_\parallel v_\parallel}{T_j} + \frac{\tilde{T}_j}{T_j}\left(\hat{\varepsilon} - \frac{3}{2}\right)\right]. \tag{G7}$$

Here, $u_\parallel$ is... Can this be generalized, and used?[80,81]

2. Lorentz Collisionality

A potentially more accurate way of including collisionality is through a Lorentz operator with complex energy dependence, and a pitch-angle dependence[82,83]:

$$\nu_3(\varepsilon,\chi,\Psi) = \frac{1}{2}\nu_2\epsilon_r\left[Z_{\text{eff}} + \frac{1}{\sqrt{\pi\hat{\varepsilon}}}e^{-\hat{\varepsilon}} + \frac{1}{\sqrt{\pi}}\left(2-\hat{\varepsilon}^{-1}\right)\int_0^{\sqrt{\hat{\varepsilon}}} e^{-t^2}dt\right]\frac{\partial}{\partial\chi}\left(1-\chi^2\right)\frac{\partial}{\partial\chi}. \tag{G8}$$

This method has been used previously by Fu et al.[83] to study the effect of electron collisionality on the stability of toroidicity-inducted Alfvén eigenmodes (TAEs). Note that the inclusion of the inverse aspect ratio in ν_1 and ν_2 has been removed here (the ϵ_r in Eq. G8 cancels with the ϵ_r^{-1} in Eq. G2). Instead, the above expression represents a bounce-averaged pitch angle scattering operator.

If we now define the bracketed quantity in Eq. G8 as Π_ε, and take a bounce-average of the drift kinetic equation,

$$\left\langle\frac{\partial\tilde{f}_j}{\partial t}\right\rangle + \left\langle\left[\frac{\tilde{\mathbf{F}}}{m}\right]\boldsymbol{\nabla}_{\boldsymbol{v}} f_j\right\rangle = -\frac{1}{2}\nu_2\epsilon_r\Pi_\varepsilon\frac{\partial}{\partial\chi}\left(1-\chi^2\right)\frac{\partial\tilde{f}_j}{\partial\chi}, \tag{G9}$$

the result is a differential equation which is best solved by a numerical, Monte Carlo calculation[82] of $\tilde{f}_j$. This approach may be the subject of future studies.

Appendix H: The Integral Over the Unperturbed Orbits

Consider the full form of the perturbation of position of the plasma in time and space:

$$\boldsymbol{\xi}_\perp = \tilde{\boldsymbol{\xi}}_\perp e^{-i\omega t - in\phi} = \sum_m \tilde{\boldsymbol{\xi}}_\perp^m e^{-i\omega t}e^{i(m\theta(t)-n\phi(t))}e^{\nu_{\text{eff}}^j t}, \tag{H1}$$

where we have decomposed $\boldsymbol{\xi}_\perp$ into Fourier harmonics in the poloidal angle θ with poloidal mode number m, and added an explicit collisionality dependence as discussed in appendix G (with $\nu_{\text{eff}}^j = \nu_0$, ν_1, or ν_2). Now, if we are to integrate the above term, then:

$$\int_{-\infty}^{t} \boldsymbol{\xi}_\perp dt' = \int_{-\infty}^{t} \sum_m \tilde{\boldsymbol{\xi}}_\perp^m e^{-i\omega t} e^{i(m\theta(t) - n\phi(t))} e^{\nu_{\text{eff}}^j t} dt'. \tag{H2}$$

Note that $e^{-i\omega t} e^{\nu_{\text{eff}}^j t} = e^{-i\omega_r t} e^{\left(\gamma + \nu_{\text{eff}}^j\right)t}$. Therefore a condition on the convergence of this integral is $\gamma + \nu_{\text{eff}}^j > 0$. Otherwise at $t = -\infty$ the integrand is infinite. With sufficiently high ν_{eff}^j, this is not an issue, even with a small negative γ. But when the plasma is considered to be collisionless, one must take care[20]. If a calculation for $\gamma \leq 0$ is required, analytic continuation of the integral must be performed[84].

Let us now change the integration to $\tau = t' - t$. Then:

$$\int_{-\infty}^{t} \boldsymbol{\xi}_\perp dt' = \sum_m \tilde{\boldsymbol{\xi}}_\perp^m e^{-i\omega t} e^{i(m\theta(t) - n\phi(t))} e^{\nu_{\text{eff}}^j t} \int_{-\infty}^{0} e^{-i\omega\tau} e^{\nu_{\text{eff}}\tau} e^{im(\theta(t') - \theta(t))} e^{-in(\phi(t') - \phi(t))} d\tau \tag{H3}$$

$$= \int_{-\infty}^{0} e^{-i(\omega + i\nu_{\text{eff}}^j)\tau} e^{-in(\phi(t') - \phi(t))} \sum_m \tilde{\boldsymbol{\xi}}_\perp^m e^{im(\theta(t') - \theta(t))} d\tau. \tag{H4}$$

Now we must find expressions for $\theta(t') - \theta(t)$ and $\phi(t') - \phi(t)$ in terms of τ. Let us write expressions for $d\theta/dt$ and $d\phi/dt$.

$$\frac{d\theta}{dt} = \cancel{\frac{\partial\theta}{\partial t}} + \mathbf{v_g} \cdot \boldsymbol{\nabla}\theta, \tag{H5}$$

$$\frac{d\phi}{dt} = \cancel{\frac{\partial\phi}{\partial t}} + \mathbf{v_g} \cdot \boldsymbol{\nabla}\phi, \tag{H6}$$

where $\mathbf{v_g}$ is the guiding center velocity, given by:

$$\mathbf{v_g} = v_\parallel \hat{\mathbf{b}} + \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{v_\parallel^2 \hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \boldsymbol{\nabla})\hat{\mathbf{b}} + v_\perp^2 \hat{\mathbf{b}} \times \boldsymbol{\nabla} B/(2B)}{\omega_c} \tag{H7}$$

$$= v_\parallel \hat{\mathbf{b}} + \mathbf{v_{E\times B}} + \mathbf{v_D}, \tag{H8}$$

with $\omega_c = eB/m_j$ and $\hat{\mathbf{b}} = \mathbf{B}/B$. The precession drift velocity is defined here as the sum of the curvature and ∇B drifts.

Now, let us relate the two parameters by writing:

$$\frac{d\phi}{d\theta} = \frac{v_\parallel \hat{\mathbf{b}} \cdot \boldsymbol{\nabla}\phi + (\mathbf{v_{E\times B}} + \mathbf{v_D}) \cdot \boldsymbol{\nabla}\phi}{v_\parallel \hat{\mathbf{b}} \cdot \boldsymbol{\nabla}\theta + (\mathbf{v_{E\times B}} + \mathbf{v_D}) \cdot \boldsymbol{\nabla}\theta}. \tag{H9}$$

We can now consider the $E \times B$ and drift velocities to be small compared to the parallel velocity. Then this fraction has the form $(x+\epsilon_1)/(y+\epsilon_2)$, where the ϵ terms are small compared to x and y. Then we can write $(x+\epsilon_1)(y-\epsilon_2)/(y^2+\epsilon_2^2)$. Keeping only quantities of first order in ϵ and using

$$q = \frac{\hat{\mathbf{b}} \cdot \boldsymbol{\nabla}\phi}{\hat{\mathbf{b}} \cdot \boldsymbol{\nabla}\theta}, \tag{H10}$$

we can write

$$d\phi = q d\theta + \frac{(\mathbf{v_{E\times B}} + \mathbf{v_D}) \cdot \boldsymbol{\nabla}\phi - q\,(\mathbf{v_{E\times B}} + \mathbf{v_D}) \cdot \boldsymbol{\nabla}\theta}{v_\parallel \hat{\mathbf{b}} \cdot \boldsymbol{\nabla}\theta} d\theta. \tag{H11}$$

Now let us define,

$$\phi(t') - \phi(t) = -\omega_D \tau - \omega_E \tau, \tag{H12}$$

where the $\mathbf{E} \times \mathbf{B}$ frequency is:

$$\omega_E = \frac{1}{\tau} \langle \mathbf{v_E} \cdot \boldsymbol{\nabla} (q\theta - \phi) \rangle = -\frac{1}{\tau} \int_{\theta(t)}^{\theta(t')} \frac{\mathbf{v_E} \cdot (\boldsymbol{\nabla}\phi - q\boldsymbol{\nabla}\theta)}{v_\parallel \hat{\mathbf{b}} \cdot \boldsymbol{\nabla}\theta} d\theta, \tag{H13}$$

and the magnetic precession drift frequency is[11]:

$$\omega_D = \frac{1}{\tau} \langle \mathbf{v_D} \cdot \boldsymbol{\nabla} (q\theta - \phi) \rangle - \frac{1}{\tau} \int_{\theta(t)}^{\theta(t')} q d\theta = -\frac{1}{\tau} \int_{\theta(t)}^{\theta(t')} \frac{\mathbf{v_D} \cdot (\boldsymbol{\nabla}\phi - q\boldsymbol{\nabla}\theta)}{v_\parallel \hat{\mathbf{b}} \cdot \boldsymbol{\nabla}\theta} d\theta - \frac{1}{\tau} \int_{\theta(t)}^{\theta(t')} q d\theta. \tag{H14}$$

Further derivation of ω_D and ω_E from this point are given in appendices J and K. Note that sometimes ω_E is kept together with ω_D[12,13,15](ie. the velocities are not separated in Eq. H8). Here they are separated to make the dependence on ω_E more clear.

So, returning to Eq. H4

$$\int_{-\infty}^{t} \boldsymbol{\xi}_\perp dt' = \int_{-\infty}^{0} e^{-i(\omega + i\nu_{\text{eff}}^j - n\omega_D - n\omega_E)\tau} \sum_m \tilde{\boldsymbol{\xi}}_\perp^m e^{im\left(\theta(t') - \theta(t)\right)} d\tau. \tag{H15}$$

Now, somehow:

$$\sum_m \tilde{\boldsymbol{\xi}}_\perp^m e^{im\left(\theta(t') - \theta(t)\right)} = \boldsymbol{\xi}_\perp \sum_{l=-\infty}^{\infty} e^{il\omega_b \tau}, \tag{H16}$$

for trapped particles. Make general for trapped or circulating particles. Here l is the bounce harmonic and the bounce frequency is given by:

$$\omega_b = . \tag{H17}$$

The bounce frequency is derived further in appendix I. Look carefully at Ref. [12], Eqs. 13 and 14.

Finally,

$$\int_{-\infty}^{t} \boldsymbol{\xi}_\perp dt' = \boldsymbol{\xi}_\perp \sum_l \int_{-\infty}^{0} e^{-i(\omega + i\nu_{\text{eff}}^j - n\omega_D - l\omega_b - n\omega_E)\tau} d\tau \tag{H18}$$

$$= \sum_l \frac{\boldsymbol{\xi}_\perp}{i(n\omega_D + l\omega_b - i\nu_{\text{eff}}^j + n\omega_E - \omega)}. \tag{H19}$$

Appendix I: Bounce Frequency

Starting from Eq. H17 for ω_b, we rewrite:

$$\omega_b = . \tag{I1}$$

$$\omega_b = . \tag{I2}$$

1. Large Aspect Ratio (Cylindrical) Approximation

In the large aspect ratio limit, the particle bounce frequency can be written[11,35,85]:

$$\frac{\omega_b}{\sqrt{2\varepsilon/m_i}} = \frac{\sqrt{2\epsilon_r\Lambda}}{4qR_0}\frac{\pi}{K(k)} \quad \text{(trapped)}, \tag{I3}$$

$$\frac{\omega_b}{\sqrt{2\varepsilon/m_i}} = \frac{\sqrt{1-\Lambda+\epsilon_r\Lambda}}{2qR_0}\frac{\pi}{K(1/k)} \quad \text{(circulating)}. \tag{I4}$$

where $\Lambda = \mu B_0/\varepsilon$, $\epsilon_r = r/R_0$, K is the complete elliptic integral of the first kind, and

$$k = \left[\frac{1-\Lambda+\epsilon_r\Lambda}{2\epsilon_r\Lambda}\right]^{\frac{1}{2}}. \tag{I5}$$

Reference [86] shows the ion bounce frequency calculated from Eq. I2 by `MISK` compared to the large aspect ratio approximation, using an analytical Solov'ev equilibrium also used in Ref. [12].

Appendix J: Magnetic Precession Drift Frequency

Starting from Eq. H14 for ω_D, we rewrite:

$$\omega_D = -\frac{1}{\tau}\int_{\theta(t)}^{\theta(t')}\frac{\mathbf{v_D}\cdot(\boldsymbol{\nabla}\phi - q\boldsymbol{\nabla}\theta)}{v_\parallel\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\theta}d\theta - \frac{1}{\tau}\int_{\theta(t)}^{\theta(t')} q d\theta \tag{J1}$$

$$= -\frac{1}{\tau}\int\frac{1}{v_\parallel}\mathbf{v_D}\cdot(\boldsymbol{\nabla}\phi - q\boldsymbol{\nabla}\theta)\,d\boldsymbol{\ell} - \frac{1}{\tau}\int_{\theta(t)}^{\theta(t')} q d\theta, \tag{J2}$$

where we have used $d\boldsymbol{\ell} = d\theta/(\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\theta)$.

Now let us consider $\mathbf{v_D}\cdot\boldsymbol{\nabla}(q\theta-\phi)$, using $\mathbf{v_D}$ from Eq. H7.

$$\mathbf{v_D}\cdot\boldsymbol{\nabla}(q\theta-\phi) = \frac{1}{\omega_c}\left(v_\parallel^2\hat{\mathbf{b}}\times\boldsymbol{\kappa} + \frac{1}{2}v_\perp^2\frac{1}{B}\hat{\mathbf{b}}\times\nabla\mathbf{B}\right)\cdot\boldsymbol{\nabla}(q\theta-\phi) \tag{J3}$$

$$= -\frac{1}{\omega_c}\left(\hat{\mathbf{b}}\times\boldsymbol{\nabla}(q\theta-\phi)\right)\cdot\left(\frac{1}{2}v_\perp^2\frac{\boldsymbol{\nabla}_\perp B}{B} + v_\parallel^2\boldsymbol{\kappa}\right) \tag{J4}$$

$$= -\frac{1}{\omega_c B}\left(q\mathbf{B}\times\boldsymbol{\nabla}\theta - \mathbf{B}\times\boldsymbol{\nabla}\phi\right)\cdot\left(\frac{1}{2}v_\perp^2\frac{\boldsymbol{\nabla}_\perp B}{B} + v_\parallel^2\boldsymbol{\kappa}\right) \tag{J5}$$

$$= -\frac{1}{\omega_c B}\left(qB_\phi\hat{\mathbf{e}}_\phi\times\boldsymbol{\nabla}\theta - \mathbf{B}_\theta\times\boldsymbol{\nabla}\phi\right)\cdot\left(\frac{1}{2}v_\perp^2\frac{\boldsymbol{\nabla}_\perp \mathbf{B}}{B} + v_\parallel^2\boldsymbol{\kappa}\right). \tag{J6}$$

Now using the definition of q from Eq. H10 and also $\mathbf{B}_\theta = (\boldsymbol{\nabla}\Psi\times\hat{\mathbf{e}}_\phi)/R$, we have

$$\mathbf{v_D}\cdot\boldsymbol{\nabla}(q\theta-\phi) = -\frac{1}{Z_jeB^2}\left(\frac{\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\phi B_\phi\hat{\mathbf{e}}_\phi\times\boldsymbol{\nabla}\theta}{\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\theta} - \frac{1}{R}(\boldsymbol{\nabla}\Psi\times\hat{\mathbf{e}}_\phi)\times\boldsymbol{\nabla}\phi\right)\cdot\left(\frac{1}{2}m_jv_\perp^2\frac{\boldsymbol{\nabla}_\perp B}{B} + m_jv_\parallel^2\boldsymbol{\kappa}\right) \tag{J7}$$

$$= -\frac{1}{Z_jeB^2}\left(\frac{\mathbf{B}_\phi\cdot\boldsymbol{\nabla}\phi B_\phi\hat{\mathbf{e}}_\phi\times\boldsymbol{\nabla}\theta}{\mathbf{B}_\theta\cdot\nabla\boldsymbol{\theta}} - \frac{1}{R}(\boldsymbol{\nabla}\Psi\times\hat{\mathbf{e}}_\phi)\times\boldsymbol{\nabla}\phi\right)\cdot\left(\frac{1}{2}m_jv_\perp^2\frac{\boldsymbol{\nabla}_\perp B}{B} + m_jv_\parallel^2\boldsymbol{\kappa}\right) \tag{J8}$$

$$= -\frac{1}{Z_jeB^2}\left(\frac{B_\phi^2\left(\frac{\hat{\mathbf{e}}_\phi}{R}\right)\hat{\mathbf{e}}_\phi\times\boldsymbol{\nabla}\theta}{|\mathbf{B}_\theta||\boldsymbol{\nabla}\theta|} - \frac{1}{R}(\boldsymbol{\nabla}\Psi\times\hat{\mathbf{e}}_\phi)\times\left(\frac{\hat{\mathbf{e}}_\phi}{R}\right)\right)\cdot\left(\frac{1}{2}m_jv_\perp^2\frac{\boldsymbol{\nabla}_\perp B}{B} + m_jv_\parallel^2\boldsymbol{\kappa}\right), \tag{J9}$$

where we have used $\boldsymbol{\nabla}\phi = \hat{\mathbf{e}}_\phi/R$. Now, since $\boldsymbol{\nabla}\theta/|\boldsymbol{\nabla}\theta| = \mathbf{B}_\theta/|\mathbf{B}_\theta|$, we have

$$\mathbf{v_D}\cdot\boldsymbol{\nabla}\left(q\theta-\phi\right) = -\frac{1}{Z_j e R^2 B^2}\left(\frac{B_\phi^2}{B_\theta^2}R\hat{\mathbf{e}}_\phi\times\mathbf{B}_\theta+\boldsymbol{\nabla}\Psi\right)\cdot\left(\frac{1}{2}m_j v_\perp^2\frac{\boldsymbol{\nabla}_\perp B}{B}+m_j v_\parallel^2\boldsymbol{\kappa}\right) \tag{J10}$$

$$= -\frac{1}{Z_j e R^2 B^2}\left(\frac{B_\phi^2}{B_\theta^2}\hat{\mathbf{e}}_\phi\times\left(\boldsymbol{\nabla}\Psi\times\hat{\mathbf{e}}_\phi\right)+\boldsymbol{\nabla}\Psi\right)\cdot\left(\frac{1}{2}m_j v_\perp^2\frac{\boldsymbol{\nabla}_\perp B}{B}+m_j v_\parallel^2\boldsymbol{\kappa}\right) \tag{J11}$$

$$= -\frac{1}{Z_j e R^2 B_\theta^2}\left(\frac{B_\phi^2+B_\theta^2}{B^2}\right)\boldsymbol{\nabla}\Psi\cdot\left(\frac{1}{2}m_j v_\perp^2\frac{\boldsymbol{\nabla}_\perp B}{B}+m_j v_\parallel^2\boldsymbol{\kappa}\right) \tag{J12}$$

$$= -\frac{1}{Z_j e R^2 B_\theta^2}\boldsymbol{\nabla}\Psi\cdot\left(\frac{1}{2}m_j v_\perp^2\frac{\boldsymbol{\nabla}_\perp B}{B}+m_j v_\parallel^2\boldsymbol{\kappa}\right). \tag{J13}$$

Let us now replace $\boldsymbol{\kappa}$, using Eq. E9, the equilibrium relation, to find

$$\boldsymbol{\kappa} = \frac{1}{B^2}\boldsymbol{\nabla}_\perp\left(\mu_0 p_\sigma+\frac{B^2}{2}\right), \tag{J14}$$

so that

$$\mathbf{v_D}\cdot\boldsymbol{\nabla}\left(q\theta-\phi\right) = -\frac{1}{Z_j e R^2 B_\theta^2}\boldsymbol{\nabla}\Psi\cdot\left(\frac{1}{2}m_j v_\perp^2\frac{\boldsymbol{\nabla}_\perp B}{B}+m_j v_\parallel^2\left(\boldsymbol{\nabla}_\perp\left(\frac{\mu_0 p_\sigma}{B^2}\right)+\frac{1}{B^2}\boldsymbol{\nabla}_\perp\frac{B^2}{2}\right)\right) \tag{J15}$$

$$= -\frac{1}{Z_j e R^2 B_\theta^2}\boldsymbol{\nabla}\Psi\cdot\left(\frac{\boldsymbol{\nabla}_\perp B}{B}\left(\frac{1}{2}m_j v_\perp^2+m_j v_\parallel^2\right)+\frac{\mu_0}{B^2}\frac{dp_\sigma}{d\Psi}\boldsymbol{\nabla}\Psi m_j v_\parallel^2\right) \tag{J16}$$

$$= -\frac{1}{Z_j e R^2 B_\theta^2}\left(\frac{\boldsymbol{\nabla}\Psi\cdot\boldsymbol{\nabla}_\perp B}{B}\left(\mu B+2\left(\varepsilon-\mu B\right)\right)+\frac{\mu_0}{B^2}\frac{dp_\sigma}{d\Psi}|\boldsymbol{\nabla}\Psi|^2 2\left(\varepsilon-\mu B\right)\right). \tag{J17}$$

Now writing in terms of $\Lambda = \mu B_0/\varepsilon$, and with $|\boldsymbol{\nabla}\Psi|^2 = R^2 B_\theta^2$, we can write

$$\mathbf{v_D}\cdot\boldsymbol{\nabla}\left(q\theta-\phi\right) = -\frac{\varepsilon}{Z_j e|\boldsymbol{\nabla}\Psi|^2}\left(\frac{\boldsymbol{\nabla}\Psi\cdot\boldsymbol{\nabla}B}{B}\left(\frac{\Lambda B}{B_0}+2\left(1-\frac{\Lambda B}{B_0}\right)\right)+\frac{2\mu_0}{B^2}\frac{dp_\sigma}{d\Psi}|\boldsymbol{\nabla}\Psi|^2\left(1-\frac{\Lambda B}{B_0}\right)\right) \tag{J18}$$

$$= -\frac{\varepsilon}{Z_j e}\left(\frac{\boldsymbol{\nabla}\Psi\cdot\boldsymbol{\nabla}B}{|\boldsymbol{\nabla}\Psi|^2}\left(\frac{2}{B}-\frac{\Lambda}{B_0}\right)+\frac{2\mu_0}{B}\frac{dp_\sigma}{d\Psi}\left(\frac{1}{B}-\frac{\Lambda}{B_0}\right)\right) \tag{J19}$$

Therefore

$$\omega_D = -\frac{1}{\tau}\frac{\varepsilon}{Z_j e}\int\frac{d\boldsymbol{\ell}}{v_\parallel}\left[\frac{\boldsymbol{\nabla}\Psi\cdot\boldsymbol{\nabla}B}{|\boldsymbol{\nabla}\Psi|^2}\left(\frac{2}{B}-\frac{\Lambda}{B_0}\right)+\frac{2\mu_0}{B}\frac{dp_\sigma}{d\Psi}\left(\frac{1}{B}-\frac{\Lambda}{B_0}\right)\right]-\frac{1}{\tau}\int_{\theta(t)}^{\theta(t')}q\,d\theta \tag{J20}$$

Now let us examine the second term. We note that q is a function of Ψ and θ, $q(\Psi,\theta)$. Let us then write Ψ in terms of the constant of motion, P_ϕ. Then

$$q(\Psi,\theta) = q\left(\frac{P_\phi}{Z_j e}-\frac{m_j R v_\phi}{Z_j e},\theta\right). \tag{J21}$$

Taylor expanding this expression, we find

$$q \approx q\left(\frac{P_\phi}{Z_j e},\theta\right)-\frac{\partial q}{\partial\Psi}\frac{m_j R}{Z_j e}v_\phi. \tag{J22}$$

Then

$$\frac{1}{\tau}\int_{\theta(t)}^{\theta(t')} q d\theta \approx \frac{1}{\tau}\int_{\theta(t)}^{\theta(t')} q\left(\frac{P_\phi}{Z_j e},\theta\right) d\theta - \frac{1}{\tau}\int_{\theta(t)}^{\theta(t')} \frac{\partial q}{\partial \Psi}\frac{m_j R}{Z_j e} v_\phi d\theta. \tag{J23}$$

The first term is zero over a complete bounce because P_ϕ is constant along the particle's motion. The second term is non-zero because v_ϕ change sign between the bounce motion in one direction and the return. Now, finally

$$\omega_D = -\frac{1}{\tau}\frac{\varepsilon}{Z_j e}\int \frac{d\boldsymbol{\ell}}{v_\parallel}\left[\frac{\boldsymbol{\nabla}\Psi\cdot\boldsymbol{\nabla}B}{|\boldsymbol{\nabla}\Psi|^2}\left(\frac{2}{B}-\frac{\Lambda}{B_0}\right)+\frac{2\mu_0}{B}\frac{dp_\sigma}{d\Psi}\left(\frac{1}{B}-\frac{\Lambda}{B_0}\right)\right]+\frac{1}{\tau}\frac{\varepsilon}{Z_j e}\int_{\theta(t)}^{\theta(t')}\frac{\partial q}{\partial\Psi}2R\frac{v_\phi}{v^2}d\theta. \tag{J24}$$

An alternative formula for the bounce-averaged magnetic precession drift frequency (used in MARS-K) comes from Ref. [87], and is given by:

$$\langle \omega_D \rangle = \frac{1}{Z_j e}\frac{\partial \mathcal{J}/\partial\Psi}{\partial\mathcal{J}/\partial\varepsilon}, \tag{J25}$$

where

$$\mathcal{J} = \int m_j v_\parallel dl, \tag{J26}$$

is the equilibrium longitudinal invariant of the particle parallel motion.

1. Implementation in MISK

The MISK code solves ω_D in the form of Eq. J24, with the following definition:

$$\omega_{B1} = -\frac{\boldsymbol{\nabla}\Psi\cdot\boldsymbol{\nabla}B}{|\boldsymbol{\nabla}\Psi|^2} = -\frac{\boldsymbol{\nabla}\Psi\cdot\boldsymbol{\nabla}B^2}{2B|\boldsymbol{\nabla}\Psi|^2}, \tag{J27}$$

so that

$$\omega_D = \frac{1}{\tau v}\frac{\varepsilon}{Z_j e}\int \frac{d\boldsymbol{\ell}}{(v_\parallel/v)}\left[\omega_{B1}\left(\frac{2}{B}-\frac{\Lambda}{B_0}\right)-\frac{2\mu_0}{B}\frac{dp_\sigma}{d\Psi}\left(\frac{1}{B}-\frac{\Lambda}{B_0}\right)\right]+\frac{1}{\tau v}\frac{\varepsilon}{Z_j e}\int_{\theta(t)}^{\theta(t')}\frac{\partial q}{\partial\Psi}2R\frac{v_\phi}{v}d\theta. \tag{J28}$$

In PEST, the magnetic field is defined by[43]:

$$\mathbf{B} = B_0\left[f\boldsymbol{\nabla}\phi\times\boldsymbol{\nabla}\Psi_P|^2 + R_0 g\boldsymbol{\nabla}\phi\right], \tag{J29}$$

and therefore

$$B^2 = \frac{1}{X^2}\left[f^2|\nabla\Psi_P|^2 + R_0^2 g^2\right]. \tag{J30}$$

where Ψ_P is the PEST Ψ, which is defined by $\nabla\Psi_P = 2\pi f\nabla\Psi$.
Using

$$v_\phi = v_\parallel\frac{|\mathbf{B}_\phi|}{|\mathbf{B}|} = v_\parallel\sqrt{\frac{B_\phi^2}{B^2}} = v_\parallel\sqrt{\frac{R_0^2 g^2/X^2}{\left(f^2|\nabla\Psi_P|^2+R_0^2g^2\right)/X^2}}, \tag{J31}$$

$$\omega_D = \frac{1}{\tau v}\frac{\varepsilon}{Z_j e}\int \frac{d\boldsymbol{\ell}}{(v_\parallel/v)}\left[\omega_{B1}\left(\frac{2}{B}-\frac{\Lambda}{B_0}\right)-\frac{2}{B}2\pi\left(\frac{1}{2\pi}\frac{d\left(\mu_0 p_\sigma\right)}{d\Psi}\right)\left(\frac{1}{B}-\frac{\Lambda}{B_0}\right)\right]+\frac{1}{\tau v}\frac{\varepsilon}{Z_j e}\int_{\theta(t)}^{\theta(t')} f\left(\frac{1}{f}\frac{\partial q}{\partial \Psi}\right)\frac{v_\parallel}{v}\frac{2X}{\sqrt{1+\frac{f^2|\Psi_P|^2}{R_0^2 g^2}}}d\theta. \tag{J32}$$

Now we can write ω_{B1} in terms of the `PEST` Ψ,

$$\omega_{B1} = -\frac{2\pi f\boldsymbol{\nabla}\Psi_P}{2B|\boldsymbol{\nabla}\Psi_P|^2}\cdot\boldsymbol{\nabla}\left(\frac{1}{X^2}\left[f^2|\nabla\Psi_P|^2+R_0^2g^2\right]\right) \tag{J33}$$

$$= -\frac{2\pi f\boldsymbol{\nabla}\Psi_P}{2B|\boldsymbol{\nabla}\Psi_P|^2}\left(-\frac{2}{X^3}\left[f^2|\nabla\Psi_P|^2+R_0^2g^2\right]+\frac{1}{X^2}\left[2|\nabla\Psi_P|^2f\nabla f+2R_0^2g\nabla g+f^2\nabla\left(|\nabla\Psi_P|^2\right)\right]\right) \tag{J34}$$

$$= -\frac{2\pi f\boldsymbol{\nabla}\Psi_P}{2B|\boldsymbol{\nabla}\Psi_P|^2}\left(-\frac{2}{X}B^2+\frac{1}{X^2}\left[2|\nabla\Psi_P|^2f\frac{df}{d\Psi_P}\nabla\Psi_P+2R_0^2g\frac{dg}{d\Psi_P}\nabla\Psi_P+f^2\frac{d|\nabla\Psi_P|^2}{d\Psi_P}\nabla\Psi_P+f^2\frac{d|\nabla\Psi_P|^2}{d\theta}\nabla\theta\right]\right) \tag{J35}$$

$$=2\pi f\left(\frac{2B^2\nabla\Psi_P}{2B|\nabla\Psi_P|^2X}-\frac{\nabla\Psi_P}{2B|\nabla\Psi_P|^2X^2}\left[2|\nabla\Psi_P|^2f\frac{df}{d\Psi_P}\nabla\Psi_P+2R_0^2g\frac{dg}{d\Psi_P}\nabla\Psi_P+f^2\frac{d|\nabla\Psi_P|^2}{d\Psi_P}\nabla\Psi_P+f^2\frac{d|\nabla\Psi_P|^2}{d\theta}\nabla\theta\right]\right) \tag{J36}$$

$$=2\pi f\left(\frac{\nabla\Psi_P B}{X|\nabla\Psi_P|^2}-\frac{|\nabla\Psi_P|^2}{BX^2}f\frac{df}{d\Psi_P}-\frac{R_0^2}{BX^2}g\frac{dg}{d\Psi_P}-\frac{f^2}{2BX^2}\frac{d|\nabla\Psi_P|^2}{d\Psi_P}-\frac{f^2\nabla\theta\nabla\Psi_P}{2BX^2|\nabla\Psi_P|^2}\frac{d|\nabla\Psi_P|^2}{d\theta}\right) \tag{J37}$$

$$=2\pi f^2\left(\frac{\nabla\Psi_P B}{Xf|\nabla\Psi_P|^2}-\frac{1}{BfX^2}\left(R_0^2g\frac{dg}{d\Psi_P}+f\frac{df}{d\Psi_P}|\nabla\Psi_P|^2+\frac{1}{2}f^2\frac{d|\nabla\Psi_P|^2}{d\Psi_P}\right)-\frac{1}{B\left(2X^2|\nabla\Psi_P|^2\right)}f\frac{d|\nabla\Psi_P|^2}{d\theta}|\nabla\Psi_P\nabla\theta|\right) \tag{J38}$$

$$=\frac{(2\pi f)^2\left(f\nabla\Psi_P\right)B}{Xf\left(2\pi f|\nabla\Psi_P|^2\right)}-\frac{1}{BfX^2}\left(R_0^2g\left(f\frac{dg}{d\Psi_P}\right)(2\pi f)+f\left(f\frac{df}{d\Psi_P}\right)\left(2\pi f|\nabla\Psi_P|^2\right)+\frac{1}{2}f^2\left(2\pi f\left(f\frac{d|\nabla\Psi_P|^2}{d\Psi_P}\right)\right)\right)$$
$$-\frac{1}{B\left(2X^2\left(2\pi f|\nabla\Psi_P|^2\right)\right)}f\left(2\pi f\left(f\frac{d|\nabla\Psi_P|^2}{d\theta}\right)\right)\left(2\pi f|\nabla\Psi_P\nabla\theta|\right) \tag{J39}$$

2. Large Aspect Ratio (Cylindrical) Approximation

The large aspect ratio precession drift frequency is[11,57,85,88–91]:

$$\frac{\langle\omega_D\rangle}{\varepsilon/e} = \frac{2q\Lambda}{R_0^2\epsilon_r B_0}\left[(2s+1)\frac{E\left(k^2\right)}{K\left(k^2\right)}+2s\left(k^2-1\right)-\frac{1}{2}\right], \tag{J40}$$

where $s=(r/q)(dq/dr)$ is the magnetic shear and E is the complete elliptic integral of the second kind. Reference [86] shows the ion precession drift frequency calculated from Eq. J24 by `MISK` compared to the large aspect ratio approximation, using an analytical Solov'ev equilibrium also used in Ref. [12].

3. Drift Reversal

When ω_D is negative, particles precess in the opposite direction of the plasma current, a phenomenon known as drift reversal.

(More)[88]

Appendix K: $\boldsymbol{E}\times\boldsymbol{B}$ Frequency

Beginning with Eq. H13 for ω_E, we can see that the derivation follows the same pattern as that of ω_D outlined in appendix J. If we write $\mathbf{v_E}$ in the form

$$\mathbf{v_E} = \hat{\mathbf{b}} \times \left(-\frac{\mathbf{E}}{B}\right), \tag{K1}$$

then we can start from Eq. J13 and replace the quantity in parentheses with $-Z_j e B(\mathbf{E}/B)$. Then

$$\omega_E = -\frac{1}{\tau}\int \frac{1}{v_\parallel}\left(-\frac{1}{Z_j e R^2 B_\theta^2}\nabla\mathbf{\Psi}\cdot(-Z_j e\mathbf{E})\right) d\boldsymbol{\ell} \tag{K2}$$

$$= -\frac{1}{\tau}\int \frac{d\boldsymbol{\ell}}{v_\parallel}\left(\frac{\nabla\mathbf{\Psi}\nabla\mathbf{\Phi}}{|\nabla\mathbf{\Psi}|^2}\right) \tag{K3}$$

$$= -\frac{d\Phi}{d\Psi}. \tag{K4}$$

Appendix L: Self-adjointness of the Anisotropic δW_F

XXX

Appendix M: Derivation of the CGL Perturbed Pressures using the Bi-Maxwellian Distribution Function

The perturbed pressures, $\tilde{p}_\perp$ and $\tilde{p}_\parallel$ can be derived in the same manner as $\tilde{p}$ in Eq. 44 in Sec. II, but in a more general way by using double-polytropic laws[92,93] as replacements for the adiabatic equation. Instead of Eq. 38, we will use:

$$\frac{d}{dt}\left(\frac{p_\parallel B^{\gamma_\parallel - 1}}{\rho^{\gamma_\parallel}}\right) = 0, \tag{M1}$$

$$\frac{d}{dt}\left(\frac{p_\perp B^{1-\gamma_\perp}}{\rho}\right) = 0. \tag{M2}$$

The Chew-Golberger-Low (CGL)[94] double-adiabatic equations[33,55,95,96], which are derived from the first and second adiabatic invariants[97] under the assumption of negligible heat flux[98,99] have $\gamma_\parallel = 3$ and $\gamma_\perp = 2$. The isothermal case is recovered with $\gamma_\parallel = 1$, $\gamma_\perp = 1$.

From Eq. M1, we find[93,99]:

$$\frac{\partial p_\parallel}{\partial t} + \mathbf{v}\cdot\nabla p_\parallel = -\left(\gamma_\parallel - 1\right)\frac{p_\parallel}{B}\hat{\mathbf{b}}\cdot\left(\frac{\partial \mathbf{B}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{B}\right) + \gamma_\parallel\frac{p_\parallel}{\rho}\left(\frac{\partial\rho}{\partial t} + \mathbf{v}\cdot\nabla\rho\right) \tag{M3}$$

$$\left(-i\omega + \mathbf{v_0}\cdot\nabla\right)\tilde{p}_\parallel + \tilde{\mathbf{v}}\cdot\nabla p_\parallel = -\left(\gamma_\parallel - 1\right)\frac{p_\parallel}{B}\hat{\mathbf{b}}\cdot\left(\left(-i\omega + \mathbf{v_0}\cdot\nabla\right)\tilde{\mathbf{B}} + \tilde{\mathbf{v}}\cdot\nabla\mathbf{B_0}\right) + \gamma_\parallel\frac{p_\parallel}{\rho}\left(\left(-i\omega + \mathbf{v_0}\cdot\nabla\right)\tilde{\rho} + \tilde{\mathbf{v}}\cdot\nabla\rho_0\right) \tag{M4}$$

$$\left(-i\omega + \mathbf{v_0}\cdot\nabla\right)\left(\tilde{p}_\parallel + \boldsymbol{\xi}_\perp\cdot\nabla p_\parallel\right) = \left(-i\omega + \mathbf{v_0}\cdot\nabla\right)\left[-\left(\gamma_\parallel - 1\right)\frac{p_\parallel}{B}\hat{\mathbf{b}}\cdot\left(\tilde{\mathbf{B}} + \boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B_0}\right) + \gamma_\parallel\frac{p_\parallel}{\rho}\left(\tilde{\rho} + \boldsymbol{\xi}_\perp\cdot\nabla\rho_0\right)\right] \tag{M5}$$

Now using Eqs. 25 for $\tilde{\mathbf{B}}$ and 34 for $\tilde{\rho}$, we have:

$$\tilde{p}_\parallel = -\boldsymbol{\xi}_\perp\cdot\nabla p_\parallel - \left(\gamma_\parallel - 1\right)\frac{p_\parallel}{B}\hat{\mathbf{b}}\cdot\left(-\mathbf{B_0}\left(\nabla\cdot\boldsymbol{\xi}_\perp\right) + \boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B_0}\right) + \gamma_\parallel\frac{p_\parallel}{\rho}\left(-\rho_0\nabla\cdot\boldsymbol{\xi}_\perp\right) \tag{M6}$$

$$= -\boldsymbol{\xi}_\perp\cdot\nabla p_\parallel - \left(\gamma_\parallel - 1\right)p_\parallel\left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp - \nabla\cdot\boldsymbol{\xi}_\perp\right) - \gamma_\parallel p_\parallel\nabla\cdot\boldsymbol{\xi}_\perp \tag{M7}$$

$$= -\boldsymbol{\xi}_\perp\cdot\nabla p_\parallel - p_\parallel\nabla\cdot\boldsymbol{\xi}_\perp + \left(1 - \gamma_\parallel\right)p_\parallel\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp. \tag{M8}$$

Similarly, for $p_\perp$,

$$\tilde{p}_\perp = -\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} p_\perp - (1-\gamma_\perp)\, p_\perp \left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp - \boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp\right) - p_\perp \boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp \tag{M9}$$

$$= -\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} p_\perp - \gamma_\perp p_\perp \boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + (\gamma_\perp - 1)\, p_\perp \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp. \tag{M10}$$

For the CGL case this leads finally to:

$$\tilde{p}_\parallel = -\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} p_\parallel - p_\parallel \boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp - 2p_\parallel \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp, \tag{M11}$$

$$\tilde{p}_\perp = -\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} p_\perp - 2p_\perp \boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + p_\perp \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp, \tag{M12}$$

while for the isothermal case we recover the result of Eq. 44.

These expressions can also be recovered from Eq. 72 (neglecting the electrostatic contribution):

$$\tilde{\mathbb{P}} = \sum_j m_j \int \mathbf{vv}\left(\tilde{f}_j + \frac{\partial f_j}{\partial B}\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla}\mathbf{B}\right) d^3\mathbf{v}, \tag{M13}$$

using a bi-Maxwellian equilibrium distribution function and our form of $\tilde{f}_j$ from Eq. 137, with certain assumptions.

Let us now examine $\tilde{f}_j$ in the limit of large ω (which really pertains to high frequency modes, not the RWM). Then, in Eq. 137, $\omega(\partial f_j/\partial\varepsilon) \gg n(\partial f_j/\partial P_\phi)$, and $\langle s_j\rangle \approx \langle HT_j\rangle/(im_j\omega)$, so that:

$$\tilde{f}_j^{\omega\to\infty} = -\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} f_j - \frac{\partial f_j}{\partial\varepsilon}\left(\langle HT_j\rangle - Z_j e\left(\tilde{\boldsymbol{\Phi}} + \boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla}\Phi_0\right)\right) - \mu\frac{\tilde{\mathbf{B}}_\parallel}{B}\frac{\partial f_j}{\partial\mu}. \tag{M14}$$

Now

$$\tilde{f}_j^{\omega\to\infty} + \frac{\partial f_j}{\partial B}\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla}\mathbf{B} = \tilde{f}_j^{\omega\to\infty} - \frac{\mu}{B}\frac{\partial f_j}{\partial\mu}\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla}\mathbf{B} \tag{M15}$$

$$= -\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} f_j - \frac{\partial f_j}{\partial\varepsilon}\left(\langle HT_j\rangle - Z_j e\left(\tilde{\boldsymbol{\Phi}} + \boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla}\Phi_0\right)\right) - \frac{\mu}{B}\frac{\partial f_j}{\partial\mu}\left(\tilde{\mathbf{B}}_\parallel + \boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla}\mathbf{B}\right) \tag{M16}$$

$$= -\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} f_j - m_j\frac{\partial f_j}{\partial\varepsilon}\left(\frac{1}{2}v_\perp^2 \boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \frac{1}{2}v_\perp^2 \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp - v_\parallel^2 \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right) - \mu\frac{\partial f_j}{\partial\mu}\left(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp + \boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp\right), \tag{M17}$$

where in the last line we have used Eqs. 204 and 141, for $\langle H\rangle$ and $\tilde{\mathbf{B}}_\parallel$.

From Eq. 251, the bi-Maxwellian distribution has the form:

$$f_j^{bM} = n_j\left(\frac{m_j}{2\pi}\right)^{\frac{3}{2}}\frac{1}{T_{j\perp}T_{j\parallel}^{\frac{1}{2}}}e^{-\frac{\varepsilon-\mu B}{T_{j\parallel}}}e^{-\frac{\mu B}{T_{j\perp}}}, \tag{M18}$$

so that

$$\frac{\partial f_j}{\partial\varepsilon} = -\frac{f_j}{T_{j\parallel}}, \tag{M19}$$

and

$$\frac{\partial f_j}{\partial\mu} = -f_j B\left(\frac{1}{T_{j\perp}} - \frac{1}{T_{j\parallel}}\right). \tag{M20}$$

Now we can make substitutions, using Eqs. M19, and M20, and we find that:

$$\tilde{f}_j^{\omega\to\infty} + \frac{\partial f_j}{\partial B}\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla}\mathbf{B} = -\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} f_j + f_j m_j \frac{1}{T_{j\|}}\left(\frac{1}{2}v_\perp^2 \boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp + \frac{1}{2}v_\perp^2 \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp - v_\|^2 \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right)$$

$$+ f_j m_j \left(\frac{1}{T_{j\perp}} - \frac{1}{T_{j\|}}\right)\left(\frac{1}{2}v_\perp^2 \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp + \frac{1}{2}v_\perp^2 \boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp\right) \tag{M21}$$

$$= -\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} f_j + f_j m_j \left[\frac{1}{T_{j\|}}\left(-v_\|^2 \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp\right) + \frac{1}{T_{j\perp}}\left(\frac{1}{2}v_\perp^2 \boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp - \frac{1}{2}v_\perp^2 \boldsymbol{\nabla}\cdot\boldsymbol{\xi}_\perp\right)\right]. \tag{M22}$$

We then define the quantities R_1, R_2, and R_3 as in Ref. [100]:

$$R_1 = \sum_j m_j \int v_\|^4 f_j d^3\mathbf{v} \tag{M23}$$

$$= 2\pi m_j \int\int v_\|^4 v_\perp n_j \left(\frac{m_j}{2\pi}\right)^{\frac{3}{2}} \frac{1}{T_{j\perp}T_{j\|}^{\frac{1}{2}}} e^{-\frac{m_j}{2}\left(\frac{v_\|^2}{T_\|}+\frac{v_\perp^2}{T_\perp}\right)} dv_\| dv_\perp \tag{M24}$$

$$= \frac{m_j^{\frac{5}{2}} n_j}{\sqrt{2\pi}} \frac{1}{T_{j\perp}T_{j\|}^{\frac{1}{2}}} \int_{-\infty}^{\infty} v_\|^4 e^{-\frac{m_j v_\|^2}{2T_{j\|}}} dv_\| \int_0^\infty v_\perp e^{-\frac{m_j v_\perp^2}{2T_{j\perp}}} dv_\perp \tag{M25}$$

$$= \frac{m_j^{\frac{5}{2}} n_j}{\sqrt{2\pi}} \frac{1}{T_{j\perp}T_{j\|}^{\frac{1}{2}}} \frac{3\sqrt{\pi}}{4} \left(\frac{2T_\|}{m_j}\right)^{\frac{5}{2}} \frac{1}{2}\left(\frac{2T_\perp}{m_j}\right) \tag{M26}$$

$$= \frac{3p_\|^2}{\rho}. \tag{M27}$$

$$R_2 = \frac{1}{2}\sum_j m_j \int v_\|^2 v_\perp^2 f_j d^3\mathbf{v} \tag{M28}$$

$$= \frac{m_j^{\frac{5}{2}} n_j}{2\sqrt{2\pi}} \frac{1}{T_{j\perp}T_{j\|}^{\frac{1}{2}}} \int_{-\infty}^{\infty} v_\|^2 e^{-\frac{m_j v_\|^2}{2T_{j\|}}} dv_\| \int_0^\infty v_\perp^3 e^{-\frac{m_j v_\perp^2}{2T_{j\perp}}} dv_\perp \tag{M29}$$

$$= \frac{m_j^{\frac{5}{2}} n_j}{2\sqrt{2\pi}} \frac{1}{T_{j\perp}T_{j\|}^{\frac{1}{2}}} \frac{\sqrt{\pi}}{2} \left(\frac{2T_\|}{m_j}\right)^{\frac{3}{2}} \frac{1}{2}\left(\frac{2T_\perp}{m_j}\right)^2 \tag{M30}$$

$$= \frac{p_\| p_\perp}{\rho}. \tag{M31}$$

$$R_3 = \frac{1}{2}\sum_j m_j \int v_\perp^4 f_j d^3\mathbf{v} \tag{M32}$$

$$= \frac{m_j^{\frac{5}{2}} n_j}{2\sqrt{2\pi}} \frac{1}{T_{j\perp}T_{j\|}^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-\frac{m_j v_\|^2}{2T_{j\|}}} dv_\| \int_0^\infty v_\perp^5 e^{-\frac{m_j v_\perp^2}{2T_{j\perp}}} dv_\perp \tag{M33}$$

$$= \frac{m_j^{\frac{5}{2}} n_j}{2\sqrt{2\pi}} \frac{1}{T_{j\perp}T_{j\|}^{\frac{1}{2}}} \sqrt{\pi} \left(\frac{2T_\|}{m_j}\right)^{\frac{1}{2}} \left(\frac{2T_\perp}{m_j}\right)^3 \tag{M34}$$

$$= \frac{4p_\perp^2}{\rho}. \tag{M35}$$

Now from Eq. M13

$$\tilde{p}_\parallel = \sum_j m_j \int v_\parallel^2 \left(\tilde{f}_j^{\omega\to\infty} + \frac{\partial f_j}{\partial B} \boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} \mathbf{B} \right) d^3\mathbf{v} \tag{M36}$$

$$= -\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} p_\parallel + \sum_j m_j \int v_\parallel^2 f_j m_j \left[\frac{1}{T_{j\parallel}} \left(-v_\parallel^2 \boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp \right) + \frac{1}{T_{j\perp}} \left(\frac{1}{2} v_\perp^2 \boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp - \frac{1}{2} v_\perp^2 \boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp \right) \right] d^3\mathbf{v} \tag{M37}$$

$$= -\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} p_\parallel - \frac{m_j}{T_{j\parallel}} R_1 \boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp + \frac{m_j}{T_{j\perp}} R_2 \boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp - \frac{m_j}{T_{j\perp}} R_2 \boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp \tag{M38}$$

$$= -\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} p_\parallel - 3p_\parallel \boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp + p_\parallel \boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp - p_\parallel \boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp \tag{M39}$$

$$= -\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} p_\parallel - p_\parallel \boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp - 2p_\parallel \boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp, \tag{M40}$$

and

$$\tilde{p}_\perp = \sum_j \frac{1}{2} m_j \int v_\perp^2 \left(\tilde{f}_j^{\omega\to\infty} + \frac{\partial f_j}{\partial B} \boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} \mathbf{B} \right) d^3\mathbf{v} \tag{M41}$$

$$= -\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} p_\perp + \sum_j \frac{1}{2} m_j \int v_\perp^2 f_j m_j \left[\frac{1}{T_{j\parallel}} \left(-v_\parallel^2 \boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp \right) + \frac{1}{T_{j\perp}} \left(\frac{1}{2} v_\perp^2 \boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp - \frac{1}{2} v_\perp^2 \boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp \right) \right] d^3\mathbf{v} \tag{M42}$$

$$= -\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} p_\perp - \frac{m_j}{T_{j\parallel}} R_2 \boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp + \frac{m_j}{2T_{j\perp}} R_3 \boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp - \frac{m_j}{2T_{j\perp}} R_3 \boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp \tag{M43}$$

$$= -\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} p_\perp - p_\perp \boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp + 2p_\perp \boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp - 2p_\perp \boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp \tag{M44}$$

$$= -\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} p_\perp - 2p_\perp \boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp + p_\perp \boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp, \tag{M45}$$

which are the same as Eqs. M11 and M12. Note that when $T_{j\parallel} = T_{j\perp}$ the bi-Maxwellian distribution reduces to a Maxwellian, and $p_\perp = p_\parallel$, but the above expressions for $\tilde{p}_\parallel$ and $\tilde{p}_\perp$ do not equal each other, nor do they reduce to Eq. 44, because of the starting point of Eqs. M1 and M2 instead of Eq. 38.

Appendix N: FLR

Finite orbit widths are captured by `MISHKA`[24?] .

$$\begin{aligned}
\tilde{s}_j = &- \frac{v_\parallel}{\omega_{cj}} \left[\hat{\mathbf{b}} \times \mathbf{v}_\perp \cdot \left(\hat{\mathbf{b}} \cdot \boldsymbol{\nabla} \right) \boldsymbol{\xi}_\perp + \hat{\mathbf{b}} \cdot \left(\hat{\mathbf{b}} \times \mathbf{v}_\perp \right) \cdot \boldsymbol{\nabla} \boldsymbol{\xi}_\perp \right] \\
&+ \frac{1}{4\omega_{cj}} \left[\mathbf{v}_\perp \times \hat{\mathbf{b}} \cdot \mathbf{v}_\perp \cdot \boldsymbol{\nabla} \boldsymbol{\xi}_\perp + \mathbf{v}_\perp \cdot \left(\mathbf{v}_\perp \times \hat{\mathbf{b}} \right) \cdot \boldsymbol{\nabla} \boldsymbol{\xi}_\perp \right] \\
&- \frac{v_\perp^2}{2\omega_{cj}} \hat{\mathbf{b}} \cdot \boldsymbol{\nabla} \times \boldsymbol{\xi}_\perp \\
&+ \int_{-\infty}^{t} dt' \left[\left(\frac{1}{2} v_\perp^2 - v_\parallel^2 \right) \boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp + \frac{1}{2} v_\perp^2 \boldsymbol{\nabla} \cdot \boldsymbol{\xi}_\perp - \frac{Z_j e}{m_j} \left(\tilde{\boldsymbol{\Phi}} + \boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} \Phi_0 \right) \right. \\
&\qquad \left. + \frac{v_\parallel v_\perp^2}{2\omega_{cj}} \boldsymbol{\nabla} \cdot \left[\frac{1}{2} \left(\hat{\mathbf{b}} \cdot \boldsymbol{\nabla} \ln B \right) \left(\hat{\mathbf{b}} \times \boldsymbol{\xi}_\perp \right) - \boldsymbol{\xi}_\perp \times \hat{\mathbf{b}} \cdot \boldsymbol{\nabla} \hat{\mathbf{b}} \right] \right]
\end{aligned}$$

- Write abstract.
- Write discussion and conclusions.
- Complete the derivation of the integral over the unperturbed orbits. Make it general for trapped or circulating particles, and then update the denominators in the rest of the text to reflect that.
- What are the finite Larmor radius terms? Make an appendix about them?
- $\delta I/\delta W$ comparison in Eq. 77. In Ref. [12] Yueqiang writes, "the plasma inertia effect... can often be neglected as long as the resistive wall time is orders of magnitude larger than the Alfvén time." And in Ref. [63] he writes, "the inertia is neglected assuming that the amplitude of the mode eigenvalue γ is much smaller than the Alfvén frequency ω_A."
- What about the extra δW term in Sec. **??**?
- Centrifugal terms.
- Complete drift reversal subsection **??**.
- Write appendix about rational surfaces and consider effective magnetic shear?
- Look at Menard's APS 2010 poster for formulation of radial force balance for rotation terms. Also look at Ron's paper.

[1] E. Strait, T. Taylor, A. Turnbull, J. Ferron, L. Lao, B. Rice, O. Sauter, S. Thompson, and D. Wroblewski, Physical Review Letters **74**, 2483 (1995).
[2] S. A. Sabbagh, A. C. Sontag, J. M. Bialek, D. A. Gates, A. H. Glasser, J. E. Menard, W. Zhu, M. G. Bell, R. E. Bell, A. Bondeson, C. E. Bush, J. D. Callen, M. S. Chu, C. C. Hegna, S. M. Kaye, L. L. Lao, B. P. LeBlanc, Y. Q. Liu, R. Maingi, D. Mueller, K. C. Shaing, D. Stutman, K. Tritz, and C. Zhang, Nuclear Fusion **46**, 635 (2006).
[3] A. Bondeson and D. Ward, Physical Review Letters **72**, 2709 (1994).
[4] R. Betti and J. P. Freidberg, Physical Review Letters **74**, 2949 (1995).
[5] R. Fitzpatrick and A. Y. Aydemir, Nuclear Fusion **36**, 11 (1996).
[6] A. C. Sontag, S. A. Sabbagh, W. Zhu, J. E. Menard, R. E. Bell, J. M. Bialek, M. G. Bell, D. A. Gates, A. H. Glasser, B. P. LeBlanc, K. C. Shaing, D. Stutman, and K. Tritz, Nuclear Fusion **47**, 1005 (2007).
[7] H. Reimerdes, A. M. Garafalo, G. L. Jackson, M. Okabayashi, E. J. Strait, M. S. Chu, Y. In, R. J. La Haye, M. Lanctot, Y. Liu, G. Navratil, W. Solomon, H. Takahashi, and R. Groebner, Physical Review Letters **98**, 055001 (2007).
[8] H. Reimerdes, A. M. Garafalo, M. Okabayashi, E. J. Strait, R. Betti, M. S. Chu, B. Hu, Y. In, G. L. Jackson, R. La Haye, M. Lanctot, Y. Liu, G. Navratil, W. Solomon, H. Takahashi, R. Groebner, and the DIII-D team, Plasma Physics and Controlled Fusion **49**, B349 (2007).
[9] B. Hu and R. Betti, Physical Review Letters **93**, 105002 (2004).
[10] B. Hu, R. Betti, and J. Manickam, Physics of Plasmas **12**, 057301 (2005).
[11] Y. Liu, M. S. Chu, C. G. Gimblett, and R. J. Hastie, Physics of Plasmas **15**, 092505 (2008).
[12] Y. Liu, M. S. Chu, I. T. Chapman, and T. C. Hender, Physics of Plasmas **15**, 112503 (2008).
[13] Y. Liu, M. S. Chu, I. T. Chapman, and T. C. Hender, Nuclear Fusion **49**, 035004 (2009).
[14] I. T. Chapman, V. G. Igochine, J. P. Graves, S. D. Pinches, A. Gude, I. Jenkins, M. Maraschek, and G. Tardini, Nuclear Fusion **49**, 035006 (2009).
[15] I. T. Chapman, C. G. Gimblett, M. P. Gryaznevich, T. C. Hender, D. F. Howell, Y. Q. Liu, and S. D. Pinches, Plasma Physics and Controlled Fusion **51**, 055015 (2009).
[16] Y. Liu, I. T. Chapman, M. S. Chu, H. Reimerdes, F. Villone, R. Albanese, G. Ambrosino, A. M. Garofalo, C. G. Gimblett, R. J. Hastie, T. C. Hender, G. L. Jackson, R. J. La Haye, M. Okabayashi, A. Pironti, A. Portone, G. Rubinacci, and E. J. Strait, Physics of Plasmas **16**, 056113 (2009).
[17] S. A. Sabbagh, J. W. Berkery, R. E. Bell, J. M. Bialek, S. P. Gerhardt, J. E. Menard, R. Betti, D. A. Gates, B. Hu, O. N. Katsuro-Hopkins, B. P. LeBlanc, F. M. Levinton, J. Manickam, K. Tritz, and H. Yuh, Nuclear Fusion **50**, 025020 (2010).
[18] J. W. Berkery, S. A. Sabbagh, R. Betti, B. Hu, R. E. Bell, S. P. Gerhardt, J. Manickam, and K. Tritz, Physical Review Letters **104**, 035003 (2010).
[19] J. W. Berkery, S. A. Sabbagh, H. Reimerdes, R. Betti, B. Hu, R. E. Bell, S. P. Gerhardt, J. Manickam, and M. Podesta, Physics of Plasmas **17**, 082504 (2010).
[20] J. W. Berkery, S. A. Sabbagh, R. Betti, R. E. Bell, S. P. Gerhardt, B. P. LeBlanc, and H. Yuh, Physical Review Letters **106**, 075004 (2011).
[21] G. Z. Hao, Y. Q. Liu, A. K. Wang, H. B. Jiang, G. Lu, H. D. He, and X. M. Qiu, Physics of Plasmas **18**, 032513 (2011).
[22] L. Guazzotto, R. Betti, J. W. Berkery, and S. A. Sabbagh, "Poster: A comprehensive model for the kinetic stability of axisymmetric plasmas," International Sherwood Fusion Theory Conference, Austin, TX, May 2-4 (2011).
[23] H. Reimerdes, J. W. Berkery, M. J. Lanctot, A. M. Garofalo, J. M. Hanson, Y. In, M. Okabayashi, S. A. Sabbagh, and E. J. Strait, Physical Review Letters **106**, 215002 (2011).
[24] I. T. Chapman, C. G. Gimblett, M. P. Gryaznevich, T. C. Hender, D. F. Howell, Y. Q. Liu, and S. D. Pinches, Plasma Physics and Controlled Fusion **53**, 065022 (2011).
[25] E. Frieman and M. Rotenberg, Reviews of Modern Physics **32**, 898 (1960).
[26] B. Hu, R. Betti, and J. Manickam, Physics of Plasmas **13**, 112505 (2006).
[27] J. Freidberg, Reviews of Modern Physics **54**, 801 (1982).
[28] M. S. Chu, J. M. Greene, T. H. Jensen, R. L. Miller, A. Bondeson, R. W. Johnson, and M. E. Mauel, Physics of Plasmas **2**, 2236 (1995).
[29] T. Antonsen and Y. Lee, Physics of Fluids **25**, 132 (1982).
[30] S. Preische, J. Manickam, and J. Johnson, Computer Physics Communications **76**, 318 (1993).
[31] J. Freidberg, *Ideal Magnetohydrodynamics* (Springer, 1987).
[32] R. Betti and J. P. Freidberg, Physics of Fluids B **4**, 1465 (1992).
[33] J. W. Van Dam, Journal of the Korean Physical Society **31**, S93 (1997).
[34] J. Van Dam, M. Rosenbluth, and Y. Lee, Physics of Fluids **25**, 1349 (1982).
[35] A. Bondeson and M. S. Chu, Physics of Plasmas **3**, 3013 (1996).
[36] H. J. de Blank, Transactions of Fusion Science and Technology **53**, 122 (2008).
[37] R. Betti and J. P. Freidberg, Physics of Fluids B **3**, 538 (1991).
[38] S. Jardin, *Computational Methods in Plasma Physics* (CRC Press, 2010).
[39] Y. Liu, M. S. Chu, W. F. Guo, F. Villone, R. Albanese, G. Ambrosino, M. Baruzzo, T. Bolzonella, I. T. Chapman, A. M. Garofalo, C. G. Gimblett, R. J. Hastie, T. C. Hender, G. L. Jackson, R. J. La Haye, M. J. Lanctot, Y. In, G. Marchiori, M. Okabayashi, R. Paccagnella, M. Furno Palumbo, A. Pironti, H. Reimerdes, G. Rubinacci, A. Soppelsa, E. J. Strait, S. Ventre, and D. Yadykin, Plasma Physics and Controlled Fusion **52**, 104002 (2010).
[40] M. S. Chu and M. Okabayashi, Plasma Physics and Controlled Fusion **52**, 123001 (2010).
[41] T. Antonsen, "Theory of fusion plasmas," (Editrice Compositori, Bologna, 1987) Chap. Kinetic Energy Principles, pp. 161–183.
[42] T. Antonsen, B. Lane, and J. Ramos, Physics of Fluids **24**, 1465 (1981).
[43] R. Grimm, J. Greene, and J. Johnson, "Methods in computational physics, vol. 16," Chap. Computation of the Magnetohydrodynamic Spectrum in Axisymmetric Toroidal Confinement Systems.
[44] M. Chance, Physics of Plasmas **4**, 2161 (1997).
[45] J. Wesson, *Tokamaks* (Oxford: Clarendon Press, 2004).
[46] X. Wang, *Effect of Energetic Particles and Plasma Rotation on Ballooning Modes*, Ph.D. thesis, Columbia University (1989).
[47] S. A. Sabbagh, S. M. Kaye, J. Menard, F. Paoletti, M. Bell, R. E. Bell, J. M. Bialek, M. Bitter, E. D. Fredrickson, D. A. Gates, A. H.

Glasser, H. Kugel, L. L. Lao, B. P. LeBlanc, R. Maingi, R. J. Maqueda, E. Mazzucato, D. Mueller, M. Ono, S. F. Paul, M. Peng, C. H. Skinner, D. Stutman, G. A. Wurden, W. Zhu, and the NSTX Research Team, Nuclear Fusion **41**, 1601 (2001).
[48]C. Wahlberg, I. T. Chapman, and J. P. Graves, Physics of Plasmas **16**, 112512 (2009).
[49]S. Haney and J. Freidberg, Physics of Fluids B **1**, 1637 (1989).
[50]A. Glasser and M. Chance, Bulletin of the American Physical Society **42**, 1848 (1997).
[51]J. W. Berkery, R. Betti, and S. A. Sabbagh, Physics of Plasmas **18**, 072501 (2011).
[52]J. Graves, I. Chapman, S. Coda, M. Lennholm, M. Albergante, and M. Jucker, Nature Communications **3**, 624 (2012).
[53]R. Betti and S. Cowley, "Kinetic effects on the resistive wall mode," Tech. Rep. (Institute for Theoretical Physics, U.C. Santa Barbara, 1995).
[54]R. Betti and J. P. Freidberg, "Destabilization of the internal kink by energetic-circulating ions," Tech. Rep. (University of Rochester, 1993).
[55]A. Bondeson and R. Iacono, Physics of Fluids B **1**, 1431 (1989).
[56]W. A. Cooper, J. P. Graves, M. Jucker, and M. Y. Isaev, Physics of Plasmas **13**, 092501 (2006).
[57]M. Jucker, J. P. Graves, G. A. Cooper, and W. A. Cooper, Plasma Physics and Controlled Fusion **50**, 065009 (2008).
[58]D. Liu, W. Heidbrink, M. Podestà, R. Bell, E. Fredrickson, S. Medley, R. Harvey, and E. Ruskov, Plasma Physics and Controlled Fusion **52**, 025006 (2010).
[59]S. Coda, I. Klimanov, S. Albertu, G. Arnoux, P. Blanchard, and A. Fasoli, Plasma Physics and Controlled Fusion **48**, B359 (2006).
[60]I. Gradshteyn and I. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, 2000).
[61]G. Matsunaga, N. Aiba, K. Shinohara, Y. Sakamoto, A. Isayama, M. Takechi, T. Suzuki, N. Oyama, N. Asakura, Y. Kamada, and T. Ozeki, Physical Review Letters **103**, 045001 (2009).
[62]G. Matsunaga, K. Shinohara, N. Aiba, Y. Sakamoto, A. Isayama, N. Asakura, T. Suzuki, M. Takechi, N. Oyama, and H. Urano, Nuclear Fusion **50**, 084003 (2010).
[63]Y. Liu, Nuclear Fusion **50**, 095008 (2010).
[64]A. Polevoi, H. Shirai, and T. Takizuka, "Benchmarking of the NBI block in ASTRA code versus the OFMC calculations," Tech. Rep. (Japan Atomic Energy Research Institute, 1997).
[65]Y. Liu, "Private communication," (2010).
[66]F. Porcelli, Plasma Physics and Controlled Fusion **33**, 1601 (1991).
[67]J. Graves, Physical Review Letters **92**, 185003 (2004).
[68]I. T. Chapman, S. D. Pinches, L. C. Appel, R. J. Hastie, T. C. Hender, S. Saarelma, S. E. Sharapov, I. Voisekhovitch, and J. P. Graves, Physics of Plasmas **14**, 070703 (2007).
[69]N. Gorelenkov, H. Berk, and R. Budny, Nuclear Fusion **45**, 226 (2005).
[70]C. Angioni, A. Pochelon, N. Gorelenkov, K. McClements, O. Sauter, R. Budny, P. de Vries, D. Howell, M. Mantsinen, M. Nave, and S. Sharapov, Plasma Physics and Controlled Fusion **44**, 205 (2002).
[71]H. Berk, W. Horton, M. Rosenbluth, and P. Rutherford, Nuclear Fusion **15**, 819 (1975).
[72]M. G. von Hellermann, W. G. F. Core, J. Frieling, L. Horton, R. Konig, W. Mandl, and H. Summers, Plasma Physics and Controlled Fusion **35**, 799 (1993).
[73]W. Solomon, K. Burrell, R. Andre, L. Baylor, P. Gohil, R. Groebner, C. Holcomb, W. Houlberg, and M. Wade, Physics of Plasmas **13**, 056116 (2006).
[74]R. Bell, R. Andre, S. Kaye, R. Kolesnikovc, B. LeBlanc, G. Rewoldt, W. Wang, and S. Sabbagh, Physics of Plasmas **17**, 082507 (2010).
[75]N. Aiba, J. Shiraishi, and S. Tokuda, Physics of Plasmas **18**, 022503 (2011).
[76]R. Betti, Physics of Plasmas **5**, 3615 (1998).
[77]P. Helander and D. Sigmar, *Collisional Transport in Magnetized Plasmas* (Cambridge University Press, 2001).
[78]M. Beer, *Gyrofluids Models of Turbulent Transport in Tokamaks*, Ph.D. thesis, Princeton University (1995).
[79]M. Beer and G. Hammett, Physics of Plasmas **3**, 4046 (1996).
[80]E. Gross and M. Krook, Physical Review **102** (1956).
[81]T. Stringer and J. Connor, Physics of Fluids **14**, 2177 (1971).
[82]A. H. Boozer and G. Kuo-Petravic, Physics of Fluids **24**, 851 (1981).
[83]G. Fu, C. Cheng, and K. Wong, Physics of Fluids B **5**, 4040 (1993).
[84]M. Brambilla, *Kinetic Theory of Plasma Waves: Homogeneous Plasmas* (Oxford University Press, 1998).
[85]B. B. Kadomtsev and O. P. Pogutse, Soviet Physics JETP **24**, 1172 (1967).
[86]J. W. Berkery, "MDC-2 benchmarking," Tech. Rep. (Columbia University, 2011).
[87]M. Rosenbluth and M. L. Sloan, Physics of Fluids **14**, 1725 (1971).
[88]J. Connor, R. Hastie, and T. Martin, Nuclear Fusion **23**, 1702 (1983).
[89]B. Coppi, S. Migliuolo, F. Pegoraro, and F. Porcelli, Physics of Fluids B **2**, 927 (1990).
[90]Y. Wu, C. Cheng, and R. White, Physics of Plasmas **1**, 3369 (1994).
[91]C. Roach, J. Connor, and S. Janjua, Plasma Physics and Controlled Fusion **37**, 679 (1995).
[92]L. Hau and B. Sonnerup, Geophysical Research Letters **20**, 1763 (1993).
[93]R. Prajapati, G. Soni, and R. Chhajlani, Physics of Plasmas **15**, 062108 (2008).
[94]G. F. Chew, M. L. Goldberger, and F. E. Low, Proceedings of the Royal Society of London, Series A **236**, 112 (1956).
[95]V. Ilgisonis, Physics of Plasmas **3**, 4577 (1996).
[96]I. Grigorev and V. Pastukhov, Plasma Physics Reports **33** (2007).
[97]D. Gurnett and A.Bhattacharjee, *Introduction to Plasma Physics with Space and Laboratory Applications* (Cambridge University Press, 2005).
[98]P. Snyder, G. Hammett, and W. Dorland, Physics of Plasmas **4**, 3974 (1997).
[99]K. M. Ferrière and N. André, Journal of Geophysical Research **107**, 1349 (2002).
[100]H. Hamabata, Journal of Plasma Physics **30**, 291 (1983).