A comprehensive model for the kinetic linear stability of axisymmetric plasmas.

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The Stability of Low-Frequency MHD (Internal Kink and Resistive Wall Mode) Depends on Kinetic Physics, Equilibrium Flows and Pressure Anisotropy

The objective of this work is to develop a comprehensive linear stability theory including:

- Resonant and non-resonant particle contributions.
- Finite-Larmor radius effects at the rational surfaces.
- Finite anisotropy.
- Finite equilibrium flow.

Generalized Collisionless Electromagnetic Stability Model

Our starting equations are:

$$\begin{cases} \left(\widetilde{\mathbf{J}} \times \mathbf{B} + \mathbf{J} \times \widetilde{\mathbf{B}}\right)_\perp - \sum\limits_{j=i,e} \left(\nabla \cdot \widetilde{\mathbf{\Pi}}_j\right)_\perp + i\omega \widetilde{\mathbf{\Gamma}}_{i\perp} = 0 \\ \widetilde{N}_e = Z\widetilde{N}_i. \end{cases}$$

Perpendicular momentum (sum of exact moments of the Vlasov equation) and quasi-neutrality for a set of three equations.
The three unknowns that need to be determined are the three components of the perturbed electric field:

$$\widetilde{\mathbf{E}} \equiv i\omega \tilde{\boldsymbol{\xi}}_\perp \times \mathbf{B} - \nabla \tilde{\Phi}.$$

Notice that $\tilde{\boldsymbol{\xi}}_\perp$ is the magnetic field line displacement:

$$\widetilde{\mathbf{B}} = \nabla \times (\tilde{\boldsymbol{\xi}}_\perp \times \mathbf{B})$$

The Vlasov Equation Determines the Perturbed Distribution Function

The linearized Vlasov equation is used for the perturbed distribution function:

$$\frac{d\tilde{f}}{dt} = -\frac{Ze}{m}\left(\widetilde{\mathbf{E}} + \mathbf{v} \times \widetilde{\mathbf{B}}\right) \cdot \nabla_{\vec{v}} f.$$

Key point: use $\tilde{f}$ to calculate the density $\widetilde{N}$, the mass flux $\widetilde{\mathbf{\Gamma}}$ and the pressure tensor $\widetilde{\mathbf{\Pi}}$ but NOT the current:

$$\widetilde{N} \equiv \int d^3\mathbf{v}\tilde{f}$$

$$\widetilde{\mathbf{\Gamma}} \equiv m_i \int d^3\mathbf{v}(\mathbf{v}\tilde{f})$$

$$\widetilde{\mathbf{\Pi}} \equiv m \int d^3\mathbf{v}(\mathbf{v}\mathbf{v}\tilde{f})$$

Instead:

$$\widetilde{\mathbf{J}} = \nabla \times \widetilde{\mathbf{B}}$$

The Formulation Is Very General

Very few limitations (or constraints) are used in the present work, namely:

1. Electron inertia is neglected.
2. Quasi-neutrality is assumed.
3. Reconnecting modes are not included.

Why Not Use the Parallel Component of the Momentum Equation?

Because it is an identity!

$$\mathbf{J}\times\mathbf{B}-\sum_j\left(\nabla\cdot\mathbf{\Pi}_j-\frac{\partial\mathbf{\Gamma}_j}{\partial t}\right)=0$$

$$\Downarrow$$

$$\begin{cases}\left(\widetilde{\mathbf{J}}\times\mathbf{B}+\mathbf{J}\times\widetilde{\mathbf{B}}\right)_\perp-\sum_{j=i,e}\left(\nabla\cdot\widetilde{\mathbf{\Pi}}_j\right)_\perp+i\omega\widetilde{\mathbf{\Gamma}}_{i\perp}=0\\ \overbrace{\left[\mathbf{b}\cdot\left(\sum_j\nabla\cdot\mathbf{\Pi}_j-\frac{\partial\mathbf{\Gamma}_j}{\partial t}\right)\right]}=0\end{cases}$$

This gives $0=0$ when f from Vlasov's equation is used to calculate $\widetilde{\mathbf{\Pi}}_j$ and $\widetilde{\mathbf{\Gamma}}_i$ into the parallel component of the momentum equation.

PPPL THEORY

Solve the Linearized Vlasov Equation Including Zeroth and First Order Gyrophase-Dependent Terms (to Include FLR Effects)

We solve:

$$\begin{cases} \widetilde{\mathbf{E}} = i\omega\tilde{\boldsymbol{\xi}}_\perp \times \mathbf{B} - \nabla\widetilde{\Phi} \\ i\omega\widetilde{\mathbf{B}} = \nabla \times \widetilde{\mathbf{E}} \end{cases}$$

and

$$\frac{d\tilde{f}}{dt} = \frac{Z_j e}{m_j}\left(\widetilde{\mathbf{E}} + \mathbf{v} \times \widetilde{\mathbf{B}}\right) \cdot \nabla_{\vec{v}} f,$$

with

$$f = f(\sigma, \mathscr{E}, \mu, P_\phi),$$

$$\sigma \equiv \frac{v_\parallel}{|v_\parallel|} \text{ for circulating particles,}$$

$$\mathscr{E} \equiv \frac{1}{2}m_j v^2 + Z_j e\Phi, \qquad P_\phi \equiv Z_j e\psi + m_j R v_\phi$$

$$\mu \equiv \frac{mv_\perp^2}{2B} + \mu_1 \Rightarrow \frac{d\mu}{dt} = 0 \text{ along real particle orbits.}$$

PPPL THEORY

Solution of the Linearized Vlasov Equation

Without any approximation:

$$
\begin{cases}
\tilde{f} = -\tilde{\boldsymbol{\xi}}_\perp \cdot \nabla f + Z_j e \dfrac{\partial f}{\partial \mathscr{E}} \tilde{\zeta} + \dfrac{\partial f}{\partial \mu} \left(\tilde{\boldsymbol{\xi}}_\perp \cdot \nabla \mu - \dfrac{M(\tilde{\boldsymbol{\xi}}_\perp)}{B} \right) \\
\qquad + i m_j \left(\omega \dfrac{\partial f}{\partial \mathscr{E}} - \dfrac{\partial f}{\partial P_\phi} \right) \left[\tilde{\boldsymbol{\xi}}_\perp \cdot \mathbf{v} - \tilde{\eta} \right] \\
\tilde{\zeta} = \tilde{\Phi} + \tilde{\boldsymbol{\xi}}_\perp \cdot \nabla \Phi_0 \\
\tilde{\eta} = \displaystyle\int_{-\infty}^{t} \left[-i\omega \tilde{\boldsymbol{\xi}}_\perp \cdot \mathbf{v} + \mathbf{v} \cdot (\mathbf{v} \cdot \nabla) \tilde{\boldsymbol{\xi}}_\perp - \dfrac{Z_j e}{m_j} \tilde{\zeta} \right] dt'
\end{cases}
$$

$$
M(\tilde{\boldsymbol{\xi}}_\perp) = -i\omega \tilde{\boldsymbol{\xi}}_\perp \cdot \mathbf{v} + \mu \left(\widetilde{B_\parallel} + \frac{v_\parallel \widetilde{\mathbf{B}} \cdot \mathbf{v}_\perp}{B} \right)
$$

The difficulty is in calculating the orbit integral $\tilde{\eta}$.

Use Exact Particle Orbit and Integration by Parts in Calculating $\tilde{\eta}$

For example:

$$\tilde{\eta} \sim \int_{-\infty}^{t} \tilde{\boldsymbol{\xi}}_{\perp} \cdot \mathbf{v} dt' = \int_{-\infty}^{t} (\hat{b} \times \mathbf{v}) \times \hat{b} \cdot \tilde{\boldsymbol{\xi}}_{\perp} dt' = -\int_{-\infty}^{t} \hat{b} \times \tilde{\boldsymbol{\xi}}_{\perp} \cdot (\mathbf{v} \times \hat{b})$$

Particle orbits: $\mathbf{v} \times \hat{b} = \frac{1}{\Omega} \frac{d\mathbf{v}}{dt}$
Integration by parts:

$$\tilde{\eta} \sim \int_{-\infty}^{t} dt' \left[\frac{d}{dt} \left(\frac{\hat{b} \times \tilde{\boldsymbol{\xi}}_{\perp} \cdot \mathbf{v}}{\Omega} \right) - \mathbf{v} \cdot \frac{d}{dt} \left(\frac{\hat{b} \times \tilde{\boldsymbol{\xi}}_{\perp}}{\Omega} \right) \right] =$$

$$\tilde{\eta} \sim \underbrace{\frac{\hat{b} \times \tilde{\boldsymbol{\xi}}_{\perp} \cdot \mathbf{v}}{\Omega}}_{\text{FLR term}} - \int_{-\infty}^{t} \mathbf{v} \cdot (-i\omega + \mathbf{v} \cdot \nabla) \frac{\hat{b} \times \tilde{\boldsymbol{\xi}}_{\perp}}{\Omega} dt'$$

gyrophase dependent out of integral
need only gyrophase average $<>_{\phi}$
(electric field is kept in the actual calculation).

Important Physics Is Given by the Evaluation of the Orbit Integral.

The general solution of the linearized Vlasov equation including zeroth and first order gyrophase-dependent terms is written as

$$\tilde{f} = -\tilde{\boldsymbol{\xi}}_\perp \cdot \nabla f + Z_j e \frac{\partial f}{\partial \mathscr{E}} \tilde{\zeta} + i m_j \left(\omega \frac{\partial f}{\partial \mathscr{E}} - \frac{\partial f}{\partial P_\phi} \right) \left[\tilde{\boldsymbol{\xi}}_\perp \cdot \mathbf{v} - \tilde{\eta} \right] + \frac{\partial f}{\partial \mu} \left(\tilde{\boldsymbol{\xi}}_\perp \cdot \nabla \mu - \frac{M(\tilde{\boldsymbol{\xi}}_\perp)}{B} \right)$$

$$\tilde{\eta} = -\frac{v_\parallel}{\Omega_{cj}} [\hat{b} \times \mathbf{v}_\perp \cdot (\hat{b} \cdot \nabla) \tilde{\boldsymbol{\xi}}_\perp + \hat{b} \cdot (\hat{b} \times \mathbf{v}_\perp) \cdot \nabla \tilde{\boldsymbol{\xi}}_\perp]$$

$$+ \frac{1}{4\Omega_{cj}} [\mathbf{v}_\perp \times \hat{b} \cdot \mathbf{v}_\perp \cdot \nabla \tilde{\boldsymbol{\xi}}_\perp + \mathbf{v}_\perp \cdot (\mathbf{v}_\perp \times \hat{b}) \cdot \nabla \tilde{\boldsymbol{\xi}}_\perp] - \frac{v_\perp^2}{2\Omega_{cj}} \mathbf{B} \cdot \nabla \times \left(\frac{\tilde{\boldsymbol{\xi}}_\perp}{B} \right)$$

$$+ \int_{-\infty}^{t} dt' \left\{ \left(\frac{v_\perp^2}{2} - v_\parallel^2 \right) \boldsymbol{\kappa} \cdot \tilde{\boldsymbol{\xi}}_\perp + \frac{v_\perp^2}{2} \nabla \cdot \tilde{\boldsymbol{\xi}}_\perp - \frac{Z_j e}{m_j} \tilde{\zeta} \right.$$

$$\left. + \frac{v_\parallel v_\perp^2}{2\Omega_{cj}} \nabla \cdot \left[\frac{1}{2} (\hat{b} \cdot \nabla \ln B)(\hat{b} \times \tilde{\boldsymbol{\xi}}_\perp) - \tilde{\boldsymbol{\xi}}_\perp \times \hat{b} \cdot \nabla \hat{b} \right] \right\}$$

FLR effects

large for MHD modes at rational surfaces

PPPL THEORY

The Perturbed Distribution Function Is Used in the Stability Analysis

Before calculating pressure tensor and energy principle, it is convenient to define some quantities:

$$\tilde{s} = \int_{-\infty}^{t} dt' \left\{ \left(\frac{v_\perp^2}{2} - v_\parallel^2 \right) \boldsymbol{\kappa} \cdot \tilde{\boldsymbol{\xi}}_\perp + \frac{v_\perp^2}{2} \nabla \cdot \tilde{\boldsymbol{\xi}}_\perp - \frac{Z_j e}{m_j} \tilde{\zeta} \right\}$$

$$p_0 = \frac{p_\parallel - p_\perp}{2} \qquad\qquad \Delta = 1 - \frac{p_\parallel - p_\perp}{B^2}$$

$$p_\parallel^K = -im^2 \int d^3\mathbf{v} v_\parallel^2 \left(\omega \frac{\partial f}{\partial \mathscr{E}} - n \frac{\partial f}{\partial p_\phi} \right) \tilde{s}$$

$$p_\perp^K = -im^2 \int d^3\mathbf{v} \frac{v_\perp^2}{2} \left(\omega \frac{\partial f}{\partial \mathscr{E}} - n \frac{\partial f}{\partial p_\phi} \right) \tilde{s}$$

Mass Flow from the Perturbed Distribution Function

The perturbed distribution function is used for instance to calculate the mass flux:

$$\widetilde{\boldsymbol{\Gamma}}_\perp = m\int \tilde{f}\mathbf{v}_\perp d^3\mathbf{v} = m_i\int \left[im_i\left(\omega\frac{\partial f}{\partial \mathscr{E}} - \frac{\partial f}{\partial P_\phi}\right) + \frac{\partial f}{\partial \mu}\left(-\frac{\mu v_\parallel}{B}\widetilde{\mathbf{B}}_\perp\cdot\mathbf{v}\right)\right]\mathbf{v}_\perp d^3\mathbf{v}$$

$$= -i\rho(\omega-\omega_*)\tilde{\boldsymbol{\xi}}_\perp + \frac{\widetilde{\mathbf{B}}_\perp}{B}\rho U_\parallel,$$

where $U_\parallel$ is the equilibrium parallel velocity and

$$\omega_* = -\frac{nm}{Ze}\frac{\partial P_\perp(\psi,B)}{\partial\psi}$$

Use $\tilde{f}$ to Compute the Pressure Tensor (does not yet include FLR)

$$\widetilde{\mathbf{\Pi}} = m\int \mathbf{v}\mathbf{v}\tilde{f}\, d^3\mathbf{v} = \widetilde{\mathbf{\Pi}}_f + \widetilde{\mathbf{\Pi}}_\phi + \widetilde{\mathbf{\Pi}}_K + \widetilde{\mathbf{\Pi}}_{FLR}$$

$$\begin{aligned}
\widetilde{\mathbf{\Pi}}_f =& \hat{b}\hat{b}[-\tilde{\boldsymbol{\xi}}_\perp\cdot\nabla p_\parallel - (p_\parallel - p_\perp)(\nabla\cdot\tilde{\boldsymbol{\xi}}_\perp + \boldsymbol{\kappa}\cdot\tilde{\boldsymbol{\xi}}_\perp)] \\
&+ (\overleftrightarrow{\mathbf{I}} - \hat{b}\hat{b})\left[-\tilde{\boldsymbol{\xi}}_\perp\cdot\nabla p_\perp + B^2(\nabla\cdot\tilde{\boldsymbol{\xi}}_\perp + \boldsymbol{\kappa}\cdot\tilde{\boldsymbol{\xi}}_\perp)\frac{\partial}{\partial B}\left(\frac{p_\parallel - p_\perp}{B}\right)\right] \\
&+ (\hat{\widetilde{\mathbf{B}}}_\perp + \widetilde{\mathbf{B}}_\perp\hat{b})\left(\frac{p_\parallel - p_\perp}{B}\right) \\
\widetilde{\mathbf{\Pi}}_\phi =& -Ze(\tilde{\Phi} + \tilde{\boldsymbol{\xi}}_\perp\cdot\nabla\Phi_0)N\overleftrightarrow{\mathbf{I}} \\
\widetilde{\mathbf{\Pi}}_K =& \tilde{p}_\parallel^K\hat{b}\hat{b} + \tilde{p}_\perp^K(\overleftrightarrow{\mathbf{I}} - \hat{b}\hat{b})
\end{aligned}$$

$\widetilde{\mathbf{\Pi}}_{FLR} \rightarrow$ to be calculated

Energy Principle (flow not yet included)

$$\delta W = \int d^3\mathbf{r}\left[\tilde{\boldsymbol{\xi}}_\perp^* \cdot \left(\widetilde{\mathbf{J}} \times \mathbf{B} + \mathbf{J} \times \widetilde{\mathbf{B}}\right) - \sum_{j=i,e} \tilde{\boldsymbol{\xi}}_\perp^* \cdot \nabla \cdot \widetilde{\boldsymbol{\Pi}}\right] =$$

$$= \delta W_f + \delta W_\phi + \delta W_K + \delta W_{FLR}$$

$$\delta W_f = \underbrace{\int d^3\mathbf{r}\left\{-\Delta\left[|\widetilde{\mathbf{B}}_\perp|^2 + \tilde{\boldsymbol{\xi}}_\perp^* \times \widetilde{\mathbf{B}}_\perp \cdot \hat{b} J_\parallel + B^2 \left|\nabla \cdot \tilde{\boldsymbol{\xi}}_\perp + \boldsymbol{\kappa} \cdot \tilde{\boldsymbol{\xi}}_\perp\right|^2\right] -2B\partial_B p_0 \left|\nabla \cdot \tilde{\boldsymbol{\xi}}_\perp + \boldsymbol{\kappa} \cdot \tilde{\boldsymbol{\xi}}_\perp\right|^2 + 2(\tilde{\boldsymbol{\xi}}_\perp \cdot \nabla p_0)\boldsymbol{\kappa} \cdot \tilde{\boldsymbol{\xi}}_\perp\right\}}_{\text{self-adjoint}}$$

$$\delta W_\phi = -\frac{1}{2}\sum_j \int d^3\mathbf{r}\left(\frac{Z_j e^2 N_j}{T_j}|\tilde{\zeta}|^2\right) \qquad \left[\tilde{\zeta} = \tilde{\Phi} + \tilde{\boldsymbol{\xi}}_\perp \cdot \nabla \Phi_0\right]$$

$$\delta W_K = i\sum_j m_j \int d^3\mathbf{r} \int d^3\mathbf{v}\left(\omega\frac{\partial f_j}{\partial \mathscr{E}} - n\frac{\partial f_j}{\partial p_\phi}\right)\left(\tilde{s}\frac{d\tilde{s}^*}{dt}\right)$$

δW_{FLR} $\rightarrow$ to be calculated

Resistive Wall Mode Dispersion Relation

The resistive wall mode dispersion relation is written as:

$$\gamma\tau_w = -\frac{(\delta W_F + \delta W_{vacuum}^{\infty}) + \delta W_\phi + \delta W_K + \delta W_{FLR}}{(\delta W_F + \delta W_{vacuum}^{b}) + \delta W_\phi + \delta W_K + \delta W_{FLR}}.$$

This requires the solution of quasi-neutrality to find $\tilde{\zeta}$:

$$\widetilde{N}_e = Z\widetilde{N}_i$$

Conclusions.

- Once completed, this calculation will represent a very comprehensive formulation for linear stability.
- Equilibrium flow effects and FLR effects are being calculated.
- The calculation of the perturbed electric potential $\tilde{\zeta}$ (from quasi-neutrality) requires an appropriate numerical algorithm.