Torque-consistent 3D Force Balance and Optimization of Non-resonant Fields in Tokamaks

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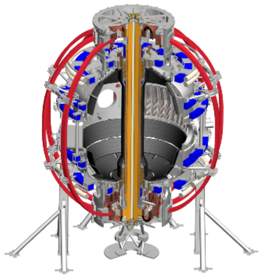


Motivation

- A small non-axisymmetric (3D) magnetic perturbation in tokamaks can significantly modify plasma performance by altering transport and stability

- It is important to control 3D field, for both resonant (RMP) and non-resonant (NRMP) parts

- NRMP can induce substantial level of non-ambipolar transport and $E\times B$ modification
  - As well known by neoclassical toroidal viscosity (NTV) and magnetic braking of toroidal rotation
  - NRNP optimization is critical to control NTV in RMP/EF application, and also rotation control

- NTV evaluation requires 3D equilibrium, but NTV creates currents associated with torque and can eventually modify 3D equilibrium – need self-consistent formulation

- This talk will describe a method of self-consistent NTV calculations and development of general perturbed equilibrium code (GPEC), which solves kinetic Euler-Lagrange equation

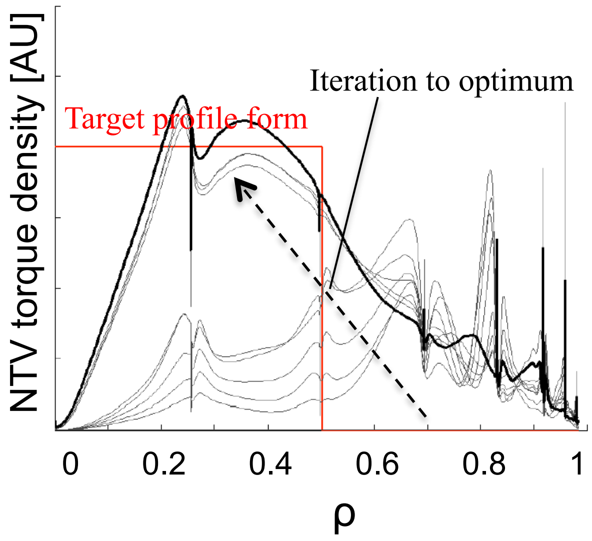
Motivation to improve non-resonant field optimization

- Non-resonant field optimization was actively investigated for Non-axisymmetric control coil (NCC) design in NSTX-U, by adopting advanced stellarator optimizers and IPEC-PENT model for NTV
- However, this smart optimizer even requires up to 100-1000 code runs to approach to a desired solution, and even that solution may be not a global optimum
- Important questions in optimization (given an NTV model):
  - What is the maximum or minimum torque, given a power of field?
  - What are the external fields to generate such an optimal torque?
  - What are the external fields to maximize a local torque, when the total integrated torque is fixed, or under more complicated constraints?

<u>IPECOPT optimization</u>

To maximize core n=1 torque (ψ<0.5), while minimizing others (NSTX-U I_P=1.6MA, β_N=3.1, q_{95}=8.2)

S. Lazerson, PPCF 2015



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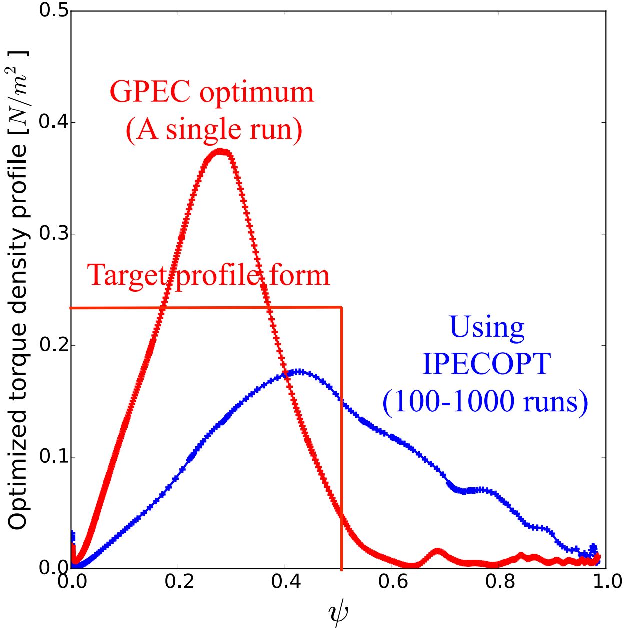
  - What are the external fields to generate such an optimal torque?
  - What are the external fields to maximize a local torque, when the total integrated torque is fixed, or under more complicated constraints
- <u>GPEC provides a systematic way to answer all of these questions by constructing non-Hermitian plasma response matrix including torque response</u>

GPEC vs. IPECOPT NTV optimization
(NSTX-U I_P=2.0MA, β_N=3.4, q_{95}=6.6)



Outline

- Theory and formulation of 3D force balance with anisotropic pressure tensor
- Derivation of modified kinetic Euler-Lagrange (Newcomb) equation

$$\left(F\Xi_\psi{}' + K_R\Xi_\psi\right)' - \left(K_L^\dagger\Xi_\psi{}' + G\Xi_\psi\right) = 0$$

- General perturbed equilibrium code (GPEC) and applications to kinetic energy principle

$$\delta W_{ideal} < \delta W_{KO} < \delta W_{Kinetic} < \delta W_{CGL}$$

- Characteristics of kinetic plasma response and torque
- Torque response matrix and optimization of non-resonant fields

$$\tau_\varphi(\psi) = \Phi^{x\dagger}\mathrm{T}(\psi)\Phi^x$$

- Summary and future work

Force balance with tensor pressure

- A single-fluid description, with quasi-neutrality for small gyro-radius:

$$\vec{\nabla}\cdot\overleftrightarrow{T}=\vec{\nabla}\cdot\overleftrightarrow{P} \quad \text{where} \quad \overleftrightarrow{T}=\vec{B}\vec{B}-B^2\overleftrightarrow{I}/2 \quad \text{and} \quad \overleftrightarrow{P}=(p_{\parallel}-p_{\perp})\hat{b}\hat{b}+p_{\perp}\overleftrightarrow{I}$$

with kinetic approaches: $p_{\parallel}=\int d^3vMv_{\parallel}^2 f$ and $p_{\perp}=\int d^3v\frac{1}{2}Mv_{\perp}^2 f$

- <u>We need to directly solve force balance, since the force operator is not self-adjoint due to the torque</u>

I. Parallel force balance: $\vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{P}=0$

II. Toroidal force balance: $\vec{j}\cdot\vec{\nabla}\psi_P=\frac{\partial\vec{x}}{\partial\varphi}\cdot\vec{\nabla}\cdot\overleftrightarrow{P}$

which implies radial currents associated with toroidal torque

III. Radial force balance: $\vec{\nabla}_{\perp}\left(p_{\perp}+\frac{B^2}{2}\right)=\vec{\kappa}\left(B^2+p_{\perp}-p_{\parallel}\right)$

Force balance with tensor pressure

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- <u>We need to directly solve force balance, since the force operator is not self-adjoint due to the torque</u>

I. Parallel force balance: $\vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{P} = 0$

*Neoclassical parallel viscosity: $\sum_{i,e}\left\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{P}\right\rangle = 0$ K. Shaing, PF 1983

II. Toroidal force balance: $\vec{j}\cdot\vec{\nabla}\psi_P = \frac{\partial\vec{x}}{\partial\varphi}\cdot\vec{\nabla}\cdot\overleftrightarrow{P}$

*Neoclassical Toroidal Viscosity: $q\Gamma_{NA}^{\psi} = \left\langle \frac{\partial\vec{x}}{\partial\tilde{\varphi}}\cdot\vec{\nabla}\cdot\overleftrightarrow{P}\right\rangle$

which implies radial currents associated with toroidal torque

III. Radial force balance: $\vec{\nabla}_\perp\left(p_\perp + \frac{B^2}{2}\right) = \vec{\kappa}\left(B^2 + p_\perp - p_\parallel\right)$

"Perturbed" force balance with tensor pressure to include non-axisymmetric magnetic perturbation

- Perturbed force balance with

$$f = f_0 + \delta f \ \text{ on Unperturbed magnetic coordinates } \vec{x} = (\psi_0, \theta_0, \varphi_0)$$

- "Lagrangian" correction is required in Eulerian Formulation:

$$\delta B_L \sim \delta B(\vec{x}) + \vec{\xi} \cdot \vec{\nabla} B_0(\vec{x})$$

$$\delta f_L = f_0\left(\vec{x} + \vec{\xi}\right) + \delta f\left(\vec{x} + \vec{\xi}\right) - f_0(\vec{x}) \sim \delta f(\vec{x}) + \vec{\xi} \cdot \vec{\nabla} f_0(\vec{x})$$

- Perturbed tensor pressure equilibrium on Eulerian frame:

$$\delta\vec{j} \times \vec{B} + \vec{j} \times \delta\vec{B} + \vec{\nabla} \cdot \left(\vec{\xi} \cdot \vec{\nabla}\vec{P}\right) = \vec{\nabla} \cdot \left(\delta p_\perp \vec{I} + (\delta p_\parallel - \delta p_\perp)\hat{b}\hat{b}\right) + \vec{\nabla} \cdot \left(\frac{\delta\vec{B}_\perp \hat{b} + \hat{b}\delta\vec{B}_\perp}{B}\left(p_\parallel - p_\perp\right) + \left[\delta B_\parallel \frac{\partial}{\partial B} + \delta\Phi \frac{\partial}{\partial \Phi}\right]\left(p_\parallel - p_\perp\right)\right)$$

"Perturbed" force balance with tensor pressure to include non-axisymmetric magnetic perturbation

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- Perturbed tensor pressure equilibrium on Eulerian frame, from Maxwellian: $f_0 = f_M$ then $\vec{j} \times \vec{B} = \vec{\nabla} p$, and

$$\delta \vec{j} \times \vec{B} + \vec{j} \times \delta \vec{B} + \vec{\nabla} \cdot \left(\vec{\xi} \cdot \vec{\nabla} \vec{P}\right) = \vec{\nabla} \cdot \left(\delta p_\perp \vec{I} + (\delta p_\parallel - \delta p_\perp) \hat{b}\hat{b}\right) + \vec{\nabla} \cdot \left(\frac{\delta \vec{B}_\perp \hat{b} + \hat{b} \delta \vec{B}_\perp}{B}\left(p_\parallel - p_\perp\right) + \left[\delta B_\parallel \frac{\partial}{\partial B} + \delta \Phi \frac{\partial}{\partial \Phi}\right]\left(p_\parallel - p_\perp\right)\right)$$

$$\delta \overleftrightarrow{\Pi} \equiv \delta p_\perp \vec{I} + (\delta p_\parallel - \delta p_\perp) \hat{b}\hat{b} \quad \text{where} \quad \delta p_\parallel = \int d^3 v M v_\parallel^2 \delta f_L \quad \text{and} \quad \delta p_\perp = \int d^3 v \frac{1}{2} M v_\perp^2 \delta f_L$$

Parallel, toroidal, radial force balance with non-axisymmetric magnetic perturbation

I. Parallel force balance – Automatically satisfied by orbit averaging

$$\hat{b}\cdot\vec{\nabla}\cdot\delta\vec{\vec{\Pi}}=0,\ \text{which holds for orbit-averaged}\ \delta f_b=\oint\frac{dl}{v_\parallel}\delta f\Big/\oint\frac{dl}{v_\parallel}$$

II. Toroidal force balance – First-order radial currents associated with toroidal torque

$$\chi'\delta\vec{j}^{\psi}=\frac{\partial\vec{x}}{\partial\varphi}\cdot\left(-\vec{j}\times\delta\vec{B}-\vec{\nabla}\left(\vec{\xi}\cdot\vec{\nabla}p\right)+\vec{\nabla}\cdot\delta\vec{\vec{\Pi}}\right)$$

*Note first-order $\left\langle\frac{\partial\vec{x}}{\partial\varphi}\cdot\vec{\nabla}\cdot\delta\vec{\vec{\Pi}}\right\rangle=0$, but second-order $\left\langle\frac{\partial\vec{x}}{\partial\tilde{\varphi}}\cdot\vec{\nabla}\cdot\delta\vec{\vec{\Pi}}\right\rangle$, on perturbed $(\tilde{\psi},\tilde{\theta},\tilde{\varphi})$

III. Radial force balance – First-order pressure-tension force balance

$$\frac{\partial}{\partial\psi}\left(\delta p_\perp-\vec{\xi}\cdot\vec{\nabla}p+\vec{B}\cdot\delta\vec{B}\right)=B^2\delta\kappa_\psi+\left(2\vec{B}\cdot\delta\vec{B}-(\delta p_\parallel-\delta p_\perp)\right)\kappa_\psi+(\hat{b}\cdot\vec{\nabla})\left(\delta p_\perp+\vec{B}\cdot\delta\vec{B}\right)\hat{b}_\psi$$

Parallel, toroidal, radial force balance with non-axisymmetric magnetic perturbation

$\longrightarrow$ These two equations, with $\delta\vec{j} = \vec{\nabla}\times\delta\vec{B}$ and $\delta\vec{B} = \vec{\nabla}\times\left(\vec{\xi}\times\vec{B}\right)$, determines

$\vec{\xi}\cdot\vec{\nabla}\psi$ and $\vec{\xi}\cdot\vec{\nabla}\alpha$ (where $\alpha \equiv q\theta - \varphi$)

II. Toroidal force balance – First-order radial currents and toroidal torque

$$\chi'\delta\vec{j}^{\psi} = \frac{\partial\vec{x}}{\partial\varphi}\cdot\left(-\vec{j}\times\delta\vec{B} - \vec{\nabla}\left(\vec{\xi}\cdot\vec{\nabla}p\right) + \vec{\nabla}\cdot\delta\vec{\vec{\Pi}}\right)$$

*Note first-order $\left\langle \frac{\partial\vec{x}}{\partial\varphi}\cdot\vec{\nabla}\cdot\delta\vec{\vec{\Pi}} \right\rangle = 0$, but second-order $\left\langle \frac{\partial\vec{x}}{\partial\tilde{\varphi}}\cdot\vec{\nabla}\cdot\delta\vec{\vec{\Pi}} \right\rangle$, on perturbed $(\tilde{\psi},\tilde{\theta},\tilde{\varphi})$

III. Radial force balance – First-order pressure-tension force balance

$$\frac{\partial}{\partial\psi}\left(\delta p_{\perp} - \vec{\xi}\cdot\vec{\nabla}p + \vec{B}\cdot\delta\vec{B}\right) = B^2\delta\kappa_{\psi} + \left(2\vec{B}\cdot\delta\vec{B} - (\delta p_{\parallel} - \delta p_{\perp})\right)\kappa_{\psi} + (\hat{b}\cdot\vec{\nabla})\left(\delta p_{\perp} + \vec{B}\cdot\delta\vec{B}\right)\hat{b}_{\psi}$$

Formulation with orbit-averaged distribution function and bounce-Harmonic Fourier representation

- Fourier representation of displacement and orbit-averaged perturbed distribution function:

$$\xi(\psi,\theta,\varphi)=\sum \Xi_{mn}(\psi)e^{i(m\theta-n\varphi)} \quad \text{and} \quad \delta f_{Lb}(\psi,\varphi,E,\mu)=\sum \delta f_{\pm 1\ell n}(\psi,E,\mu)e^{in\alpha-i(\ell-\sigma nq)h(\sigma,\theta)}$$

F. Porcelli, POP 1994

- Perturbed distribution function for collisionless plasma, and collisional plasmas with Krook operator:

N. Logan, POP 2013

$$\delta f_{\pm 1\ell n}=\frac{n\omega_b/e}{(\ell-\sigma nq)\omega_\ell-n(\omega_E+\omega_B)+i\nu_{eff}}\frac{\partial f_M}{\partial \psi_p}\delta J_{\pm 1\ell n}\equiv R_{\ell n}\delta J_{\pm 1\ell n},\ \text{where } \delta J \text{ is action variation}$$

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- Connections to well-known kinetic energy principles in collisionless limit:

I. Kruskal-Oberman for Maxwellian (MHD scale): $R_{\ell n}=-\frac{\omega_b}{T}f_M$ (Only $\ell=0$)

II. Chew-Goldberger-Low for Maxwellian (MHD scale): $R_{\ell n}=-\frac{\omega_b}{T}f_M$ (All ℓ)

III. Antonson-Lee for Maxwellian (Drift MHD scale): $R_{\ell n}=\lim_{\nu\to 0}R_{\ell n}$

Modified kinetic Euler-Lagrange equation

- Combing all the components, equations for $\left(\Xi_\psi, \Xi_\alpha\right)$ poloidal modes:

  Toroidal balance: $A\Xi_\alpha + B_R \Xi_\psi{}' + C_R \Xi_\psi = 0$ where $' \equiv \frac{\partial}{\partial \psi}$

  Radial balance: $\left(D\Xi_\psi{}' + E_R \Xi_\psi + B_L^\dagger \Xi_\alpha\right)' - \left(E_L^\dagger \Xi_\psi{}' + H\Xi_\psi + C_L^\dagger \Xi_\alpha\right) = 0$

- Leading to modified kinetic Euler-Lagrange equation:

$$\left(F\Xi_\psi{}' + K_R \Xi_\psi\right)' - \left(K_L^\dagger \Xi_\psi{}' + G\Xi_\psi\right) = 0$$

*Ξ : Poloidal mode vector for ξ

$A \equiv A_I + \int dE\, d\mu \left(W^{A\dagger} R W^A\right)$

$B_R \equiv B_I + \int dE\, d\mu \left(W^{A\dagger} R W^B\right)$

$B_L \equiv B_I + \int dE\, d\mu \left(W^{A\dagger} R^* W^B\right)$

$C_R \equiv C_I + \int dE\, d\mu \left(W^{A\dagger} R W^C\right)$

$C_L \equiv C_I + \int dE\, d\mu \left(W^{A\dagger} R^* W^C\right)$

$D \equiv D_I + \int dE\, d\mu \left(W^{B\dagger} R W^B\right)$

$E_R \equiv E_I + \int dE\, d\mu \left(W^{B\dagger} R W^C\right)$

$E_L \equiv E_I + \int dE\, d\mu \left(W^{B\dagger} R^* W^C\right)$

$H \equiv H_I + \int dE\, d\mu \left(W^{C\dagger} R W^C\right)$

$F \equiv D - B_L^\dagger A^{-1} B_R$

$K_R \equiv E_R - B_L^\dagger A^{-1} C_R$

$K_L \equiv E_L - B_R^\dagger A^{-1} C_L$

$G \equiv H - C_L^\dagger A^{-1} C_R$

$\delta J = W^A \Xi_\alpha + W^B \Xi_\psi{}' + W^C \Xi_\psi$

Modified kinetic Euler-Lagrange equation

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- Leading to modified kinetic Euler-Lagrange equation:

$$\left(F\Xi_\psi' + K_R\Xi_\psi\right)' - \left(K_L^\dagger\Xi_\psi' + G\Xi_\psi\right) = 0$$

- Ideal (and collisionless) Euler-Lagrange (DCON) equation:

$$\left(F_I\Xi_\psi' + K_I\Xi_\psi\right)' - \left(K_I^\dagger\Xi_\psi' + G_I\Xi_\psi\right) = 0$$

A. Glasser, APS 1997

$*F_I, G_I$ becomes Hermitian, and $K_R = K_L = K_I$

- Ideal matrices and Euler-Lagrange equation were shown to be identical to DCON matrices and equation, showing directly:

  Ideal perturbed equilibrium = Minimum state of potential energy

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- Leading to modified kinetic Euler-Lagrange equation:

$$\left(F\Xi_\psi' + K_R \Xi_\psi\right)' - \left(K_L^\dagger \Xi_\psi' + G\Xi_\psi\right) = 0$$

- Ideal (and collisionless) Euler-Lagrange (DCON) equation:

$$\left(F_I\Xi_\psi' + K_I \Xi_\psi\right)' - \left(K_I^\dagger \Xi_\psi' + G_I\Xi_\psi\right) = 0$$ A. Glasser, APS 1997

  *F_I, G_I becomes Hermitian, and $K_R = K_L = K_I$

- Cylindrical Euler-Lagrange (Newcomb) equation:

$$\left(f\xi'\right)' - g\xi = 0$$

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$\delta J = W^A \Xi_\alpha + W^B \Xi_\psi' + W^C \Xi_\psi$

Energy and (NTV) torque consistent with derived tensor pressure equilibrium with non-axisymmetric perturbation

- Energy integration with tensor force operator (in complex representation with $\exp(in\varphi)$):

$$2\delta W + i\frac{\tau_\varphi}{n} = -\int \vec{\xi}\cdot\left(\delta\vec{j}\times\vec{B} + \vec{j}\times\delta\vec{B} + \vec{\nabla}\left(\vec{\xi}\cdot\vec{\nabla}p\right) - \vec{\nabla}\cdot\left(\delta p_\perp \vec{I} + (\delta p_\parallel - \delta p_\perp)\hat{b}\hat{b}\right)\right)dx^3$$

$$= \int d\psi\left(\Xi_\alpha^\dagger A\Xi_\alpha + \Xi_\alpha^\dagger B_R\Xi_\psi' + \Xi_\alpha^\dagger C_R\Xi_\psi + \Xi_\psi'^\dagger B_L^\dagger\Xi_\alpha + \Xi_\psi^\dagger C_L^\dagger\Xi_\alpha + \Xi_\psi'^\dagger D\Xi_\psi' + \Xi_\psi'^\dagger E_R\Xi_\psi + \Xi_\psi^\dagger E_L^\dagger\Xi_\psi' + \Xi_\psi^\dagger H\Xi_\psi\right)$$

- Hermitian part becomes perturbed energy in the system, and anti-Hermitian part is toroidal torque, which is precisely what is known as neoclassical toroidal viscosity (NTV) torque J.-K. Park, POP 2011

- Using $A\Xi_\alpha + B_R\Xi_\psi' + C_R\Xi_\psi = 0$ and integrating by parts:

$$2\delta W + i\frac{\tau_\varphi}{n} = \int d\psi\left[\Xi_\psi^\dagger\left(F\Xi_\psi' + K_R\Xi_\psi\right)\right]' - \int d\psi\left[\Xi_\psi^\dagger\left(\left(F\Xi_\psi' + K_R\Xi_\psi\right)' - \left(K_L^\dagger\Xi_\psi' + G\Xi_\psi\right)\right)\right]$$

* = 0 in equilibrium, by modified Euler-Lagrange equation

Energy and (NTV) torque consistent with derived tensor pressure equilibrium with non-axisymmetric perturbation

- Energy integration with tensor force operator (in complex representation with $\exp(in\varphi)$):

$$2\delta W + i\frac{\tau_\varphi}{n} = -\int \vec{\xi}\cdot\left(\delta\vec{j}\times\vec{B} + \vec{j}\times\delta\vec{B} + \vec{\nabla}\left(\vec{\xi}\cdot\vec{\nabla}p\right) - \vec{\nabla}\cdot\left(\delta p_\perp \vec{I} + (\delta p_\parallel - \delta p_\perp)\hat{b}\hat{b}\right)\right)dx^3$$

$$= \int d\psi\left(\Xi_\alpha^\dagger A\Xi_\alpha + \Xi_\alpha^\dagger B_R\Xi_\psi' + \Xi_\alpha^\dagger C_R\Xi_\psi + \Xi_\psi'^\dagger B_L^\dagger\Xi_\alpha + \Xi_\psi^\dagger C_L^\dagger\Xi_\alpha + \Xi_\psi'^\dagger D\Xi_\psi' + \Xi_\psi'^\dagger E_R\Xi_\psi + \Xi_\psi^\dagger E_L^\dagger\Xi_\psi' + \Xi_\psi^\dagger H\Xi_\psi\right)$$

- Hermitian part becomes perturbed energy in the system, and anti-Hermitian part is toroidal torque, which is precisely what is known as neoclassical toroidal viscosity (NTV) torque J.-K. Park, POP 2011

- Using $A\Xi_\alpha + B_R\Xi_\psi' + C_R\Xi_\psi = 0$ and integrating by parts:

$$2\delta W + i\frac{\tau_\varphi}{n} = \left[\Xi_\psi^\dagger\left(F\Xi_\psi' + K_R\Xi_\psi\right)\right]$$

* Only surface term remains

Energy and (NTV) torque consistent with derived tensor pressure equilibrium with non-axisymmetric perturbation

- Energy integration with tensor force operator (in complex representation with $\exp(in\varphi)$):

$$2\delta W + i\frac{\tau_\varphi}{n} = -\int \vec{\xi}\cdot\left(\delta\vec{j}\times\vec{B}+\vec{j}\times\delta\vec{B}+\vec{\nabla}\left(\vec{\xi}\cdot\vec{\nabla}p\right)-\vec{\nabla}\cdot\left(\delta p_\perp\vec{I}+(\delta p_\parallel-\delta p_\perp)\hat{b}\hat{b}\right)\right)dx^3$$
$$= \int d\psi\left(\Xi_\alpha^\dagger A\Xi_\alpha+\Xi_\alpha^\dagger B_R\Xi_\psi'+\Xi_\alpha^\dagger C_R\Xi_\psi+\Xi_\psi'^\dagger B_L^\dagger\Xi_\alpha+\Xi_\psi^\dagger C_L^\dagger\Xi_\alpha+\Xi_\psi'^\dagger D\Xi_\psi'+\Xi_\psi'^\dagger E_R\Xi_\psi+\Xi_\psi^\dagger E_L^\dagger\Xi_\psi'+\Xi_\psi^\dagger H\Xi_\psi\right)$$

- Hermitian part becomes perturbed energy in the system, and anti-Hermitian part is toroidal torque, which is precisely what is known as neoclassical toroidal viscosity (NTV) torque J.-K. Park, POP 2011

- Using $A\Xi_\alpha + B_R\Xi_\psi' + C_R\Xi_\psi = 0$ and integrating by parts:

$$2\delta W + i\frac{\tau_\varphi}{n} = \left[\Xi_\psi^\dagger\left(F\Xi_\psi' + K_R\Xi_\psi\right)\right] = \Xi_\psi^\dagger W_P\Xi_\psi$$

- Energy, torque, and 3D force balance are all self-consistently calculated with perturbed tensor pressure, yielding non-Hermitian plasma response matrix including torque response

Non-Hermitian Plasma Response Matrix: $W_P \equiv \left(F\Xi_\psi' + K_R\Xi_\psi\right)\Xi_\psi^{-1}$

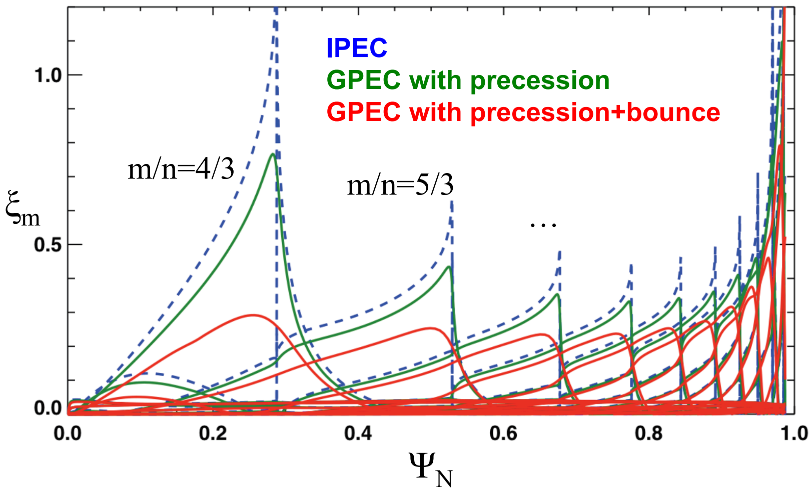
General perturbed equilibrium code (GPEC) has been successfully developed based on modified DCON and IPEC

- GPEC integrates modified kinetic Euler-Lagrange equation through modified DCON stability code

N. Logan, POP 2013

- Kinetic matrices are calculated presently by PENT, but can be flexibly extended and combined with any kinetic solver for drift-kinetic equation
- Integration from core to edge with *M* linearly independent solutions leads to non-Hermitian plasma response matrix
- Anti-Hermitian part of plasma response matrix provides all the information of torque and its profile
- It is shown that the field penetration can be strongly modified by kinetic energy and toroidal torque, throughout plasma including the neighborhood of rational surfaces

Eigenfunction for n=3 least δW mode
(DIII-D β_N=2.5, q_{95}=3.7)



Modification of field penetration in the neighborhood of singular surfaces is important for self-consistent NTV

- Ideal Euler-Lagrange equation has regular singular points at *q=m/n* surface

$$\left(F_I\Xi_\psi' + K_I\Xi_\psi\right)' - \left(K_I^\dagger\Xi_\psi' + G_I\Xi_\psi\right) = 0$$

$$F_I = Q\bar{F}_I Q\ ,\ K_I = Q\bar{K}_I \quad \text{where}\ \ Q_{mm'} = (m - nq)\delta_{mm'}$$

- Exclusion of large resonant solution in DCON means:
  - No magnetic islands and flux surfaces are nested everywhere
  - Most of complete NTV models rely on this nested flux surface condition
  - However, NTV across the singular surface is still non-integrable

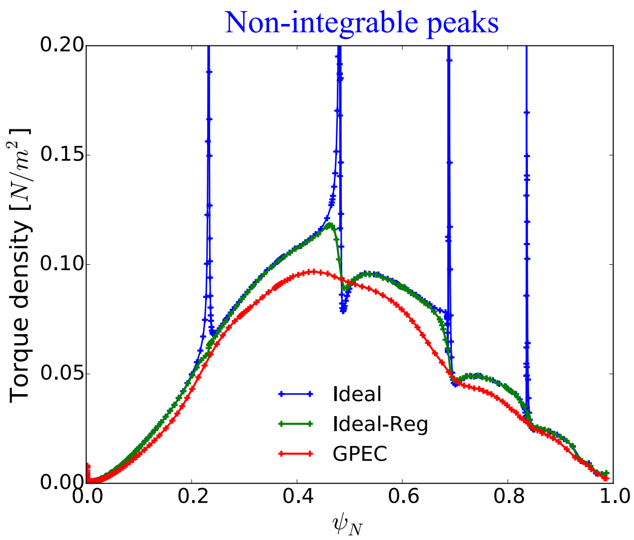
- Modified kinetic Euler-Lagrange equation is not singular as long as the toroidal torque is finite

$$\left(F\Xi_\psi' + K_R\Xi_\psi\right)' - \left(K_L^\dagger\Xi_\psi' + G\Xi_\psi\right) = 0$$

$$F = Q\bar{F}_K Q - P_L^\dagger Q - QP_R + R_1,\ K_R = Q\bar{K}_{KR} + R_2,\ K_L = \bar{K}_{KL}Q + R_3$$

- Meaning that NTV torque can be integrated across nested flux surfaces, but without adhoc dissipation model near the rational surfaces

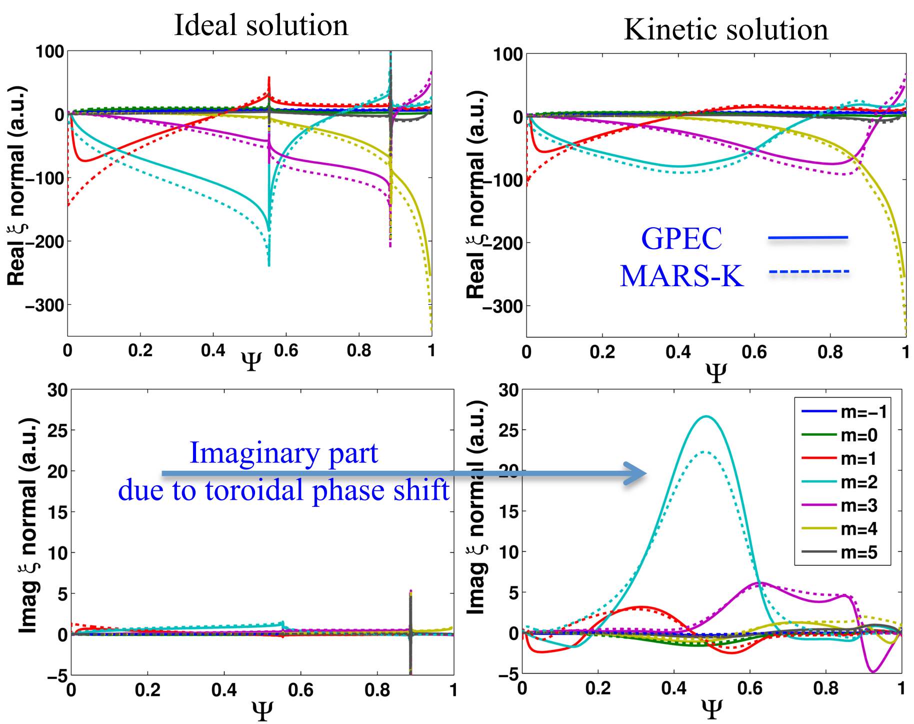
Torque density by non-resonant NCC n=1
(NSTX-U I_P=2MA, Low β_N=1.9)



Verification of GPEC solution against MARS-K in zero-frequency limit

- Present GPEC with Krook collisional operator should give identical solutions to MARS-K without fluid rotation, in zero-frequency limit
- Successful benchmark was made, and both codes captured important changes in eigenfunctions
- Two codes use distinct methods in the computation, leading to different flexibility and advantages
- GPEC is a perturbed equilibrium code working for the zero-frequency limit, but can adopt any δf-solver and give <u>the full eigenmode structure by a single run</u>, which can be used to optimize non-resonant field

(A circular plasma, A=2.8, β_N =3.3)



Kinetic energy principle by Kruskal-Oberman, CGL, and Antonson-Lee

- Kruskal-Oberman, Rosenbluth-Rostocker, and Newcomb used kinetic closure for anisotropic pressure and derived kinetic energy principle in MHD scale

- KO limit is equivalent to take: M. Kruskal, PF 1958

$$R_{\ell n} = -\frac{\omega_b}{T} f_M \text{ (Only } \ell = 0)$$

- CGL limit is equivalent to the upper bound of Schwartz inequality of K-O limit, and also is equivalent to take:

$$R_{\ell n} = -\frac{\omega_b}{T} f_M \text{ (All } \ell)$$

Chew, PRS 1956
J. Berkery, POP 2014

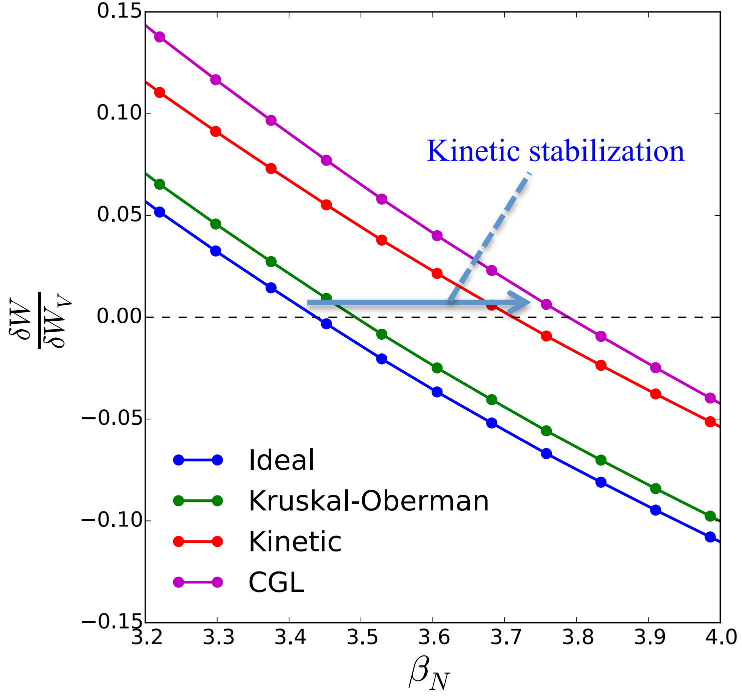
- As expected, NSTX-U target studies yielded:

$$\delta W_{ideal} < \delta W_{KO} < \delta W_{Kinetic} < \delta W_{CGL}$$

T. Antoson, PF 1982

*Where $\delta W_{Kinetic}$ is calculated by ignoring toroidal torque (Antonson-Lee), but the stability of the system should be determined by solving normal mode problems with the wall

<u>Least δW normalized to vacuum energy</u>
(NSTX-U I_P=2MA, q_{95}=6.6)

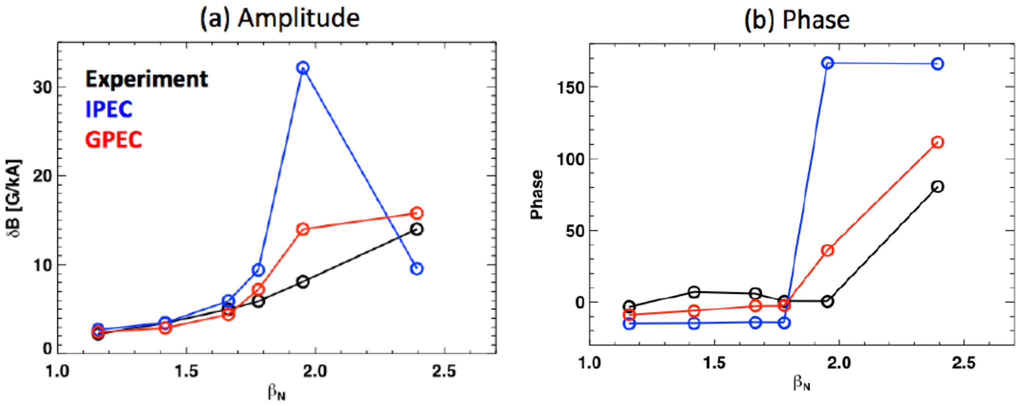


Resonant field amplification across no-wall β limit

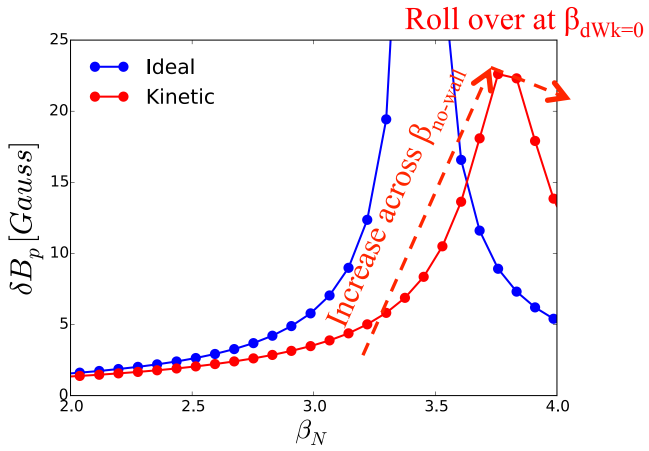
- MARS-K simulation already showed plasma response field (called RFA, resonant field amplification) can be increased almost linearly across the no-wall limit
- GPEC also reproduces the trend, as well as phase-shift due to the torque

- Furthermore, GPEC predicts that the plasma response can be eventually peaked and decreased if it crosses $\delta W_{kinetic}$=0 limit, but the peak is limited by the finite torque

Recaptured DIII-D n=1 RFA   Z. Wang, PRL 2015



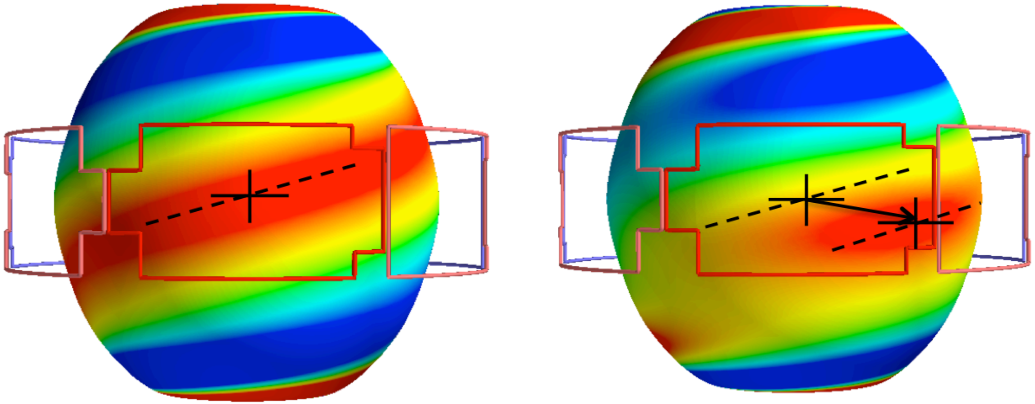
NSTX-U RFA prediction (I_P=2MA n=1)



Toroidal phase shift in plasma response and self-shielding

- Torque creates toroidal phase-shift in plasma response

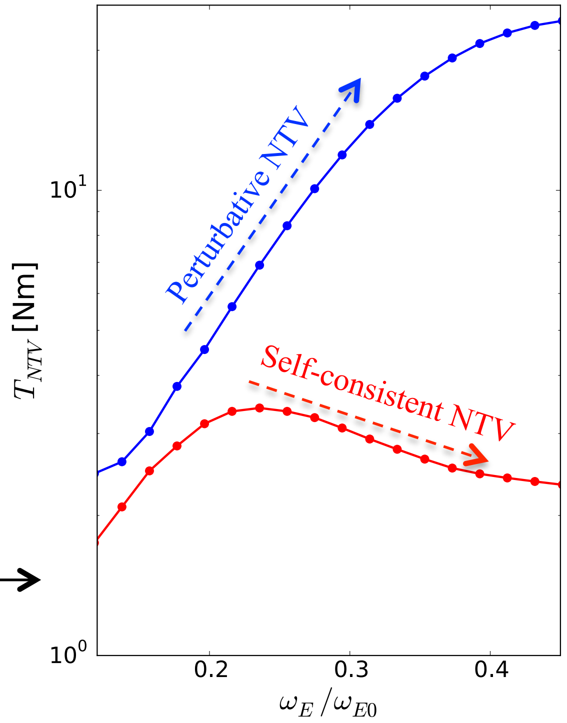
Ideal Response　　Kinetic Response



(NSTX-U I_P=2MA, β_N =3.0, n=1)

- Toroidal phase shift increases along with torque, and eventually coupling between external field and plasma can become inefficient, resulting in self-shielding process A. Boozer, PRL 2001

NTV vs. $E\times B$



(*ω_{E0} From TRANSP with 12MW NBI)

Construction of torque response matrix to external field

- Given displacement, torque is determined by imaginary part of plasma response matrix:

$$\text{Integrated torque } \tau_\varphi(\psi) = \Xi_\psi^\dagger [n \,\mathrm{Im}\, W_P] \Xi_\psi$$

- GPEC solution matrix relates displacement $\Xi_\psi(\psi)$ to field $\Phi(\psi = \psi_b)$ on the boundary

$$\text{Leading to } \tau_\varphi(\psi) = n\Phi^\dagger \left(\Lambda_T(\psi)\right)^{-1} \Phi, \text{ where } \Lambda_T(\psi) \text{ is imaginary part of inductance matrix}$$

- Virtual casing principle relates (total) field to external (vacuum) field on the boundary by:

$$\Phi = \Lambda L^{-1} \Phi^x \equiv P\Phi^x, \text{ where } L \text{ is surface inductance and } P \text{ is permeability matrix}$$

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- So one can obtain the integrated NTV torque up to any radial point, once external field is known, by

$$\tau_\varphi(\psi) = n\Phi^{x\dagger} P^\dagger \left(\Lambda_T(\psi)\right)^{-1} P\Phi^x \equiv \Phi^{x\dagger} \mathrm{T}(\psi) \Phi^x$$

$\mathrm{T}(\psi) \equiv nP^\dagger \left(\Lambda_T(\psi)\right)^{-1} P$ is Hermitian, although P (i.e. plasma response) is non-Hermitian

Theoretically maximum (or minimum) integrated torque and external field required to produce them

- Theoretical maximum (minimum) integrated torque, when power of external field is fixed, is given by the largest (smallest) eigenvalue λ of the torque response matrix

$$\tau_{\varphi,\max}(\psi) = \lambda_{\max} \ \text{ for } \ \mathrm{T}(\psi)$$

- Also, one can obtain the maximum (or minimum) torque possible for any arbitrary interval (ψ_1, ψ_2), given the total integrated torque fixed:

$$\text{Question : Maximize (or minimize) } R_{\max} = \frac{\tau_\varphi(\psi_2) - \tau_\varphi(\psi_1)}{\tau_\varphi(\psi_b)} = \frac{\Phi^{x\dagger}\left[\mathrm{T}(\Delta\psi_{12})\right]\Phi^x}{\Phi^{x\dagger}\left[\mathrm{T}(\psi_b)\right]\Phi^x}$$

$$\text{Answer : } R_{\max} = \lambda_{\max} \ \text{ for } \ \mathrm{T}^{-1}(\psi_b)\mathrm{T}(\psi_{12}) \ \text{ and its eigenvector gives required external field}$$

- In general, quadratic matrix optimizer can answer more complicated demands in NTV and non-resonant field optimization (e.g. when negative torque can exist, and when external field is limited by coils)

Theoretically maximum torque inside a given flux surface relative to the total integrated torque

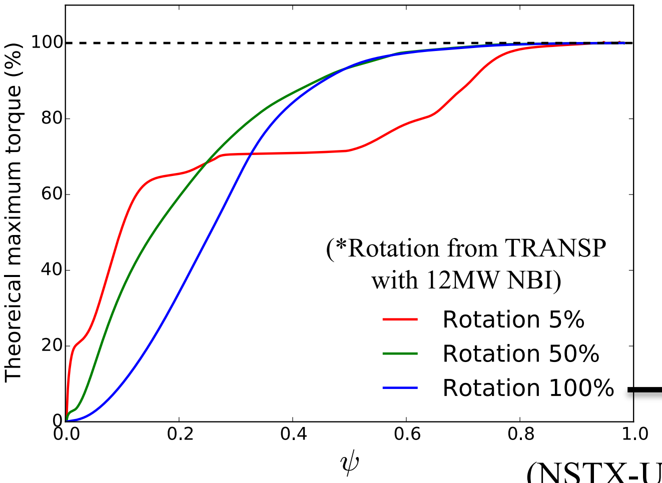
- Maximum torque ratio at a given ψ to the total integrated torque is given by:

$$R_{max} = \lambda_{max} \text{ for } T^{-1}(\psi_b)T(\psi)$$

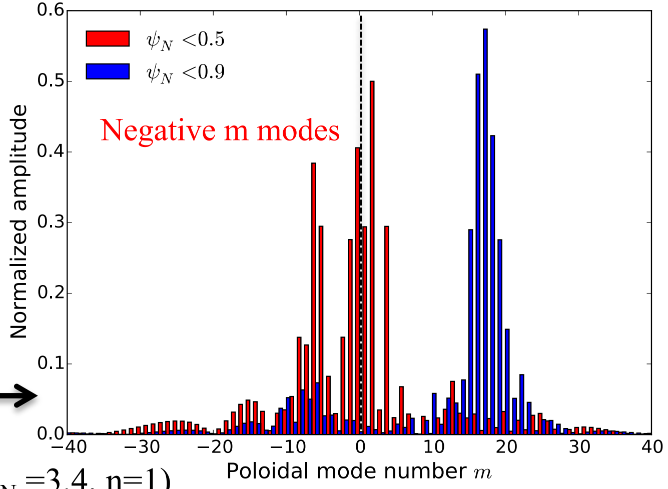
J.-K. Park, PRL 2013 for KSTAR experiments

- Eigenvector shows the importance of low and "negative m" mode (i.e. backward helicity mode), to increase the torque only in the core by deeper penetration

Maximum torque % inside a given flux surface



External field maximizing the torque



(NSTX-U I_P=2MA, β_N =3.4, n=1)

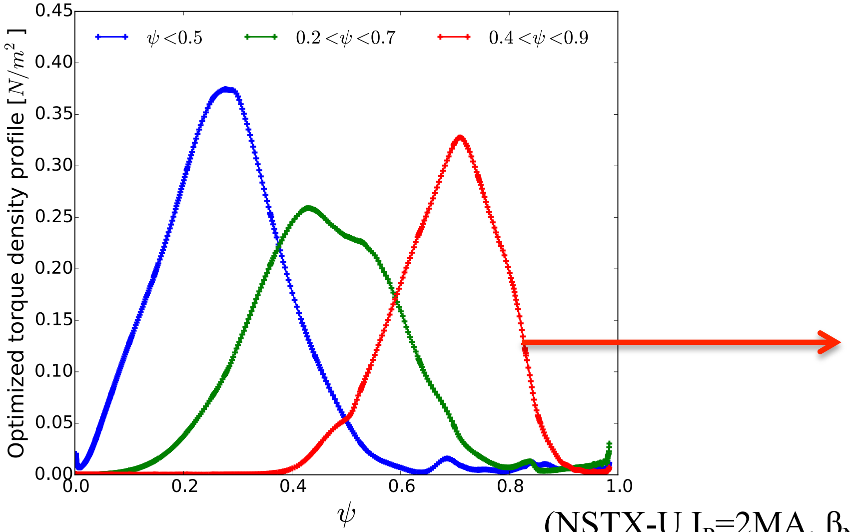
Optimization of local torque to the total integrated torque

- Maximum torque for interval (ψ_1, ψ_2) to the total integrated torque is given by:

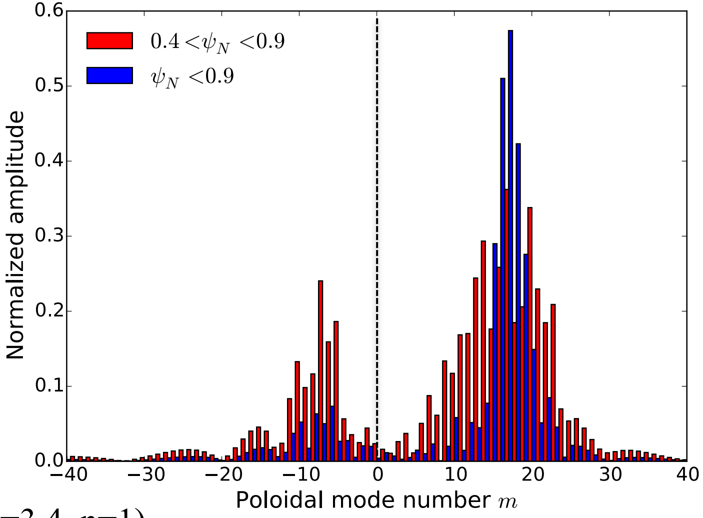
$$R_{\max} = \lambda_{\max} \ \text{for} \ \mathrm{T}^{-1}(\psi_b)\mathrm{T}(\Delta\psi)$$

- Eigenvector shows delicate compensations between negative m modes and dominant positive m modes

Optimized torque profile for given interval:



External field optimizing the local torque:



(NSTX-U I_P=2MA, β_N =3.4, n=1)

Summary and Future work

- 3D force balance with tensor pressure for Maxwellian equilibrium has been solved directly, leading to modified kinetic Euler-Lagrange equation
- General perturbed equilibrium code (GPEC) has been successfully developed to numerically integrate the new Euler-Lagrange equation, giving 3D equilibrium consistent with NTV torque
- GPEC shows various stability limits of kinetic energy principle, and reproduces RFA trends
- Self-consistent NTV can be calculated by non-Hermitian plasma response matrix including NTV torque
- Torque response matrix provides a new and systematic way of NTV and non-resonant field optimization, revealing the importance of backward helicity modes for local torque optimization
- GPEC provides all the information of self-consistent NTV torque in a matrix function form, which can be coupled to external coils and matrix optimizers to optimize local torque under various constraints