Computation of Perturbed Equilibrium with Resistive DCON

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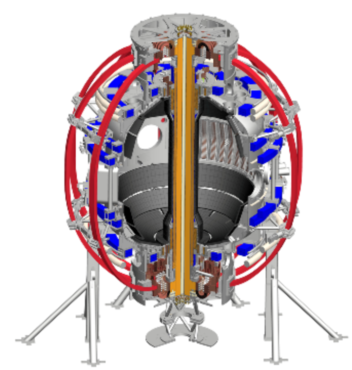


Motivation

- IPEC is successfully used for computing ideal perturbed equilibrium.
  - DCON is implemented into IPEC and provides full eigen mode spectrum in ideal MHD to couple with external magnetic perturbation.
  - No allowance of magnetic island at the singular surface.
- The resistive perturbed equilibrium, allowing island opening, requires to solve resistive MHD.
  - Resistive DCON based on the asymptotic matching method retains non-vanishing big solutions and matches to inner layer, DELTAC code.
  - Resistive DCON is coupled to IPEC module→Resistive Perturbed Equilibrium Code (RPEC)
- Resistive DCON with high efficiency and flexibility of extending inner region physics can address the boundary of resistive instabilities, in operational or parametric space for tokamaks or ITER.
  - Resistive DCON can provides full Δ' matrix in toroidal geometry to linear/non-linear TM and NTM study.
- Resistive perturbed equilibrium can improve the predictability such as error fields and neoclassical toroidal viscosity in the presence of magnetic island.

Outline

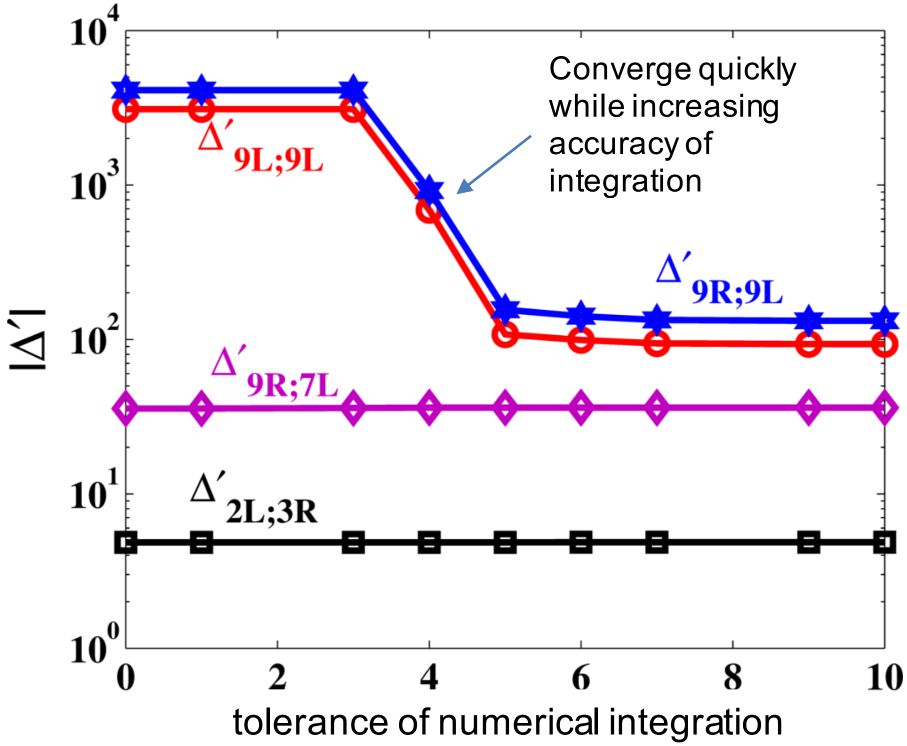
- Outer region: resistive DCON with resonant Galerkin method
  - Non-homogeneous Euler-Lagrange equation using big solution as the driven term
- Inner region: DELTAC code reliably solves GGJ model with Galerkin method
- MATCH code matches the outer region and inner region to determine the tearing mode stability
- Resistive DCON, DELTAC, MATCH couple with IPEC to obtain the resistive/relax perturbed equilibrium (RPEC)
- Summary and Discussion

Outer Region Solution (Resistive DCON code)

Convergence Has Been Greatly Improved in Resistive DCON

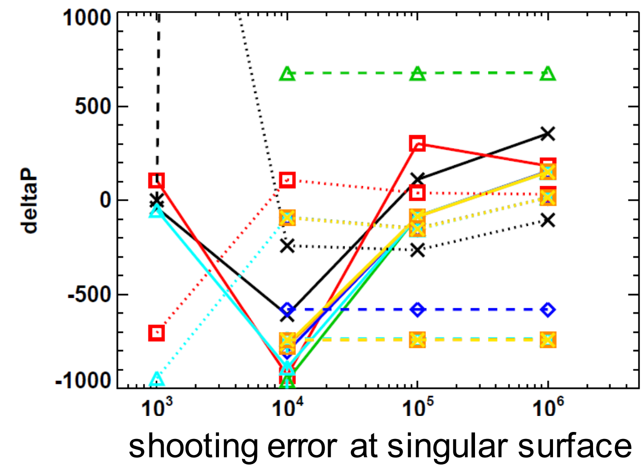
- A very robust convergence of matching data is achieved even in a challenging NSTX case

NSTX case (q_0=2.13, q_a=14.9, β_N=3.29)



The scan of integral tolerance shows the quick and stable convergence of Δ' value among q=2,3,7,9 resonant surfaces

DIII-D high β case: shot 126006 at 3600ms



The convergence of Δ' value is bad in shooting method.

Formulation for Resistive DCON in Outer Region

- In asymptotic matching method, outer region is solved independently of inner region of the layer
  - **This separation enables flexible implementation of inner-layer models**

- Outer region is modeled by zero-frequency ideal MHD：

$$\vec{j} \times \vec{B} + \vec{J} \times \vec{b} - \nabla p = 0$$

$$p = -\vec{\xi} \cdot \nabla P - \Gamma P \nabla \cdot \vec{\xi}$$

$$\vec{b} = \nabla \times (\vec{\xi} \times \vec{B})$$

$$\vec{j} = \nabla \times \vec{b}$$

$\Rightarrow$

Euler-Lagrange Equation:

$$\mathbf{L}[\Xi] = -\left(\mathbf{F}\Xi' + \mathbf{K}\Xi\right)' + \left(\mathbf{K}^{\dagger}\Xi' + \mathbf{G}\Xi\right) = 0$$

$$where \quad \Xi = \{\xi_{mn}(\psi)\} \quad and \quad \vec{\xi} \cdot \vec{\nabla}\psi(\psi,\vartheta,\varphi) = \sum_{mn} \xi_{mn}(\psi)\exp(im\vartheta - in\varphi)$$

The system of the second-order ODE is singular near the resonant surface

$$\mathbf{F} \sim |\psi - \psi_R|^2$$

and difficult to solve when coupled with regular non-resonant solutions.

Resistive DCON Solves Outer Region with C¹ Finite Element Method and Precise Frobenious Expansion Up to Arbitrary Order

Solve non-homogeneous E-L equation driven by big solutions to get outer region solution:

$$\mathbf{L}\Xi = -\mathbf{L}\Xi_{ip}^{b}$$

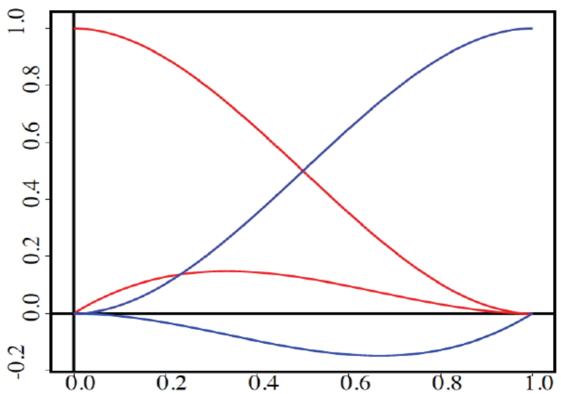
response of small solution and non-resonant solutions

driving term of big solution

i=1~N resonance surfaces,
p=R or L representing right or left side of the resonant surfaces

Boundary value problem with the system of singular ODE is solved with:

C¹ Hermite Cubics



C¹ continuity: function values and first derivatives

Used for non-resonant solutions across the singular surface

Convergent Power Series Expansion

$$\Xi = z^{\alpha}\sum_{k=0}^{K}\boldsymbol{u}^{k}z^{k} \qquad z = \begin{cases}\psi - \psi_r & (\psi \geq \psi_r)\\ \psi_r - \psi & (\psi < \psi_r)\end{cases}$$

$\alpha = -\frac{1}{2} \pm \sqrt{-D_I}$ denotes large and small resonant solutions at ψ_r^i.

Solved to arbitrarily high order N;

Automated using matrix formulation

Essential for larger values of $|D_I|$;

Resonant And Extension Elements Are Introduced to Precisely Treat Solutions Near Resonant Surfaces

- With adjustable grid packing methods, DCON solves Euler-Lagrange equation by modified "resonant" Galerkin method

$$W = \frac{1}{2}\left(\bar{\Xi}, \mathbf{L}\left[\bar{\Xi}\right]\right) - \left(\bar{\Xi}, \mathbf{L}\left[\Xi_{ip}^{b}\right]\right)$$

$$\delta W = \left(\delta\bar{\Xi}, \mathbf{L}\left[\bar{\Xi}\right]\right) - \left(\delta\bar{\Xi}, \mathbf{L}\left[\Xi_{ip}^{b}\right]\right) = 0$$

- Resonant element includes small resonant solutions as extra basis functions and extension element connects solutions smoothly to normal Hermite cubic element

Coupling coefficients from small to big solutions

Big solutions driving others

Small solutions

$$\Xi_{ip} = \Xi_{ip}^{b} + \underbrace{\Delta'_{ip;jq}\,\Xi_{jq}^{s}}_{\text{Resonant element}} + \underbrace{\Xi_{ip}^{n}}_{\text{Hermite cubic element}}$$

Non-Resonant (Regular) solutions

Resonant element **Hermite cubic element**

N	...	N	E	R	R	E	N	...	N

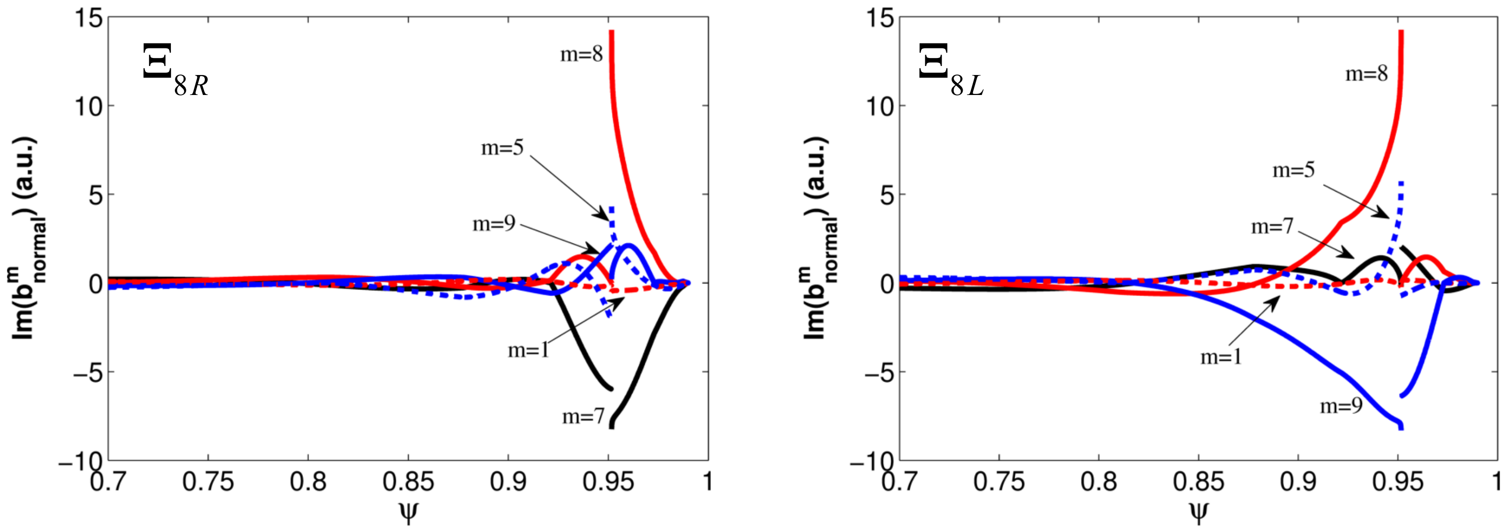
Extension element

Big Solution Can Drive Small Solutions in All Other Surfaces As Well As Non-Resonant Solutions

- Each of 2N big resonant solution can drive all other 2R small resonant solutions as well as non-resonant solutions

$$\Xi_{ip} = \Xi^{b}_{ip} + \Delta'_{ip;jq}\,\Xi^{s}_{jq} + \Xi^{n}_{ip}$$

NSTX high-β example using Resistive DCON



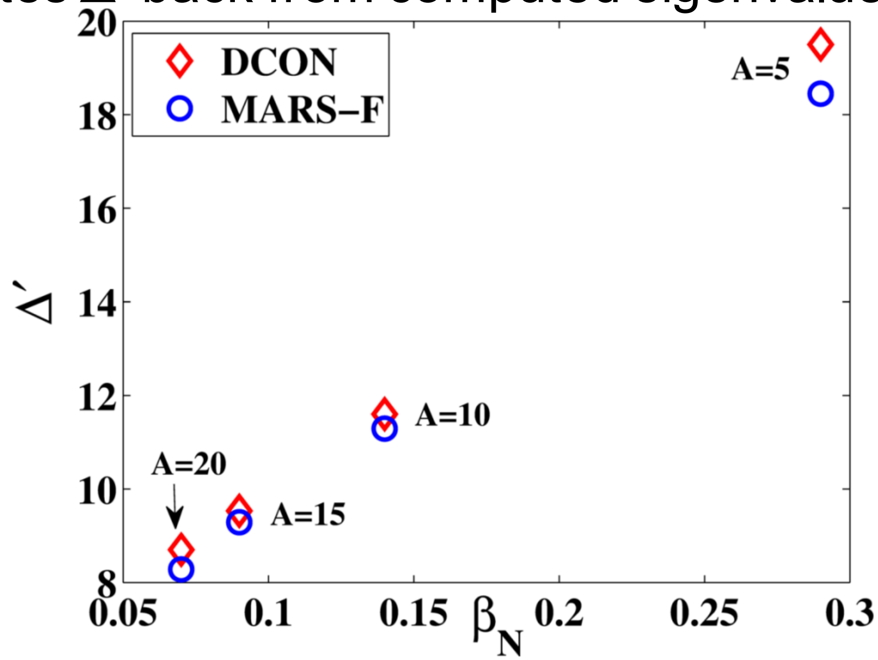
- Note resistive instability is determined by the stability index matrix $\Delta'_{ip;jq}$ rather than a scalar tearing mode index Δ'

Successful Δ’ Benchmark Between Resistive DCON and MARS-F

- A cylindrical Δ’ can be obtained ignoring other surfaces by

PEST 3 ←
$$\begin{aligned} \boxed{\Gamma'} &= \boxed{\Delta_{RR} + \Delta_{RL} - \Delta_{LR} - \Delta_{LL}} \\ \boxed{\Delta'} &= \boxed{\Delta_{RR} - \Delta_{RL} - \Delta_{LR} + \Delta_{LL}} \end{aligned}$$
→ DCON

MARS-F calculates Δ’ back from computed eigenvalue and GGJ model



Quantitative agreement has been obtained in various cases

Δ' has been compare between Resistive DCON and PEST3 (Kim, JP12.00099, Tuesday, 2:00pm)

Inner Region Solution (DELTAC code)

Inner Region: DELTAC Code Has Been Developed to Solve Inner Layer Δ'

- The inner layer model in [Glasser, Jardin and Tesauro, Phys. Fluids 1984] is solved by the resonant Galerkin method.

$$\Psi_{XX} - H\Upsilon_X = Q(\Psi - X\Xi)$$

GGJ model:

$$Q^2\Xi_{xx} - QX^2\Xi + QX\Psi + (E+F)\Upsilon + H\Psi_X = 0$$

$$Q\Upsilon_{XX} - X^2\Upsilon + X\Psi + Q^2\left[G(\Xi - \Upsilon) - K(E\Xi + F\Upsilon + H\Psi_X)\right] = 0$$

GGJ matrix form:

$$\mathbf{L\Psi} = \mathbf{\Psi}'' - \mathbf{V\Psi}' - \mathbf{U\Psi} = 0, \qquad \mathbf{\Psi} = \begin{pmatrix} \Psi \\ \Xi \\ \Upsilon \end{pmatrix}$$

$$\Psi \sim <\mathbf{b}\cdot\nabla\psi>$$
$$\Xi = <\xi\cdot\nabla\psi>$$
$$\Upsilon = <\mathbf{b}\cdot\mathbf{B}>/P'$$

- Similarly to outer region, large power solution is used to drive the response of three small solutions (two exponential small and one small power solutions) in the configuration space.

$$\mathbf{L}\left(\Delta\mathbf{\Psi}_{small} + \alpha\mathbf{\Psi}_{small,1}^{exp} + \beta\mathbf{\Psi}_{small,2}^{exp}\right) = -\mathbf{L\Psi}_{big}$$

- The resonant Galerkin method is applied to solve inner region with the adjustable grid packing.

DELTAC Shows Very Robust Numerical Convergence

Test Case in [Glasser, Jardin and Tesauro, Phys. Fluids 1984]

-1<E<1, F=G=H=K=0, 1e-5<Q<10

Parameters from tokamak equilibrium can be more challenging.

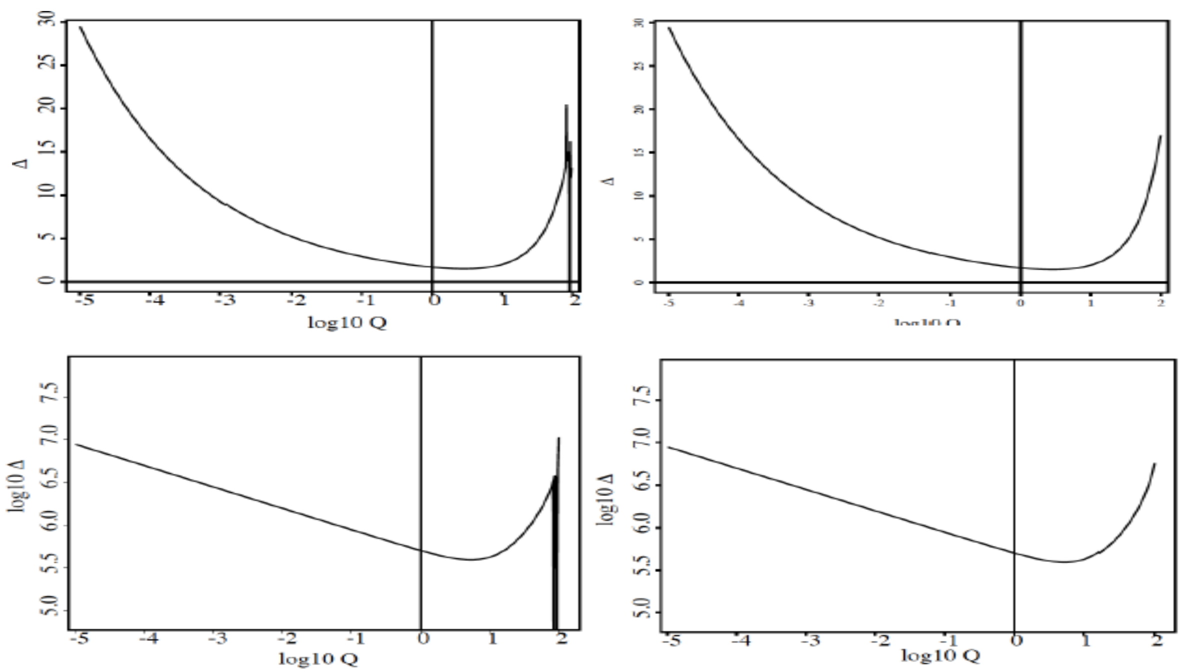
$E = -5.912\times10^{-3}$
$F = 5.480\times10^{-4}$
$G = 3.200\times10^{3}$
$H = 3.721\times10^{-3}$
$K = 1.716\times10^{3}$
$M = 19.43$

$Q = 539$

DELTAR Code | DELTAC Code

Odd parity Δ_-

Even parity Δ_+



DELTAR solves inner layer in Fourier space [Glasser, Jardin and Tesauro, Phys. Fluids 1984]

Matching Outer and Inner Region (MATCH code)

Match Outer Region Solutions With Glasser-Greene-Johnson Model For Toroidal Geometry

- GGJ theory with inertia and resistivity produces the even and odd inner region solutions as **[Glasser, Greene, and Johnson, Phys. Fluids (1975)]**

$$\Xi_{i\pm}(x) = \Xi^{b}_{i\pm}(x) + \Delta_{i\pm}(Q)\Xi^{s}_{i\pm}(x)$$

Q: Resistive scaled growth rate
x: Scale length across layer

- Outer region formulation with the left and right side solutions can be replaced by even and odd, and matching conditions are given by

big solution matching

$$C_L^j - \left(d_{j+} - d_{j-}\right) = 0$$

$$C_R^j - \left(d_{j+} + d_{j-}\right) = 0$$

small solution matching

$$\sum_{i=1}^{N}\sum_{p=L,R} C_p^i \Delta_{pL}^{ij} - \left(d_{j+}\Delta_{j+}(Q) - d_{j-}\Delta_{j-}(Q)\right) = 0$$

$$\sum_{i=1}^{N}\sum_{p=L,R} C_p^i \Delta_{pR}^{ij} - \left(d_{j+}\Delta_{j+}(Q) + d_{j-}\Delta_{j-}(Q)\right) = 0$$

Outer region

Inner region

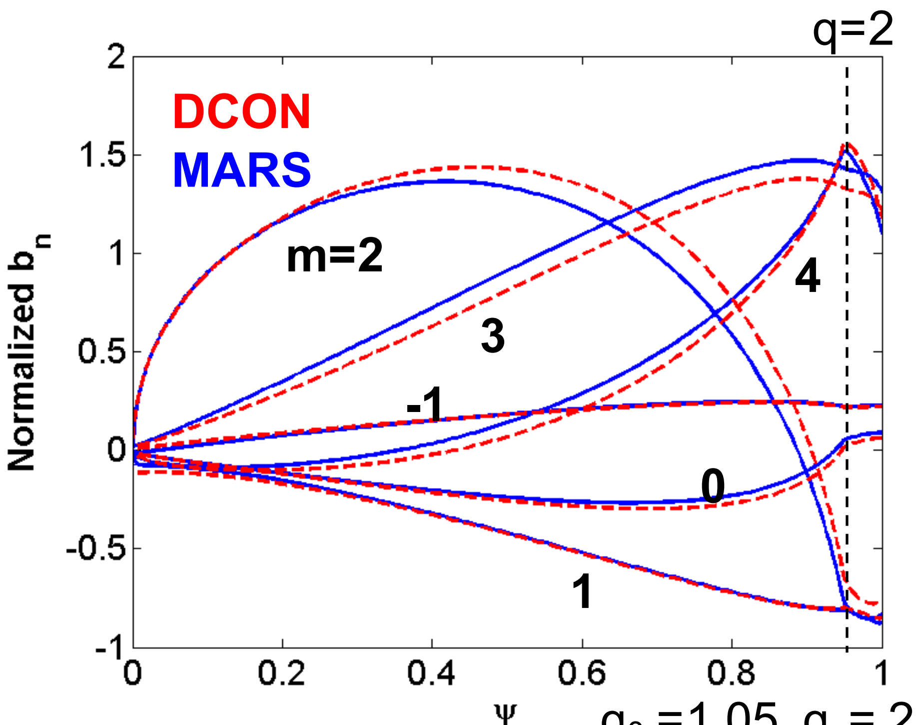
$$\Rightarrow \quad \mathbf{M}\left(Q\right)\mathbf{c} = 0$$

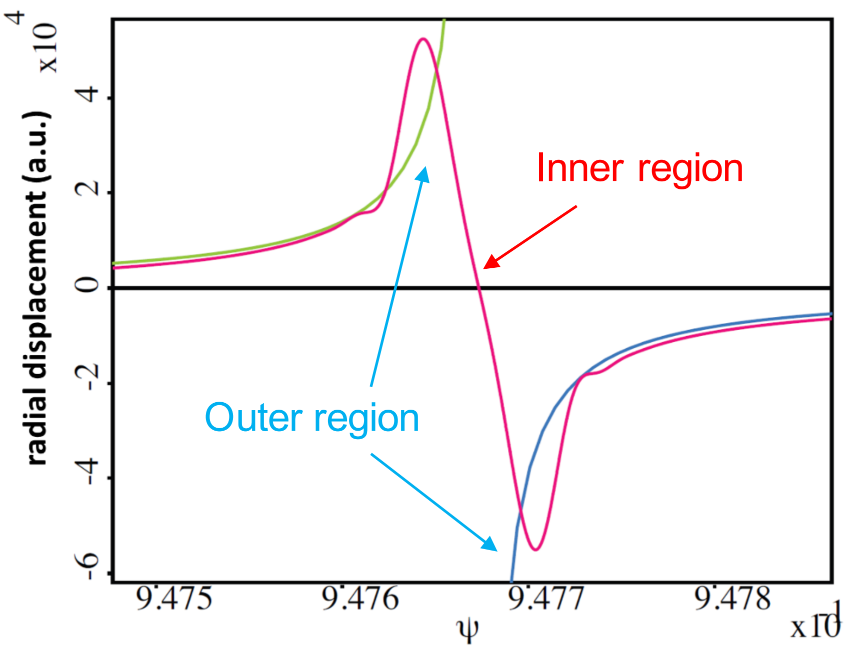
- So resistive instability and growth rate is determined by the determinant of matching matrix

$$|\mathbf{M}\left(Q\right)| = 0$$

Resistive DCON Solutions With GGJ Model Has A Good Agreement With MARS-F

- Eigenfunctions solved by matching resistive DCON solutions with GGJ model showed a good agreement with MARS-F for a single resonant surface case in D-shaped plasma.
- Inner region solution asymptotically matches to outer region in Resistive DCON.

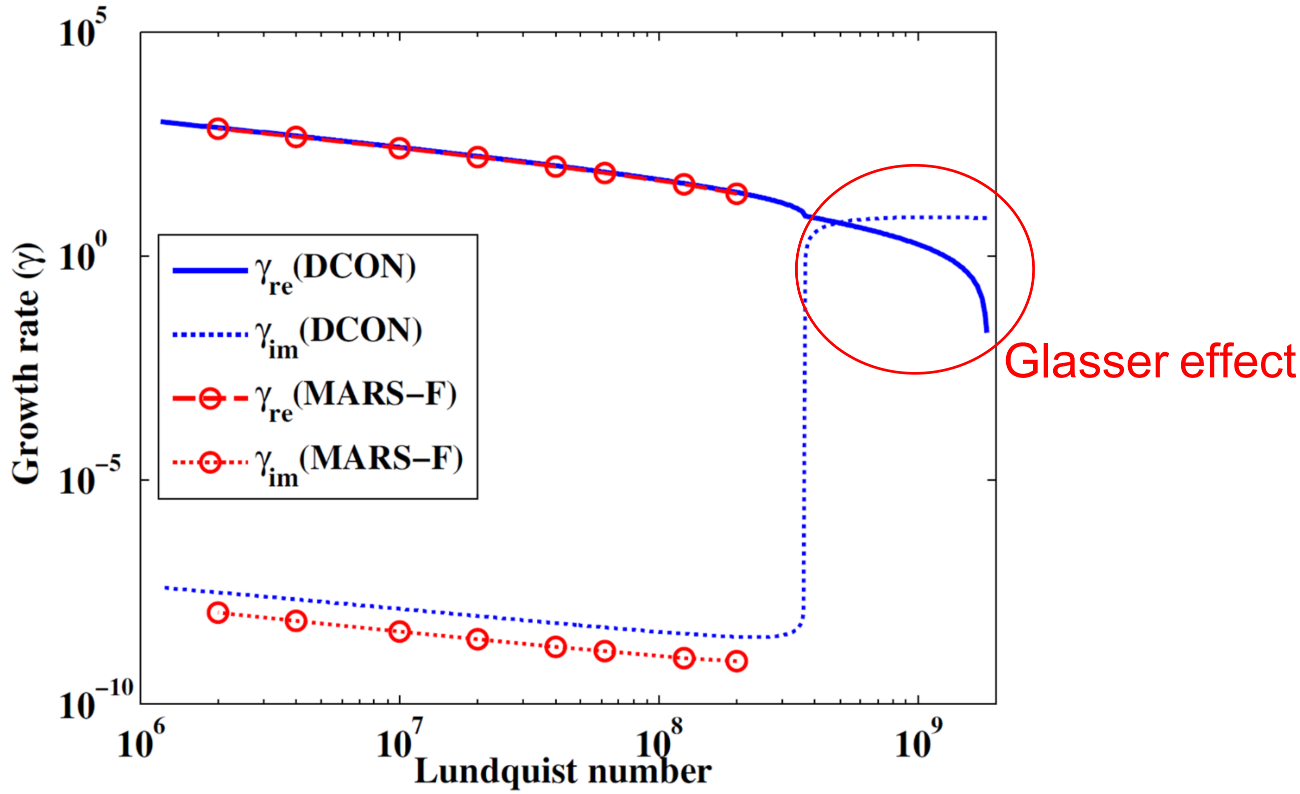




q_0 =1.05, q_a= 2.31, β_N=0.77, A=2.734

Growth Rate Comparison With MARS-F

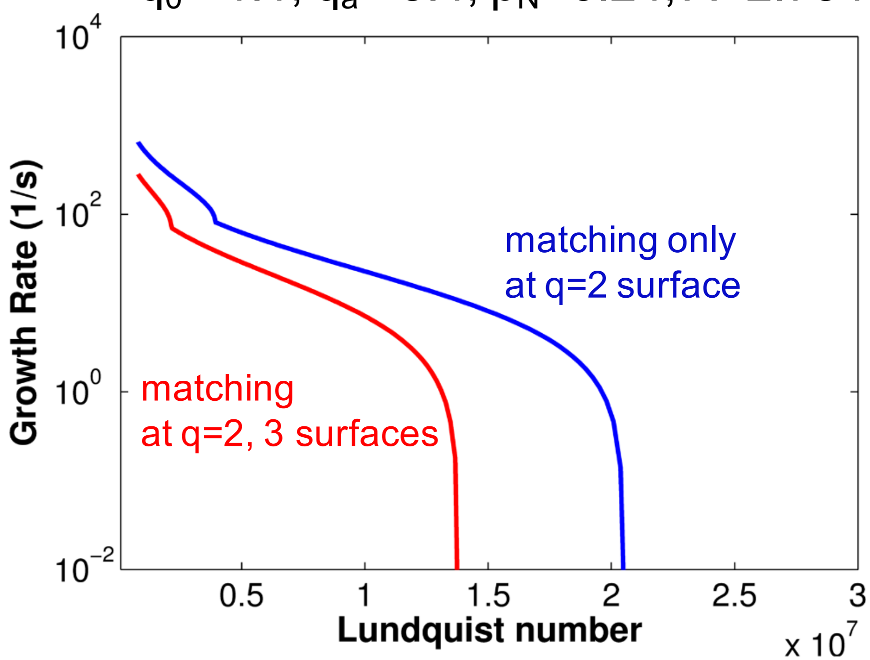
- Growth rates calculated by DCON with GGJ model and by MARS-F showed an excellent quantitative agreement, with expected scaling for tearing mode instability.



Resistive DCON Shows Underlying Information of How Singular Layers Couples Each Other to Affect Tearing Instability

- D shape low beta tokamak equilibrium shows the stabilizing effect from q=3 surface to q=2 surface in the Lundquist number scan.
- Outer region Δ'_{out} matrix clearly show the Δ'>0 at q=2 is unstable and $\Delta' < 0$ at q=3 is stable.
- It is necessary to match the full Δ' matrix between outer and inner region in toroidal geometry.

q_0 =1.1, q_a= 3.4, β_N=0.24, A=2.734



(2,1) mode
unstable

$$\Delta'_{out} = \begin{pmatrix} 7.37 & 2.24 \\ 11.06 & -1.286 \end{pmatrix}$$

(3,1) mode
stable

Resistive Perturbed Equilibrium Code (RPEC)

Perturbed Equilibrium Including Resistive Layer Can Be Solved By Modified Matching Dispersion Relation

- Resistive perturbed equilibrium allows magnetic island at singular surface.
- Resistive perturbed equilibrium can be solved through the linear combination between
  - Resistive eigenfunction with $\Xi = 0$ at plasma edge
  - Ideal response driven by the external magnetic perturbation
- Dispersion Relation matching the outer and inner region further includes the contribution of small solutions driven by external fields.

$Q = if + in\Omega$

f is applied coil frequency
Ω is flow rotation

Inner region

$$C_L^j - \left(d_{j+} - d_{j-}\right) = 0$$

$$C_R^j - \left(d_{j+} + d_{j-}\right) = 0$$

Outer region

$$\sum_{i=1}^{N}\sum_{p=L,R} C_p^i \Delta_{pL}^{ij} - \left(d_{j+}\Delta_{j+}(Q) - d_{j-}\Delta_{j-}(Q)\right) = -\Delta_L^{j,ext}$$

$$\sum_{i=1}^{N}\sum_{p=L,R} C_p^i \Delta_{pR}^{ij} - \left(d_{j+}\Delta_{j+}(Q) + d_{j-}\Delta_{j-}(Q)\right) = -\Delta_R^{j,ext}$$

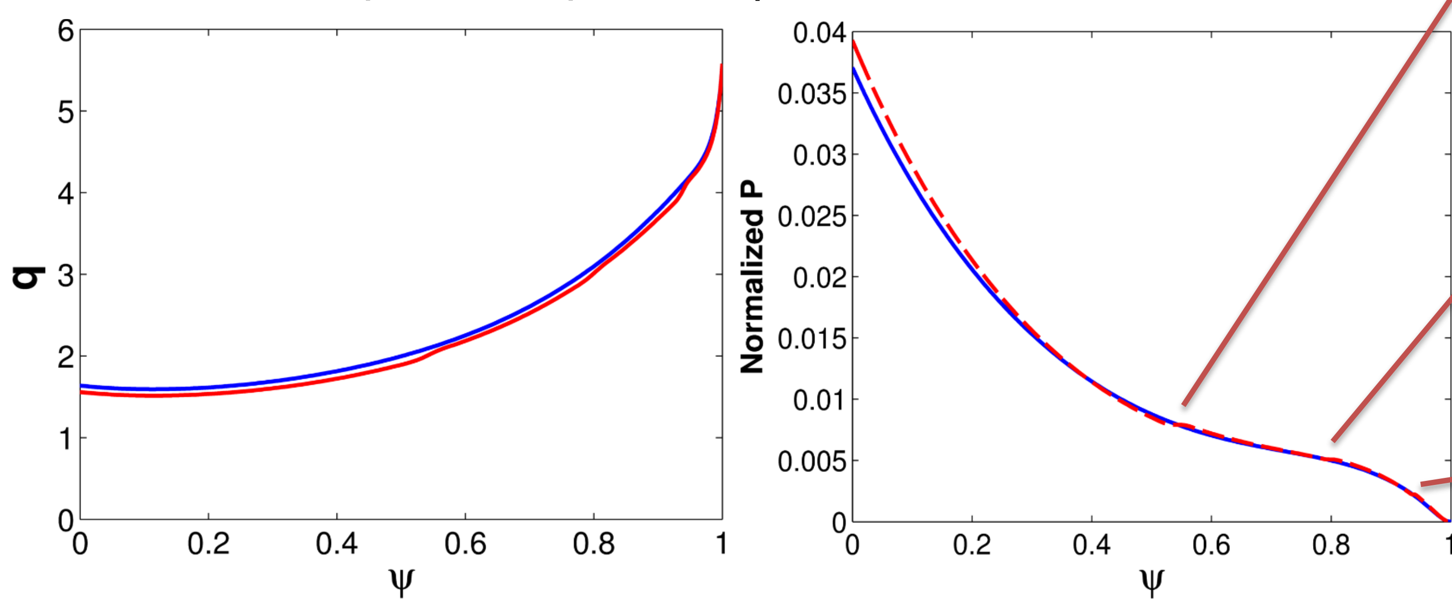
Small solutions driven by external perturbations

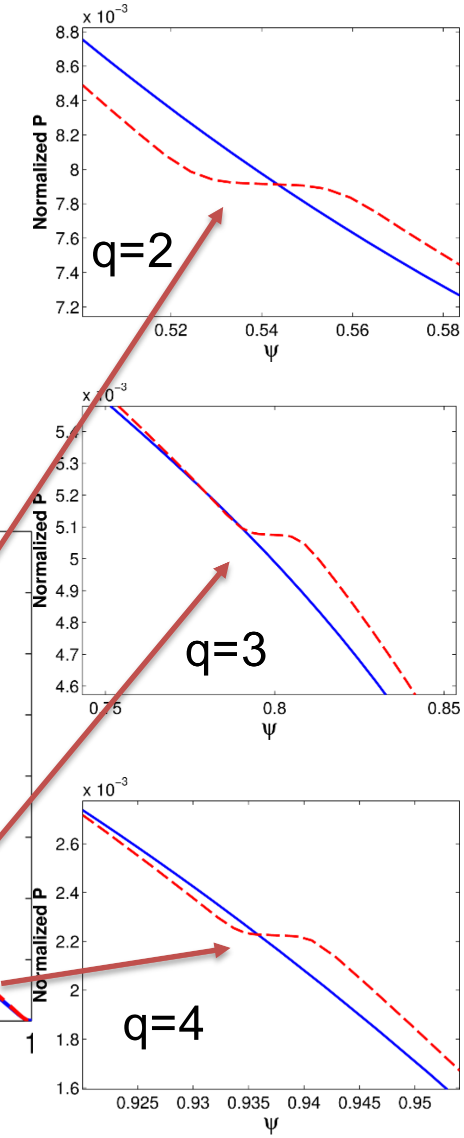
Resistive Perturbed Equilibrium Code (**RPEC**) couples resistive DCON with IPEC

Relax Perturbed Equilibrium Assumes Pressure Flattening (Decrease ∇P_0) At Resonant Surface

- DIII-D like equilibrium has been used to test resistive/perturbed equilibrium.
- The equilibrium with the flattened pressure profile at resonant surface is solved by CHEASE to simulate the relax perturbed equilibrium with RPEC.

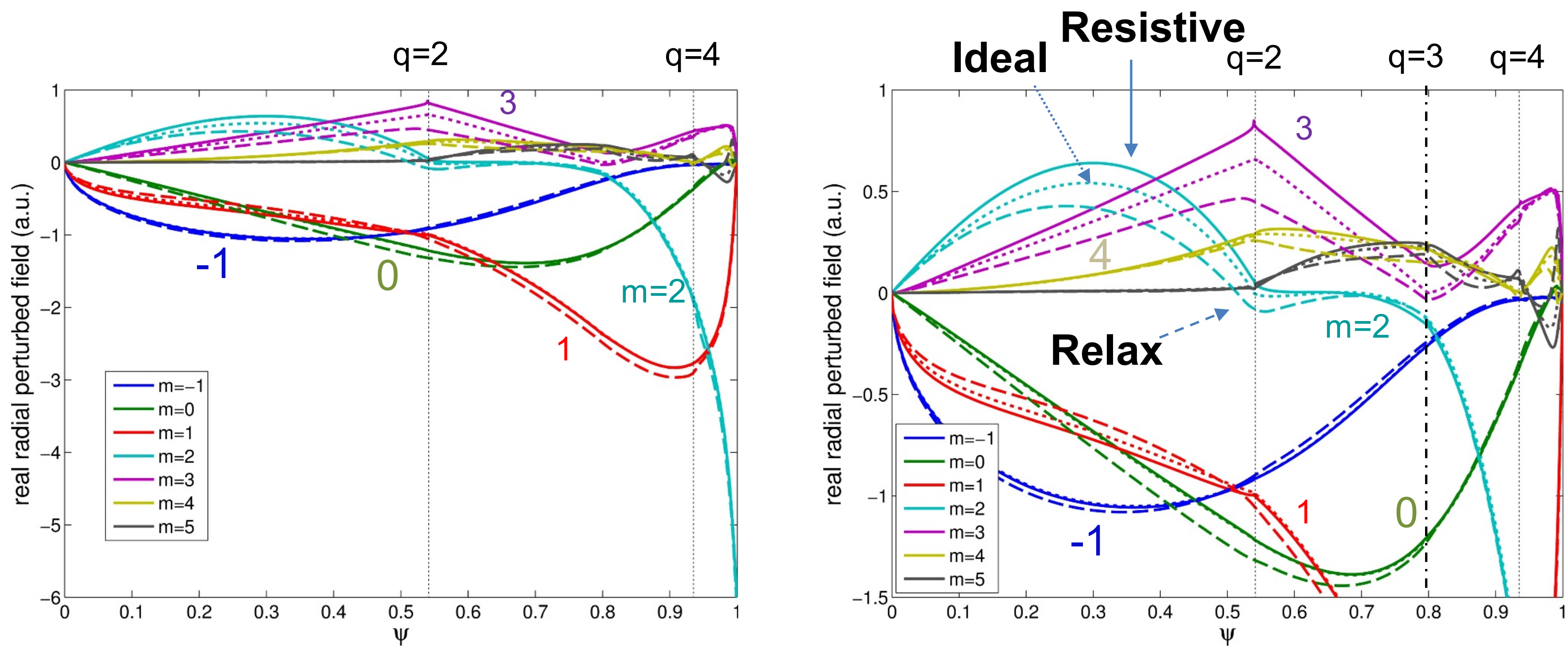
$q_0 = 1.56$, $q_a = 5.57$, $\beta_N = 1.97$, $A = 2.75$





Example of Resistive/Relaxed Perturbed Equilibrium Solved by RPEC

- Resistive/Relax perturbed equilibria clearly show finite δb_r at q=2,3 which means the magnetic islands open at the resonant surfaces while applying the external perturbations.
- The existence of island can change the magnetic perturbation globally.



Island width ~ $\sqrt{\delta b_r}$

Resistive DCON and RPEC Codes Have Been Successfully Developed

- Outer region: Resistive DCON has been successfully developed with resonant Galerkin method.
  - Provide the outer region Δ' matrix and give deep understanding of coupling effects among different resonant surfaces.
- Inner region: DELTAC code has been developed to reliably solve the inner region in configuration space instead of Fourier space.
  - Flexibility to include more inner region physics (Glasser, NP 12.00013, Wednesday 9:30am)
- Matching: MATCH code has been developed to match inner and outer region based on full matrix dispersion relation in toroidal geometry.
- Resistive DCON, DELTAC and MATCH codes have been coupled to IPEC code to solve resistive/relax perturbed equilibrium (RPEC).
- Resistive DCON and RPEC are ready to apply to study the experimental cases such as DIII-D and NSTX-U tokamaks.