Model Predictive Control with Integral Action for Current Density Profile Tracking in NSTX-U

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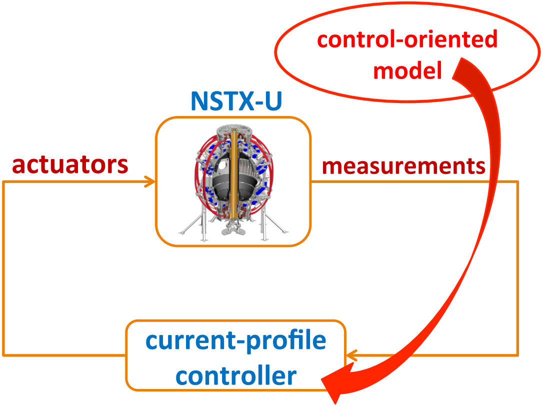
Motivation for Current Density Profile Control in NSTX-U

- **Advanced Tokamak (AT) operational goals** for the **NSTX-U** include **[1]**:
  - Non-inductive sustainment of high-β plasmas in spherical torus. (fusion power scales as $P_{fus} \approx \beta^2 B^4$)
  - High performance equilibrium scenarios with neutral beam heating.
  - Longer pulse durations.
- **Active, model-based, feedback control** of the **current density profile** evolution can be useful to achieve these AT operational goals.
- **The rotational transform (ι-profile)**, which is related to the toroidal **current density profile** in the machine, plays an important role in the stability and performance of a given magnetic configuration.
- Availability of the additional neutral beam current sources ***enables feedback control of the ι-profile*** in NSTX-U.

[1] GERHARDT, S. P. et al., Nuclear Fusion (2012).

Control-Oriented Current Profile Modeling

- ***Modeling for control design and not for physical understanding!***
- **The control-oriented model** only needs to **capture the dominant effects** of the actuators on the current profile evolution.
- Control-oriented model is **embedded in current-profile controller**.



Safety Factor, Rotational Transform, and Poloidal Flux

- Based on a magnetic description, relation between q-profile and the toroidal current density (j_ϕ) profile can be written as **[2]**

$$q(\hat{\rho},t) = \frac{\hat{\rho}^2 B_\phi}{R_0 \mu_0} \frac{1}{\int_0^{\hat{\rho}} j_\phi(\hat{\rho}',t)\hat{\rho}' d\hat{\rho}'} = \frac{2\pi \hat{\rho}^2 B_\phi}{R_0 \mu_0 I(\hat{\rho},t)} \quad (1)$$

- Using $\Phi = \pi B_{\phi,0} \rho^2$ and $\hat{\rho} = \rho/\rho_b$, where ρ_b is the mean effective minor radius of the last closed magnetic flux surface

$$q(\hat{\rho},t) = -\frac{d\Phi}{d\Psi} = -\frac{d\Phi}{2\pi d\psi} = -\frac{\frac{\partial \Phi}{\partial \rho}\frac{\partial \rho}{\partial \hat{\rho}}}{2\pi \frac{\partial \psi}{\partial \hat{\rho}}} = -\frac{B_{\phi,0}\rho_b^2 \hat{\rho}}{\partial \psi / \partial \hat{\rho}} \quad (2)$$

- Combining (1) and (2), poloidal magnetic flux profile (ψ) can be related to the toroidal current density profile (j_ϕ) through the safety factor (q) or rotational transform (ι) profile

$$\frac{\partial \psi}{\partial \hat{\rho}} \longrightarrow \boxed{\iota(\hat{\rho},t) = \frac{2\pi}{q(\hat{\rho},t)}} \longrightarrow j_\phi(\hat{\rho},t)$$

[2] J. Wesson, *Tokamaks.* Clarendon Press, Oxford, UK, 1984.

Magnetic Diffusion Equation

- The evolution of the **poloidal magnetic flux** is given by the **Magnetic Diffusion Equation [3]:**

$$\boxed{\frac{\partial \psi}{\partial t} = \frac{\eta(T_e)}{\mu_0 \rho_b^2 \hat{F}^2} \frac{1}{\hat{\rho}} \frac{\partial}{\partial \hat{\rho}} \left(\hat{\rho} D_\psi \frac{\partial \psi}{\partial \hat{\rho}} \right) + R_0 \hat{H} \eta(T_e) \frac{< \bar{j}_{NI} \cdot \bar{B} >}{B_{\phi,0}},} \qquad (3)$$

with the boundary conditions:

$$\left.\frac{\partial \psi}{\partial \hat{\rho}}\right|_{\hat{\rho}=0} = 0, \qquad \left.\frac{\partial \psi}{\partial \hat{\rho}}\right|_{\hat{\rho}=1} = -\frac{\mu_0}{2\pi} \frac{R_0}{\left.\hat{G}\right|_{\hat{\rho}=1} \left.\hat{H}\right|_{\hat{\rho}=1}} I(t), \qquad (4)$$

where $D_\psi(\hat{\rho}) = \hat{F}(\hat{\rho})\hat{G}(\hat{\rho})\hat{H}(\hat{\rho})$, and $\hat{F}$, $\hat{G}$, $\hat{H}$ are geometric factors pertaining to the magnetic configuration of a particular equilibrium.

[3] OU, Y., et al., Fusion Engineering and Design (2007).

Simplified models for

$n_e, T_e, \eta, \bar{J}_{NI}$

$$\frac{\partial \psi}{\partial t} = \frac{\eta(T_e)}{\mu_0 \rho_b^2 \hat{F}^2} \frac{1}{\hat{\rho}} \frac{\partial}{\partial \hat{\rho}} \left(\hat{\rho} D_\psi \frac{\partial \psi}{\partial \hat{\rho}} \right) + R_0 \hat{H} \eta(T_e) \frac{\langle \bar{J}_{NI} \cdot \bar{B} \rangle}{B_{\phi,0}}$$

Magnetic Diffusion Equation

First − Principles − Driven (FPD)
Control − oriented Current Profile Evolution Model

Accurate Simplified Models are Used for Plasma Parameters

- NSTX-U-tailored empirical models **[4, 5]** for the electron temperature, electron density, plasma resistivity, and noninductive current drives **[6]** takes the form

$$n_e(\hat{\rho},t) = n_e^{prof}(\hat{\rho})u_n(t) \tag{5}$$

$$T_e(\hat{\rho},t) = k_{T_e}(\hat{\rho},t_r)\frac{T_e^{prof}(\hat{\rho},t_r)}{n_e(\hat{\rho},t)}I(t)\sqrt{P_{tot}(t)} \tag{6}$$

$$\eta(T_e) = \frac{k_{sp}(\hat{\rho},t_r)Z_{eff}}{T_e(\hat{\rho},t)^{3/2}} \tag{7}$$

$$\begin{aligned}\frac{\langle \bar{j}_{ni}\cdot\bar{B}\rangle}{B_{\phi,0}} &= \sum_{i=1}^{6}\frac{\langle \bar{j}_{nbi_i}\cdot\bar{B}\rangle}{B_{\phi,0}} + \frac{\langle \bar{j}_{bs}\cdot\bar{B}\rangle}{B_{\phi,0}} \\ &= \sum_{i=1}^{6} k_i^{prof}(\hat{\rho}) j_i^{dep}(\hat{\rho})\frac{\sqrt{T_e(\hat{\rho},t)}}{n_e(\hat{\rho},t)}P_i(t) \\ &+ \frac{k_{JeV}R_0}{\hat{F}(\hat{\rho})}\left(\frac{\partial\psi}{\partial\hat{\rho}}\right)^{-1}\left[2\mathcal{L}_{31}T_e\frac{\partial n_e}{\partial\hat{\rho}} + \{2\mathcal{L}_{31}+\mathcal{L}_{32}+\alpha\mathcal{L}_{34}\}n_e\frac{\partial T_e}{\partial\hat{\rho}}\right]\end{aligned} \tag{8}$$

[4] ILHAN, Z. O. et al., 29th Symposium on Fusion Technology (SOFT) (2016)
[5] ILHAN, Z. O. et al., 55th Annual Meeting of the APS DPP (2013)
[6] SAUTER, O. et al., Physics of Plasmas (1999), (2002)

Magnetic Diffusion Equation in Control-oriented Form

$$\frac{\partial \psi}{\partial t} = \frac{\eta(T_e)}{\mu_0 \rho_b^2 \hat{F}^2} \frac{1}{\hat{\rho}} \frac{\partial}{\partial \hat{\rho}} \left(\hat{\rho} D_\psi \frac{\partial \psi}{\partial \hat{\rho}} \right) + R_0 \hat{H} \eta(T_e) \frac{< \bar{j}_{NI} \cdot \bar{B} >}{B_{\phi,0}}$$

$\Downarrow n_e, T_e, \eta, \bar{j}_{NI}$ (simplified models)

$$\frac{\partial \psi}{\partial t} = f_\eta(\hat{\rho}) u_\eta(t) \frac{1}{\hat{\rho}} \frac{\partial}{\partial \hat{\rho}} \left(\hat{\rho} D_\psi(\hat{\rho}) \frac{\partial \psi}{\partial \hat{\rho}} \right) + \sum_{i=1}^{6} f_i(\hat{\rho}) u_i(t) + f_{bs}(\hat{\rho}) u_{bs}(t) \left(\frac{\partial \psi}{\partial \hat{\rho}} \right)^{-1}, \quad (9)$$

where the boundary conditions are $\left.\frac{\partial \psi}{\partial \hat{\rho}}\right|_{\hat{\rho}=0} = 0$ and $\left.\frac{\partial \psi}{\partial \hat{\rho}}\right|_{\hat{\rho}=1} = -f_b I(t)$.

- Spatial functions $f_\eta, f_1, \ldots, f_6, f_{bs}, f_b$ are expressed in terms of the various plasma model reference profiles and constants.
- Time functions on the RHS of of (9)

$$\bar{u}(t) = [u_\eta,\ u_1,\ u_2,\ u_3,\ u_4,\ u_5,\ u_6,\ u_{bs},\ I]^T \in \mathbb{R}^{9\times 1}$$

are the nonlinear combinations of the physical actuators, i.e., $\bar{u}(t) = p(u(t))$, where

$$u(t) = [u_n,\ P_1,\ P_2,\ P_3,\ P_4,\ P_5,\ P_6,\ I]^T \in \mathbb{R}^{8\times 1}$$

Overview of Model Predictive Control (MPC)

- MPC is an **optimal control strategy** based on **numerical optimization.**
- The main advantage of MPC over PID and LQ-optimal control techniques is the explicit handling of actuator and state constraints **[7].**
- ***MPC is proactive*** **[8]** as it recalculates the optimal input sequence online at each time step by considering both input and state constraints.
- It eliminates the need for anti-windup augmentation and high level of skill and experience required for the tuning of the controllers **[9].**

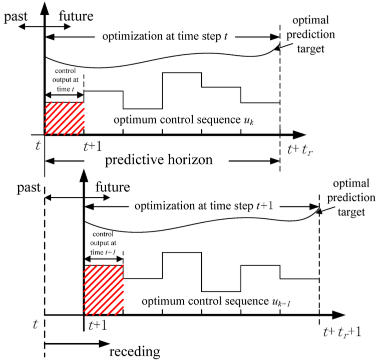
[7] CAMACHO, E. F. and BORDONS, C. *Model Predictive Control.* Springer-Verlag, London, UK (1999)

[8] MACIEJOWSKI, J. M., *Predictive Control With Constraints.* Prentice-Hall, Harlow, UK (2002)

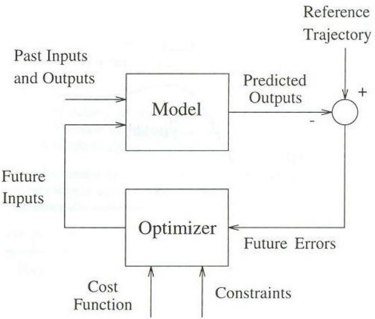
[9] STEPHENS, M. A. et al., IEEE Transactions on Industrial Informatics (2013)

MPC Strategy

1. A **dynamic model** of the system is used to predict the system output for a future time horizon.
2. Control sequence is calculated to **optimize an objective function.**
3. ***Receding strategy:*** Only first element of the control sequence is applied at each step!



Hu, C. et al., Energies (2015)



Camacho and Bordons, Springer-Verlag (1999)

PDE model is reduced and linearized for MPC formulation

- **The discrete-time, LTI model** for the ι-profile evolution in NSTX-U can be obtained by **discretizing the PDE model (9) in space,** and then **linearizing** it around the reference state (ι_r) and input (u_r) trajectories

$$\tilde{\iota}(k+1) = \bar{A}_d\,\tilde{\iota}(k) + \bar{B}_d\,\tilde{u}(k), \tag{10}$$

$$y(k) = \bar{C}_d\,\tilde{\iota}(k), \tag{11}$$

where, $\tilde{u}(k) = u(k) - u_r(k)$ and $\tilde{\iota}(k) = \iota(k) - \iota_r(k)$.

- Rewrite (10)-(11) in terms of the **state increment,** $\Delta\tilde{\iota}(k+1)$ and **output increment,** $\Delta y(k+1)$ so that input is the **control increment,** $\Delta\tilde{u}(k)$**.**

$$\begin{aligned}
\Delta\tilde{\iota}(k+1) &= \tilde{\iota}(k+1) - \tilde{\iota}(k) && (12)\\
&= \bar{A}_d\,\tilde{\iota}(k) + \bar{B}_d\,\tilde{u}(k) - \left[\bar{A}_d\,\tilde{\iota}(k-1) + \bar{B}_d\,\tilde{u}(k-1)\right] && (13)\\
&= \bar{A}_d \underbrace{\left[\tilde{\iota}(k) - \tilde{\iota}(k-1)\right]}_{\Delta\tilde{\iota}(k)} + \bar{B}_d \underbrace{\left[\tilde{u}(k) - \tilde{u}(k-1)\right]}_{\Delta\tilde{u}(k)} && (14)
\end{aligned}$$

$$\begin{aligned}
\Delta y(k+1) &= y(k+1) - y(k) && (15)\\
&= \bar{C}_d\left[\tilde{\iota}(k+1) - \tilde{\iota}(k)\right] && (16)\\
&= \bar{C}_d\bar{A}_d\Delta\tilde{\iota}(k) + \bar{C}_d\bar{B}_d\Delta\tilde{u}(k) && (17)
\end{aligned}$$

Control increment ($\Delta\tilde{u}$) is used as input for offset-free tracking

- The state-space model in incremental form becomes

$$\Delta\tilde{\iota}(k+1) = \bar{A}_d \Delta\tilde{\iota}(k) + \bar{B}_d \Delta\tilde{u}(k) \quad (18)$$

$$\underbrace{\Delta y(k+1)}_{y(k+1)-y(k)} = \bar{C}_d \bar{A}_d \Delta\tilde{\iota}(k) + \bar{C}_d \bar{B}_d \Delta\tilde{u}(k) \quad (19)$$

- Defining a new (enlarged) state variable as $x(k) = \begin{bmatrix} \Delta\tilde{\iota}(k) \\ y(k) \end{bmatrix}$, equations (18) and (19) are combined together to form

$$\underbrace{\begin{bmatrix} \Delta\tilde{\iota}(k+1) \\ y(k+1) \end{bmatrix}}_{x(k+1)} = \underbrace{\begin{bmatrix} \bar{A}_d & 0_{n\times m} \\ \bar{C}_d \bar{A}_d & I_{m\times m} \end{bmatrix}}_{\widetilde{A}} \underbrace{\begin{bmatrix} \Delta\tilde{\iota}(k) \\ y(k) \end{bmatrix}}_{x(k)} + \underbrace{\begin{bmatrix} \bar{B}_d \\ \bar{C}_d \bar{B}_d \end{bmatrix}}_{\widetilde{B}} \Delta\tilde{u}(k) \quad (20)$$

- The enlarged plant can then be written as

$$x(k+1) = \widetilde{A} x(k) + \widetilde{B} \Delta\tilde{u}(k), \quad (21)$$

$$y(k) = \widetilde{C} x(k), \quad (22)$$

where, $\widetilde{C} = \begin{bmatrix} 0_{m\times n} & I_{m\times m} \end{bmatrix}$.

Predicted control increments are related to the predicted outputs

$$y(k+1) = \widetilde{C}\widetilde{A}x(k) + \widetilde{C}\widetilde{B}\Delta\tilde{u}(k)$$

$$y(k+2) = \widetilde{C}\widetilde{A}^2x(k) + \widetilde{C}\widetilde{A}\widetilde{B}\Delta\tilde{u}(k) + \widetilde{C}\widetilde{B}\Delta\tilde{u}(k+1)$$

$$\vdots$$

$$y(k+N) = \widetilde{C}\widetilde{A}^Nx(k) + \widetilde{C}\widetilde{A}^{N-1}\widetilde{B}\Delta\tilde{u}(k) + \widetilde{C}\widetilde{A}^{N-2}\widetilde{B}\Delta\tilde{u}(k+1) + \ldots + \widetilde{C}\widetilde{B}\Delta\tilde{u}(k+N-1)$$

Prediction Model (PM):

$$\boxed{y_{k+1|N} = O_N\widetilde{A}x(k) + F_N\Delta\tilde{u}_{k|N}}\,, \tag{23}$$

$$y_{k+1|N} = \begin{bmatrix} y(k+1) \\ y(k+2) \\ \vdots \\ y(k+N) \end{bmatrix}, \quad \Delta\tilde{u}_{k|N} = \begin{bmatrix} \Delta\tilde{u}(k) \\ \Delta\tilde{u}(k+1) \\ \vdots \\ \Delta\tilde{u}(k+N-1) \end{bmatrix}, \quad O_N = \begin{bmatrix} \widetilde{C} \\ \widetilde{C}\widetilde{A} \\ \vdots \\ \widetilde{C}\widetilde{A}^{N-1} \end{bmatrix} \tag{24}$$

$$F_N = \begin{bmatrix} \widetilde{C}\widetilde{B} & 0 & 0 & 0 & \cdots & 0 \\ \widetilde{C}\widetilde{A}\widetilde{B} & \widetilde{C}\widetilde{B} & 0 & 0 & \cdots & 0 \\ \widetilde{C}\widetilde{A}^2\widetilde{B} & \widetilde{C}\widetilde{A}\widetilde{B} & \widetilde{C}\widetilde{B} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & 0 \\ \vdots & \vdots & \vdots & & \ddots & 0 \\ \widetilde{C}\widetilde{A}^{N-1}\widetilde{B} & \widetilde{C}\widetilde{A}^{N-2}\widetilde{B} & \cdots & \cdots & \widetilde{C}\widetilde{A}\widetilde{B} & \widetilde{C}\widetilde{B} \end{bmatrix}. \tag{25}$$

- Note that tracking problem for $\iota(k)$ becomes a regulation problem for $y(k)$

$$y(k) = \bar{C}_d \underbrace{\tilde{\iota}(k)}_{\iota(k)-\iota_r(k)} , \qquad \boxed{\iota(k) \rightarrow \iota_r(k) \quad \Rightarrow \quad y(k) \rightarrow 0}$$

- The performance index penalizes both the **predicted tracking error** and the **predicted changes to the control input**

$$\boxed{J(k) = \sum_{i=1}^{N} y(k+i)^T Q y(k+i) + \Delta\tilde{u}(k+i-1)^T R \Delta\tilde{u}(k+i-1)} \quad (26)$$

$$\Downarrow \quad y_{k+1|N} = O_N \widetilde{A} x(k) + F_N \Delta\tilde{u}_{k|N} \quad (\textbf{PM})$$

$$\boxed{J(k) = \Delta\tilde{u}_{k|N}^T H \Delta\tilde{u}_{k|N} + 2x^T(k) f^T \Delta\tilde{u}_{k|N} + J_0} , \quad (27)$$

where

$$H = F_N^T \widetilde{Q} F_N + \widetilde{R}, \quad (28)$$

$$f = F_N^T \widetilde{Q} O_N \widetilde{A}, \quad (29)$$

Solution of the MPC problem requires Quadratic Programming

- **Future feedback control increments** $(\Delta\tilde{u}^*_{k|N})$ are obtained by minimizing the **quadratic performance index** while satisfying the **input constraints**, i.e.,

$$\Delta\tilde{u}^*_{k|N} = \text{arg} \min_{\Delta\tilde{u}_{k|N}} \left\{ \Delta\tilde{u}^T_{k|N} H \Delta\tilde{u}_{k|N} + 2x^T(k) f^T \Delta\tilde{u}_{k|N} \right\} \quad (30)$$

$$\text{subject to} \quad \mathcal{A}\Delta\tilde{u}_{k|N} \leq b_k \quad (31)$$

- (30)-(31) define a standard **Q**uadratic **P**rogramming (**QP**) problem.
- A ***receding horizon strategy*** is used and only the first control increment $\Delta\tilde{u}^*(k)$ in the calculated $\Delta\tilde{u}^*_{k|N}$ is used for control.
- Optimal feedback control action becomes

$$\tilde{u}(k) = \Delta\tilde{u}^*(k) + \tilde{u}(k-1). \quad (32)$$

Closed-Loop Integral MPC Simulation Study

- The target state trajectory $\iota_r(\rho, t)$ is generated through an open-loop TRANSP simulation with the following constant reference inputs.

$\mathbf{n_e}(\mathbf{m^{-3}})$	5.0×10^{19}	$\mathbf{P_4}(\mathbf{W})$	0.8×10^{6}
$\mathbf{P_1}(\mathbf{W})$	0.2×10^{6}	$\mathbf{P_5}(\mathbf{W})$	1.0×10^{6}
$\mathbf{P_2}(\mathbf{W})$	0.4×10^{6}	$\mathbf{P_6}(\mathbf{W})$	1.2×10^{6}
$\mathbf{P_3}(\mathbf{W})$	0.6×10^{6}	$\mathbf{I_p}(\mathbf{A})$	0.7×10^{6}

- The prediction horizon is set to $N = 5$ to guarantee closed-loop stability.
- The initial condition perturbation rejection capability is tested by setting

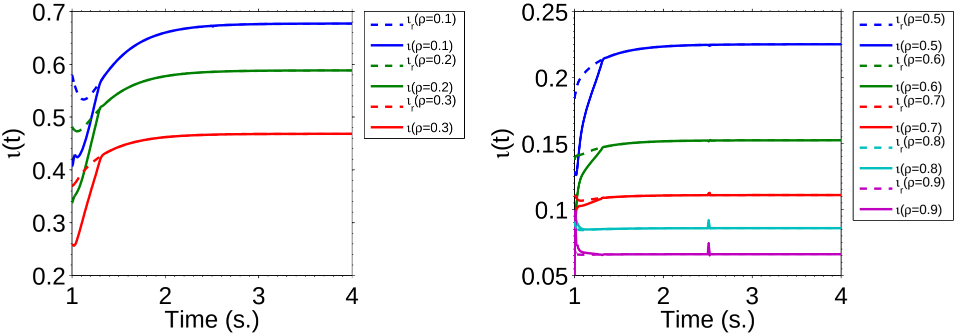
$$\iota(t_0) = \iota_r(t_0) + \delta\iota \tag{33}$$

- The controller is also tested against constant input disturbances starting from $t = 2.5$ s. i.e.,

$$\tilde{u}(k) = \begin{cases} \Delta\tilde{u}^*(k) + \tilde{u}(k-1), & t < 2.5\,\text{s.} \\ \Delta\tilde{u}^*(k) + \tilde{u}(k-1) + u_d, & t \geq 2.5\,\text{s.} \end{cases} \tag{34}$$

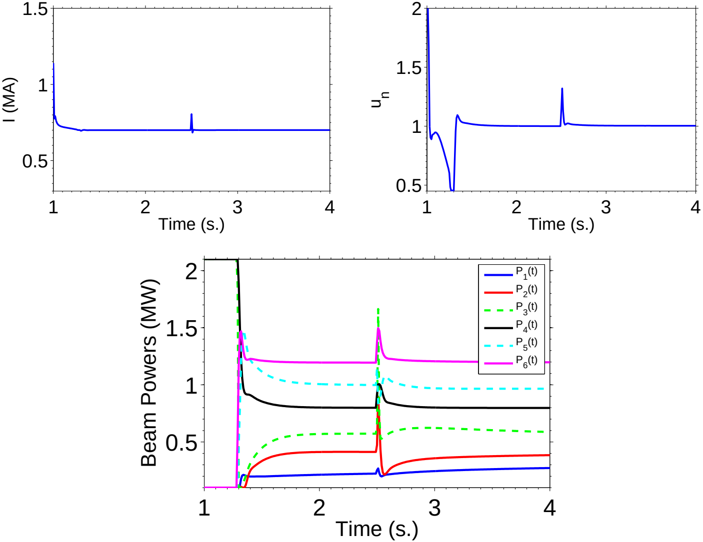
where $u_d = 0.15u_r$ stands for the constant disturbance inputs.

Numerical tests have guaranteed robust tracking performance



Figures: Time evolution of the optimal outputs (solid) with their respective targets (dashed) at selected radial locations.

Actuators instantly cancel the effect of the input disturbances



Upper Figures: Time evolution of the optimal plasma current (left), and optimal n_e regulation (right).

Lower Figure: Time evolution of the optimal neutral beam injection powers.

MPC regulates the ι-profile around a target profile in NSTX-U

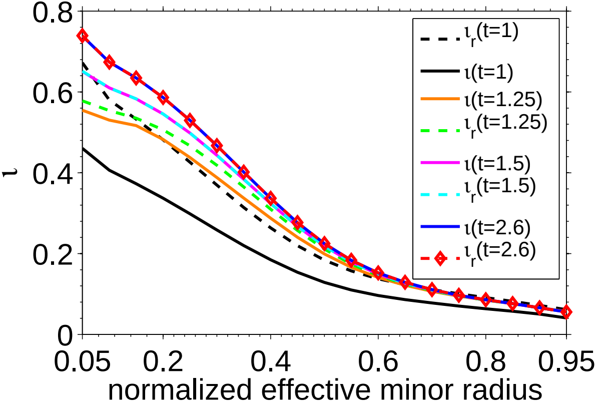


Figure: Time evolution of the rotational transform (ι-profile).

Conclusion and Future Work

- An **NSTX-U-tailored plasma response model** is obtained by combining the MDE with simplified models for various plasma variables.
- A constrained **MPC algorithm** is formulated based on the reduced-order, LTI model to regulate the rotational transform (ι-profile).
- **An integrator** is added to the MPC formulation to achieve offset-free tracking against modeling uncertainties and external disturbances.
- The proposed MPC control scheme is tested via closed-loop numerical simulations based on the control-oriented MDE solver.
- ***First MPC design for NSTX-U for current density profile control.***
  - explicitly handles input and state constraints
  - predicts plasma future state in real time based on current plasma state
  - may be crutial in achieving current profile control + MHD instability avoidance
- **Future work** includes:
  - **Refinement** of the FPD control-oriented model using actual experimental data once NSTX-U achieves relevant plasma scenarios.
  - **Implementation** of MPC algorithm in **TRANSP's Expert routine** and **PCS.**
  - **TRANSP closed-loop simulations** $\Rightarrow$ **Experimental testing in NSTX-U.**