Equilibrium and Stability Code (ESC)[1]

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Abstract

Existing 3-D equilibrium codes either use flux coordinates or rely on numerical field line tracing in describing topology of magnetic field. Assumption on existence of flux coordinates is the most week point of 3-D equilibrium theory and can be in conflict with the islands formation in the finite pressure plasma, especially in compact stellarator configurations. On the other hand, use of field line tracing creates problems with convergence in other type of equilibrium codes. These principal difficulties make 3-D equilibrium codes much less efficient than their 2-D analogs. Invention of the fast algorithm for Reference Magnetic Coordinates (RMC)[1] provides a rigorous basis for solving 3-D equilibrium problems in a similar manner as in tokamaks. Equilibrium and Stability Code (ESC) uses RMC in order to specify the current density, consistent with equilibrium constraints and magnetic topology, and to solve the field equations resulted from the perturbed equilibrium. Then, the solution for vector potential is used for advancing RMC. Use of perturbed equilibrium equations leads to the fast Newton scheme for solving nonlinear equilibrium equations (advancing RMC is already the Newton method by its nature). On the other hand, the perturbed equilibrium equations reproduce the marginal stability equations as a particular case, and, thus, unify the equilibrium and stability calculations.

[1] L. Zakharov, this meeting

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1. OUTLINE

1. Introduction.
2. 2-D theory of perturbed equilibrium.
3. 3-D Reference Magnetic Coordinates (RMC).
4. 3-D theory of perturbed equilibrium.
5. Equilibrium and Stability Code (ESC).
6. Comparison: VMEC / ESC2d.
7. 2-D free-boundary scheme.
8. Summary.

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2. Introduction

Grad-Shafranov (GSh) equation

$$\Delta^* \Psi = j(\Psi, r) \tag{2.1}$$

is a foundation of magnetic confinement. It is amasing, that while known since 50-s, this equation still has no theory for its solutions.

Presumabley, because the simplest iteration scheme (practically, the only one existing)

$$\Delta^* \Psi^{n+1} = j(\Psi^n, r) \tag{2.2}$$

converges well, there were no attemps to make such a theory.

Nethertheless, there are needs in fast and accurate equilibrium solvers, e.g.,
-for transport codes (TRANSP, ASTRA, etc)
-for extensive comparison theory models against experimental data bases
-for real time control of big machines, etc

Here, we present the theory of solving GSh equiation.

3. 2-D theory of perturbed equilibrium

The simplest (and naive) way of linearization of the GSh equation

$$\Psi \equiv \Psi_0 + \psi, \quad \Delta^* \Psi = j(\Psi_0, r) + \frac{\partial j(\Psi_0, r)}{\partial \Psi_0} \psi \tag{3.1}$$

does not represent the rigorous theory of its solution and gives only modest improvement in convergence.

In curvilinear coordinates (with toroidal topology)

$$r = r(a, \tau, \phi), \quad z = z(a, \tau, \phi) \tag{3.2}$$

the GSh equation has the form

$$\begin{aligned} &\mathcal{L}\bar{\Psi} = -rDP - \frac{D}{r}T, \quad \bar{\Psi} \equiv \frac{\Psi}{2\pi}, \quad D \equiv \frac{D(r,z)}{D(a,\tau)}, \\ &P \equiv \frac{\mu_0 dp}{d\Psi}, \quad T \equiv \frac{r B_{tor} d(r B_{tor})}{d\Psi}, \end{aligned} \tag{3.3}$$

where $\mathcal{L}$ is the partial differential operator

$$\begin{aligned} \mathcal{L} \equiv &\frac{\partial}{\partial a}\frac{g_{\tau\tau}}{rD}\frac{\partial}{\partial a} - \frac{\partial}{\partial a}\frac{g_{a\tau}}{rD}\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \tau}\frac{g_{a\tau}}{rD}\frac{\partial}{\partial a} + \frac{\partial}{\partial \tau}\frac{g_{aa}}{rD}\frac{\partial}{\partial \tau} \\ &= \frac{D}{r}\Delta^*. \end{aligned} \tag{3.4}$$

Assuming that a is close to flux coordinates, the perturbed GSh equation has the form

$$\begin{aligned}\bar{\Psi} \equiv & \bar{\Psi}_0 + \psi, \\ \mathcal{L}\bar{\Psi} = & -rDP - \frac{D}{r}T - rD\frac{P'(a)}{\bar{\Psi}'}\psi - \frac{D}{r}\frac{T'(a)}{\bar{\Psi}'}\psi \\ & - rD\delta P - \frac{D}{r}\delta T.\end{aligned} \tag{3.5}$$

Presence of two terms

$$\delta P = \delta P(a), \quad \delta T = \delta T(a)$$

in the RHS is essential for the theory of perturbed equilibrium.

Having the solution of perturbed equilibrium equation, the coordinate system can be advanced as

$$a \to a + \xi, \tag{3.6}$$

where ξ satisfies magnetic differential equation equation

$$\psi'_\tau + \Psi'_0 \xi'_\tau = 0, \tag{3.7}$$

which resembles the equation for flux coordinates

$$\Psi = const \quad \psi + \Psi'_0 \xi = const. \tag{3.8}$$

Two unknown 1D profiles δP, δT and zeroth harmonics ξ_0 of displacement (the third 1D-‘‘profile’’) represent all the freedom, the theory of perturbed equilibrium provides for formulating equilibrium problems:

Freedom in ξ_0, allows to choose different radial cordinates: geometrical a, $\sqrt{\Phi}$, $\sqrt{\Psi}$, $\sqrt{V}$, etc.

Freedom in δP, allows to imply different constraints on plasma pressure p: given $P = P(a)$, $p(a)$ or $p'(a)$, adiabatic p, etc.

Freedom in δT, allows to use different current density profiles: $T = T(a)$, $j_{\|}$, $< \mathbf{j} \cdot \mathbf{B} >$, $q(a)$, etc.

In particular, with

$$\begin{aligned} &\xi_0 = 0, \quad \delta P = -\gamma_0 \frac{(\mu_0 p \Psi_0)'}{\Psi_0'}, \\ &\delta T - \frac{(\bar{F}^2 X_0)_a'}{\bar{\Psi}_0'}, \quad X_0 \equiv \frac{\left(\frac{D\xi}{r}\right)'_{0a}}{\left(\frac{D}{r}\right)_0}, \end{aligned} \tag{3.9}$$

the theory reproduces the ideal energy principle (for n=0) with plasma inertia neglected.

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4. 3-D Reference Magnetic Coordinates (RMC)

In 3-D, the corresponding coordinate advancing scheme leads to Magnetic differential equation

$$\mathbf{B} \cdot \xi = \mathbf{B} \cdot \nabla a. \qquad (4.1)$$

Repetitive solving this equation for NON-resonant harmonics advances coordinate system to Reference Magnetic Coordinates, where the vector potential has the SIMPLEST possible representation

$$\mathbf{A} = A_a \nabla a + \bar{\Phi}_0(a) \nabla \tau + [\Psi_0(a) + \psi^r(a, \tau, \phi)] \nabla \phi \qquad (4.2)$$

with $\psi^r(a, \tau, \phi)$ containing only RESONANT harmonics.

This new notion of RMC, we introduced, allows to formulate a theory of perturbed 3D equilibria in a rigorous way.

In 2D, the pertirbed equilibrium theory may be considered as a ‘‘luxurious’’ way of solving GSh equation.

In 3D, it is the ONLY way of formulating self-consistent 3D equilibrium equations: the island size is intrinsically the eigenvalue of the perturbed equilibrium.

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5. Equilibrium and Stability Code (ESC)

ESC (‘‘escape’’) is based on the formulated theory.

- For tokamaks, ESC uses the most compact and efficient Fourier representation of flux coordinates

$$r = r_0^c + 2\sum_{m=1}^{m\le M}(r_m^c \cos m\tau + r_m^s \sin m\tau),$$
$$z = r_0^s + b\sin\tau, \quad r_m^{c,s} = r_m^{c,s}(a), \quad b = b(a) \tag{5.1}$$

- In 3D it will use 3D RMC.
- Options are to use different radial coordinates: geometrical a, $\sqrt{V}$, $\sqrt{\Phi}$, $\sqrt{\Psi}$.
- Different pressure equivalent profiles: $P(a)$, $p(a)$ are possible.
- ESC has options of use different current density equivalent profiles: $T(a)$, $q(a)$, $j_\parallel$, jB.

It uses shooting (sweeping) technique of solving ODEs for harmonics of $\Psi(a,\tau)$., which seemes to be the only possible way of solving perturbed equilibrium equations in their general form.

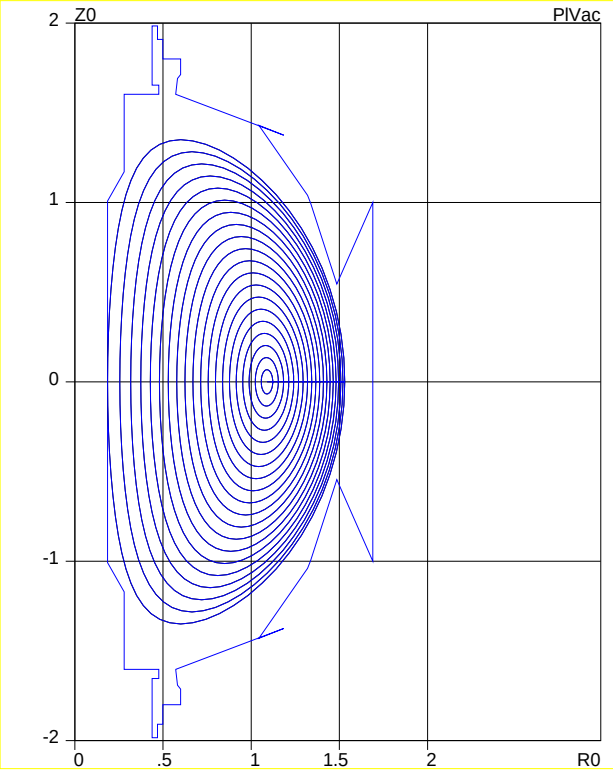
It ises spline representation for Fourier coefficients of coordinates as well as for all 1D-profiles.

ESC is a SELF-validating code, presumably the first one. As an option, it uses gridless LSODE radial solver, which gives the best test of consistency of the solution to the GSh equation.

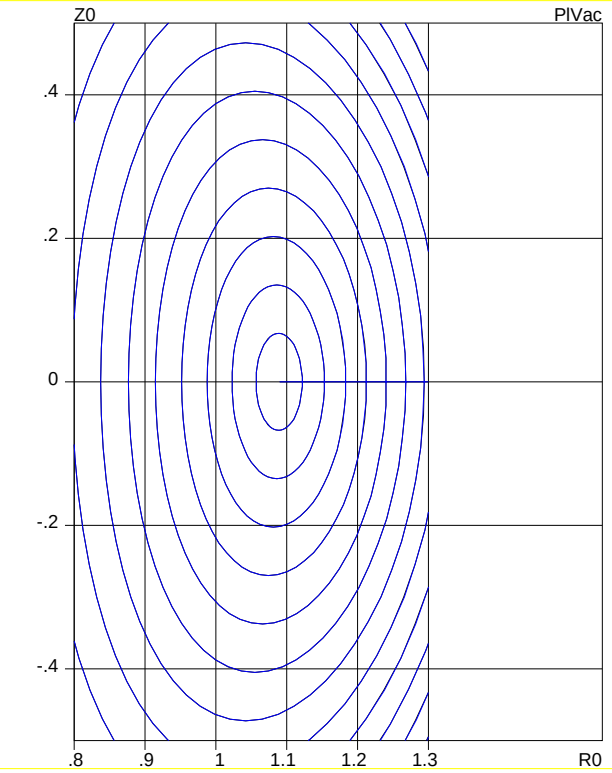
Together with its flexibility in inputs, ESC can be easily interfaced either with other codes or theory.

Figure shows TWO equilibrium configurations: the first is a Soloviev analytical solution (for NSTX aspect ratio) and the secondis ESC numerical equilibrium (with only 5 poloidal harmonics in the solution)

The solutions are undistinguishable.



Comparison of Soloviev equilibrium with ESC2d with M=5



Soloviev/ESC2d equilibrium, M=5 (calc. time=0.37 sec)

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6. Comparison: VMEC/ESC2d

Comprehensive comparison of VMEC (Vm6-version, used in TRANSP) with ESC2d has been performed and gave the following RESULTS:

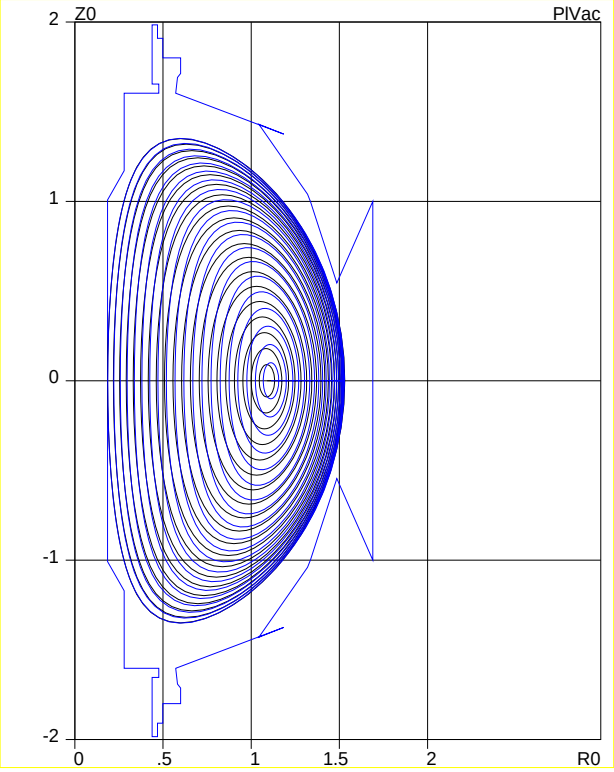
- On a set of simple Soloviev equilibria (R/a=2, b/a=1.6, d/a=0.2-0.5) ESC2d gives precise results (error is less than 1e-4 in all quantities, including first derivatives) even with only M=5 poloidal harmonics in Fourier representation. VMEC with M=10 deviates more than 1e-2 even in the shape of flux surfaces.
- For Soloviev equilibria, emulating NSTX geometry, VMEC with M=10 has shown unacceptable lack of accuracy, while ESC2d even with M=5 was precise.
- Even in the case of simple geometry, VMEC with M=10 failed to give acceptable accuracy for q-profiles, which deviates from Soloviev-like profiles.
- VMEC sometimes fails (with no warnings on failure) even in simple cases (e.g., with moderate raise in q value).
- Noise in data on p- and q-profiles leads to additional errors in VMEC. In ESC2d, the noise causes only more iterations.

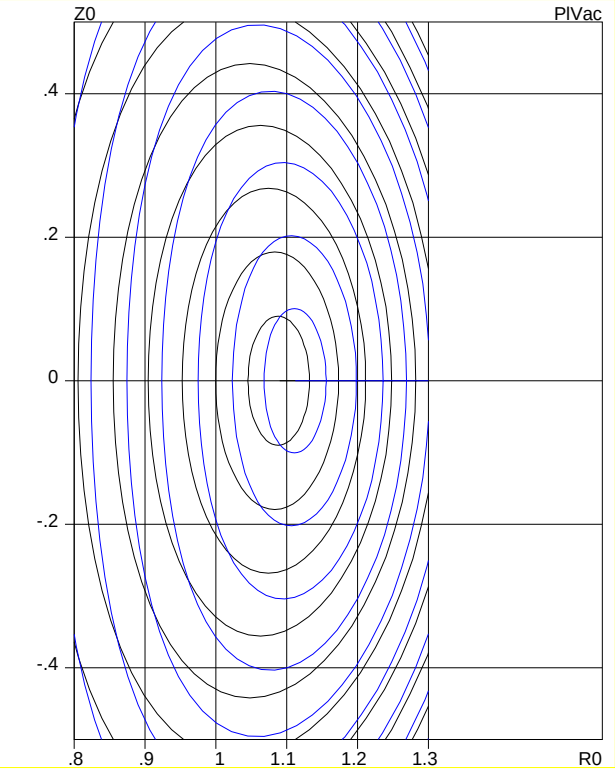
6. Comparison: VMEC/ESC2d (cont.)

- ESC2d with M=5 is at least 10 times faster than (less accurate) VMEC with M=10 (0.3 sec/3.8 sec, DEC-alpha). ESC2d with M=10 is at least 2 times faster than VMEC with M=10 (2 sec/3.8 sec).
- ESC2d has no limitations on number of harmonics. VMEC fails if M is more than 11.

Comparison of ESC2d with other codes is easily possible upon necessity.

Comparison VMEC/ESC2d on with use of exact Soloviev solution (for NSTX-like aspect ratio). Both codes use p and q as function of toroidal flux $\sqrt{\Phi}$. Esc reproduces exact solution (even with only 5 harmonics in its Fourier expansion).

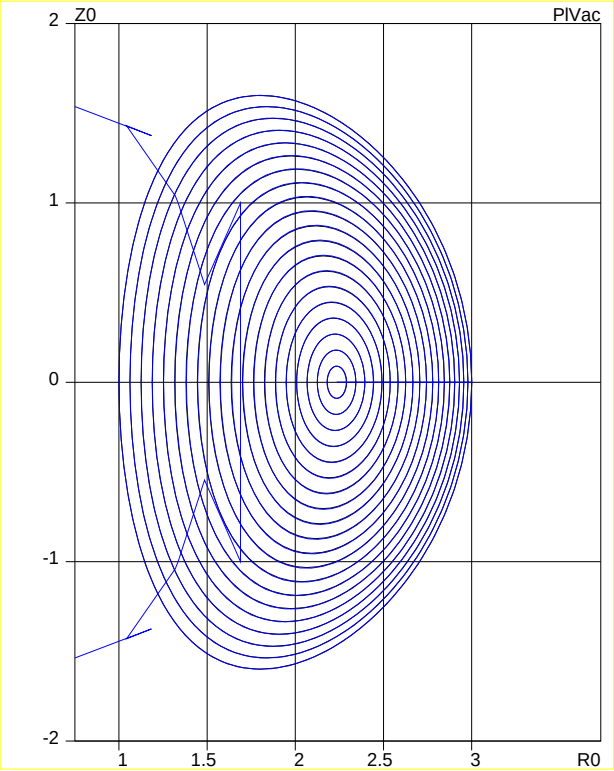




VMEC/ESC2d equilibrium (M=10) given a=sqrt(Phi), p,q

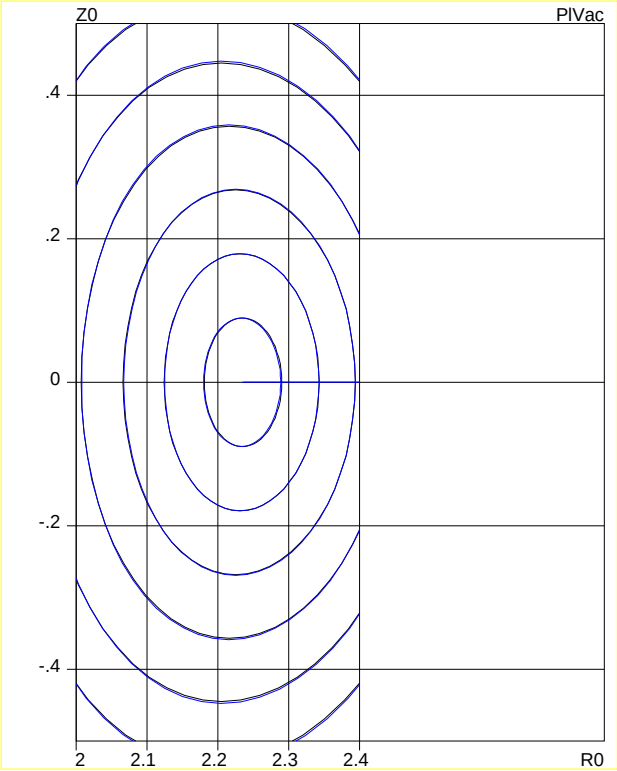
VMEC/ESC2d equilibrium (M=10, t_VMEC/t_ESC=5.3/2.1 sec)

Comparison VMEC/ESC2d on with use of exact Soloviev solution for mild shaping and aspect ratio.



VMEC(M=10, t=3.2 sec)/ESC2d(M=5 == M=10)/Soloviev #1

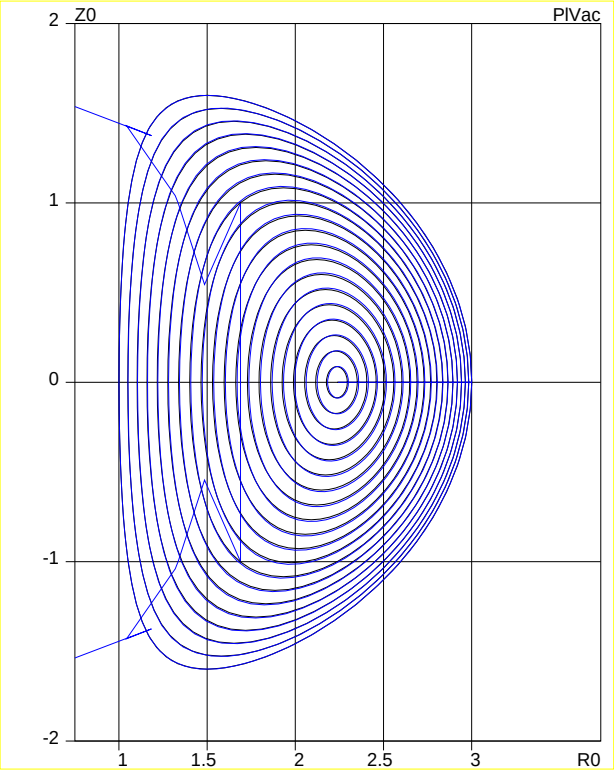
Page 2



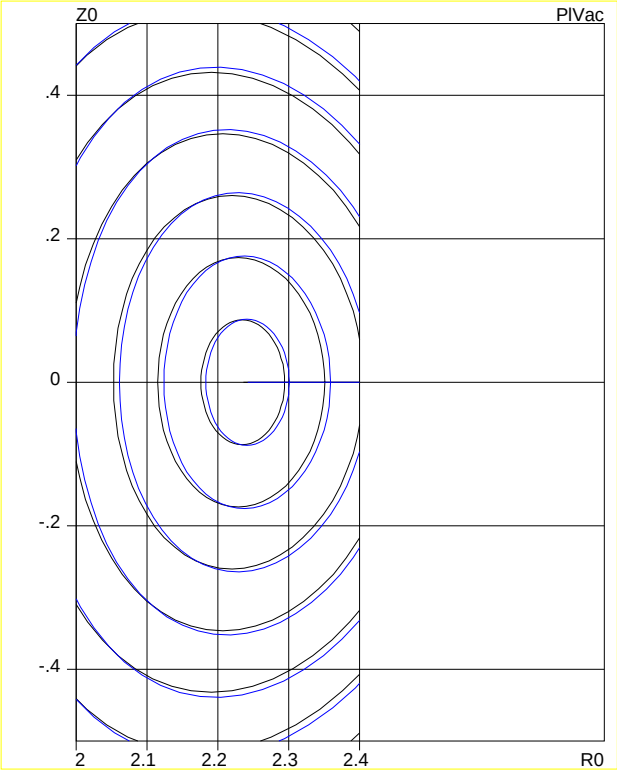
VMEC(M=10, t=3.2 sec)/ESC2d(M=5 == M=10)/Soloviev #1

Page 3

Comparison VMEC/ESC2d on with use of exact Soloviev solution for mild aspect ratio more argessive shape.

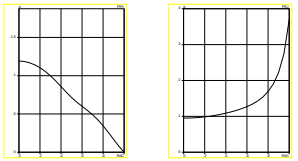


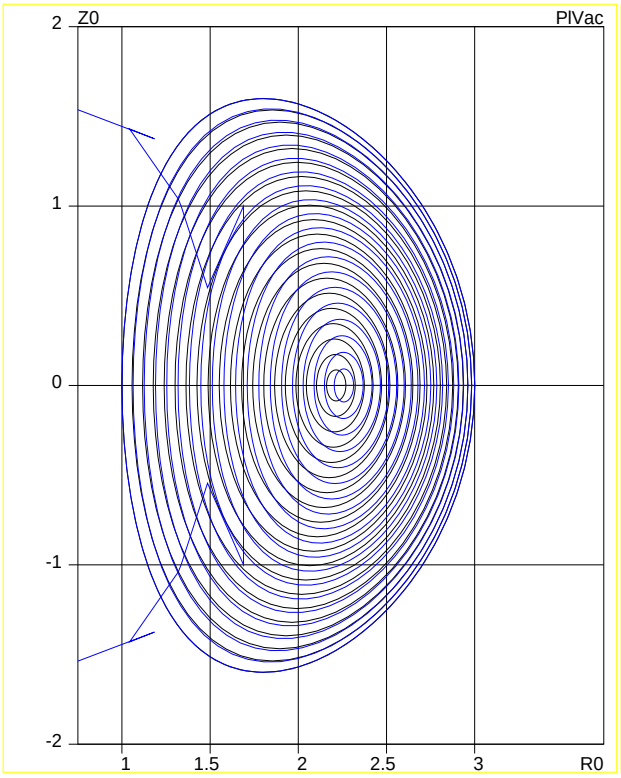
VMEC(M=10, t=3.4 sec)/ESC2d(M=5 == M=10)/Soloviev #4

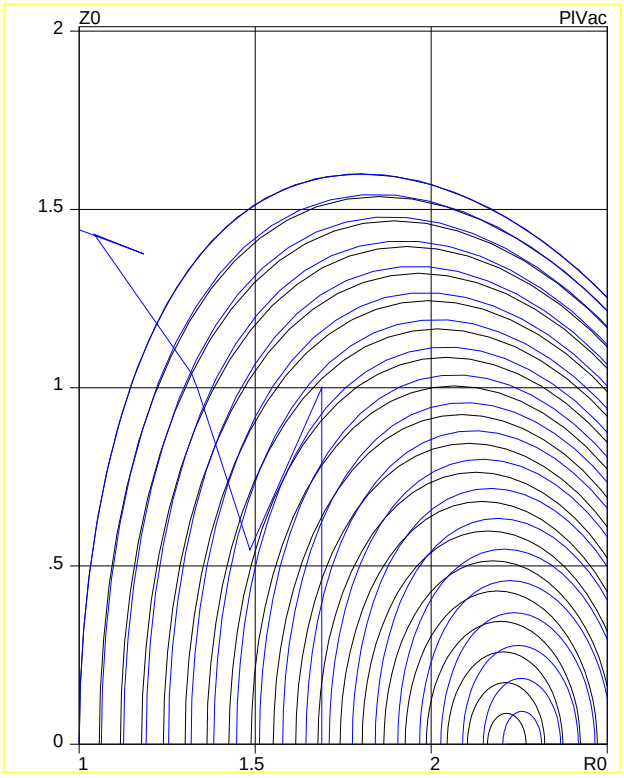


VMEC(M=10, t=3.4 sec)/ESC2d(M=5 == M=10)/Soloviev #4

Comparison VMEC/ESC2d on with use of Soloviev's shape and NON-Soloviev p and q profiles.







NON-Soloviev q-profile #5 VMEC(M=10)/ESC2d

Page 9

NON-Soloviev q-profile #5 VMEC(M=10)/ESC2d

Page 11

The CONCLUSIONS are:

- ESC2d provides prescribed accuracy of 1e-4 in the radial direction and good convergence on harmonics number M. This accuracy is sufficient for MHD and transport studies.
- M=8 in ESC2d, seems to be sufficient for present transport codes (ASTRA, TRANSP, etc) for all cases, including spherical tokamaks (some restriction may apply at the moment).
- VMEC (at least its TRANSP version) has shown unpredictable inaccuracy, which is unacceptable for both MHD and TRANSP purposes.
- ESC2d outperforms VMEC in all other aspects, such as:
  1. flexibility;
  2. compactness and precision of Fourier representation;
  3. speed;
  4. reliability (although, there are remaining problems);
  5. handling the noisy data;
  6. diagnostics of the solution.

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7. 2-D free-boundary scheme

External currents are calculated based on prescribed poloidal flux values in Control points together with minimization of the functional

$$
\begin{aligned}
W &= 2\sum_{i}^{i<I}\lambda_i\left(\sum_{k'}^{k'<K}\Psi^{*}_{ik'}\delta I_{k'}-\Psi_{cp}+\Psi_{ip}\right)\\
&+2\sum_{j}^{j<J}\lambda_{I+j}\left(\delta I_{k(j)}-\delta I^{ref}_{k(j)}\right)+2\lambda_{I+J}\left(\sum_{k'}^{k'<k}V_{2k'}\delta I_{k'}+\delta V_2^0\right)\\
&+\alpha^W_{pl}\alpha^W_{pl}\Psi_{PlV}I_{pl}+\sum_{k}^{k<K}\sum_{k'}^{k'<K}\Psi_{kk'}\alpha^W_k\alpha^W_{k'}I_{k'}I_k\\
&+2\alpha^W_{pl}\sum_{k'}^{k'<K}I_{k'}\left(\alpha^W_{k'}-\frac{1}{2}\alpha^W_{pl}\right)\left(\oint\psi_{sk'}\bar{i}_s dL_s+\sum_{k}^{k<K}I_k\frac{\delta}{\delta I_k}\oint\psi_{sk'}i_s dL_s\right)\\
&+\sum_{k}^{k<K}\left[\alpha^R_k\alpha^R_k\tau_k R_k I_k^2+\tau_k^2\frac{\alpha^V_k\alpha^V_k}{\Psi_{kk}}\left(V_k-V_k^{ref}\right)^2\right]\\
&+\sum_{k}^{k<K}\left[\alpha^I_k\alpha^I_k\Psi_{kk}\left(\delta I_k-\delta I_k^{ref}\right)^2+\frac{\alpha^\Psi_k\alpha^\Psi_k}{\Psi_{kk}}\left(\Psi_k-\Psi_k^{ref}\right)^2\right].
\end{aligned}
\tag{7.1}
$$

ESC has no problems with multiparameter control of the calculations.

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8. Summary

The perturbed equilibrium theory, presented here, constitutes the Newton (fastest) scheme for solving equilibrium equations in its explicit and the most efficient form. For evolving equilbrium configurations (typical for transport codes) it requires only one-two iterations to get the solution.

This theory unifies equilibrium and stability theory, existing at present as separate. (In particular, as a free gift, ESC2d gives comprehensive information about structure of axisymmetric modes when q is the input profile).

ESC2d opens the door for extensions of transport codes (ASTRA, TRANSP, etc). These extensions may now include the real time ideal and non-ideal stability tests.

Flexibility of ESC2d in inerfacing with other codes and its gridless representation of physical varables give the possibility for extensive comparison of experimental/transport data bases with theoretical models.