***Physics Meeting***

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# Distinct Ohmic Breakdown Physics in a Tokamak

**Min-Gu Yoo[1], Jeongwon Lee[1], Young-Gi Kim[1], Francesco Maviglia[2], Adrianus C. C. Sips[3,4], Jayhyun Kim[5], and Yong-Su Na[1*]**

***Corresponding author's E-mail: ysna@snu.ac.kr**

1) *Department of Nuclear Engineering, Seoul National University, Seoul, Korea*
2) *EURATOM-ENEA-CREATE, University of Naples Federico II, Napoli 80125, Italy*
3) *JET-EFDA, Culham Science Centre, Abingdon OX 14 3DB, U.K.*
4) *European Commission, Brussels 1049, Belgium*
5) *National Fusion Research Institute, Daejeon, Korea*

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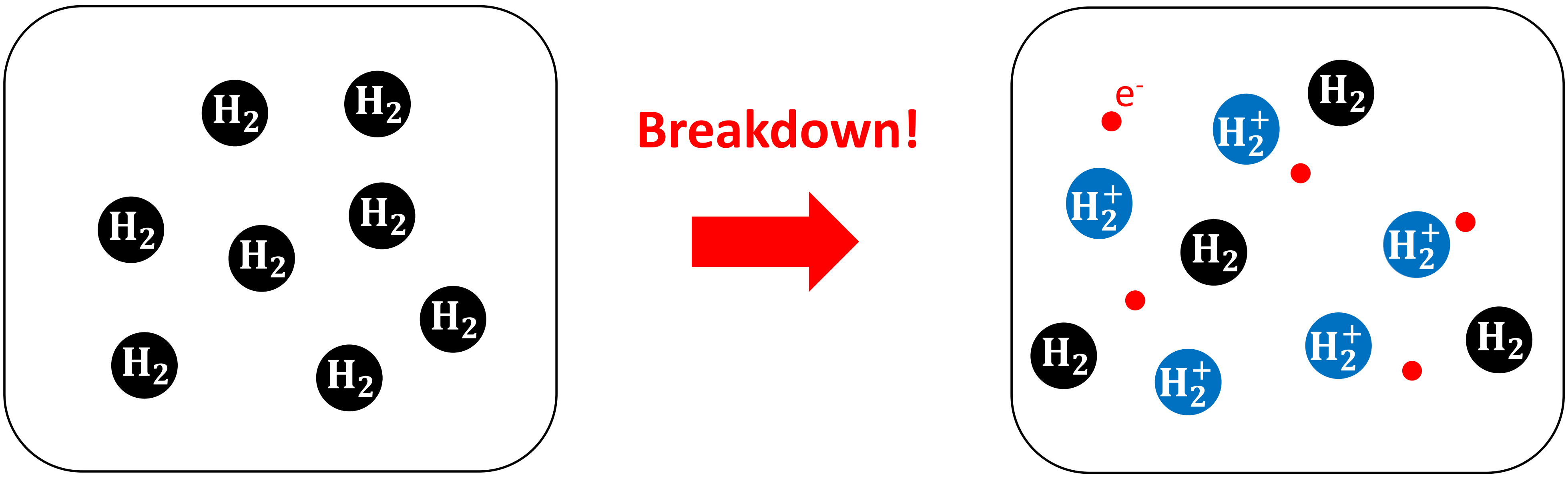
# Background

# What is a breakdown?

## Electrical breakdown

➔ Rapid reduction in the resistance of an electrical insulator




**Neutral Gas**

**(Insulating)**

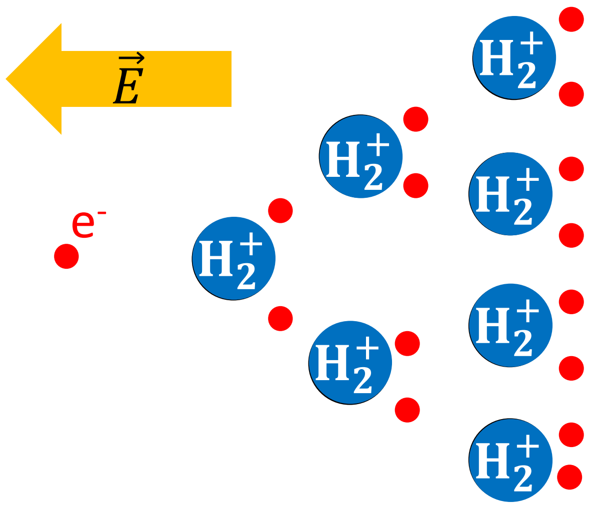
**Partially Ionized Plasma**

**(Conducting)**

# Electron Avalanche

- **Electron drift motion**

$$v_{d,e} = -\mu_e \boldsymbol{E}$$

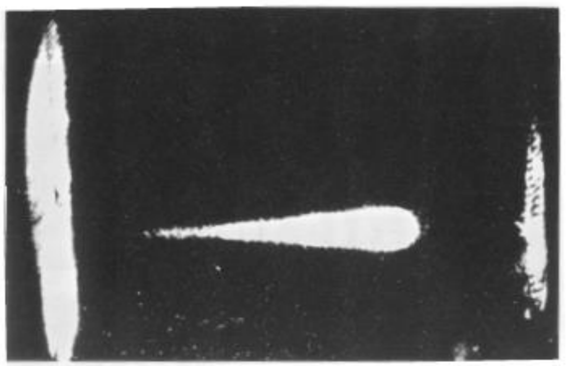



- **Townsend avalanche theory**

$$\frac{dn(x)}{dx} = \alpha n(x) \Rightarrow n(x) = n_0 \exp(\alpha x)$$

where $\alpha = Ap \exp\left(-\frac{Bp}{E}\right)$

[1]

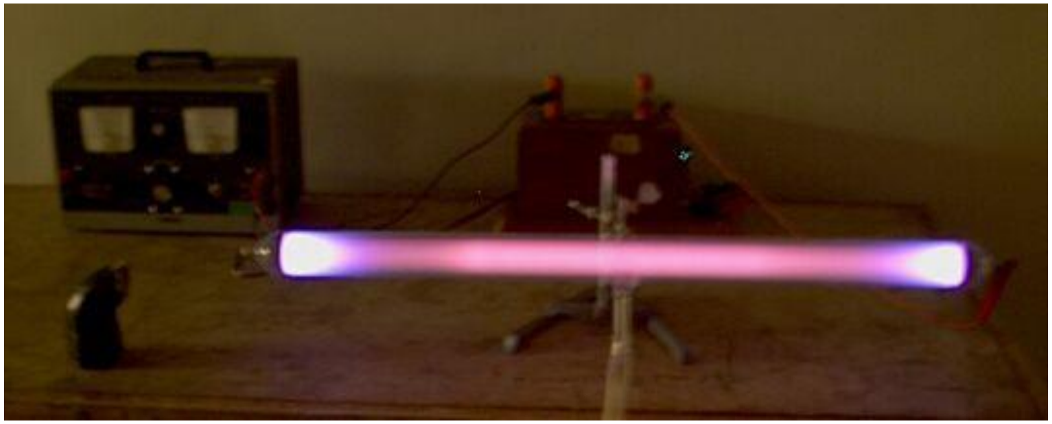


- **Characteristics of Townsend avalanche**

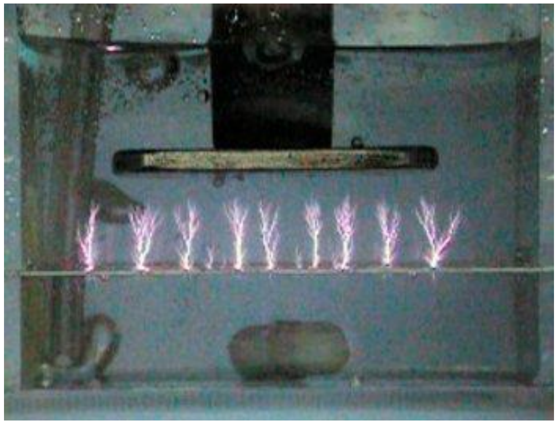
  **- External electric fields is dominant.**
  **- Transport is parallel to the electric field.**

[1] H. Raether. Die Entwicklung der Elektronenlawine in den Funkenkanal. Z. Phys., 112(7-8):464–489), 1939

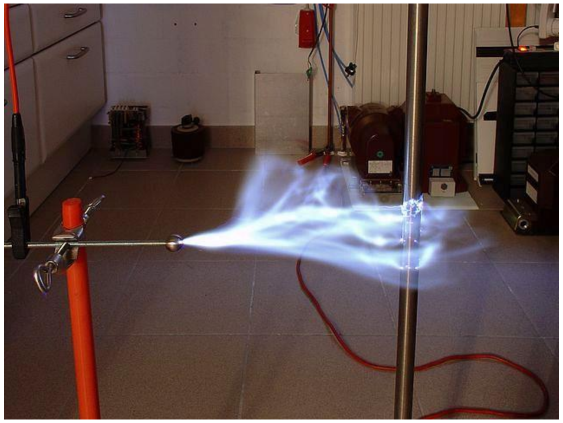
# Electric Discharges



Glow discharge



Streamer



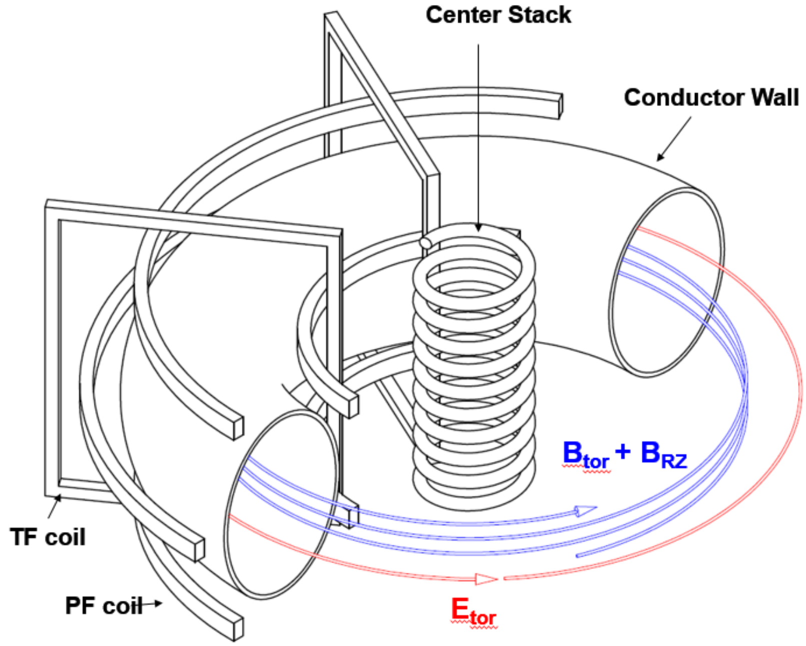
Arc discharge

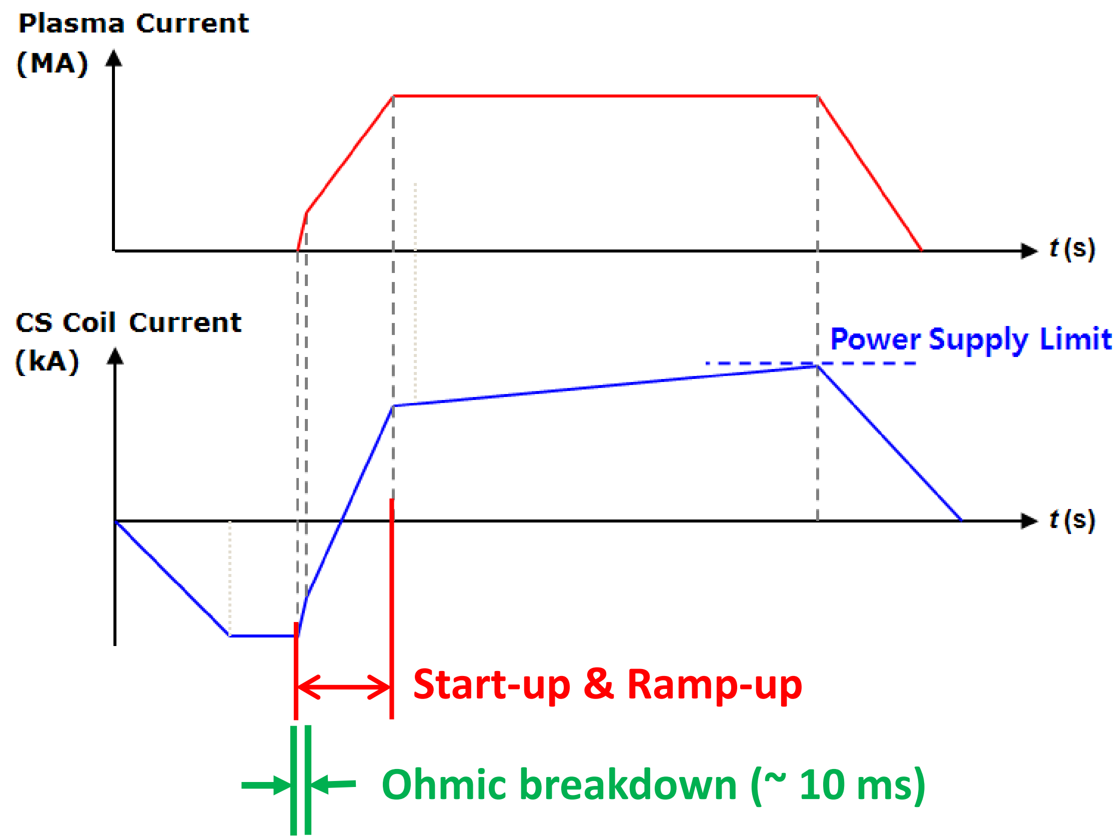

Lightning

**Electric discharges are one of most interesting physical phenomena for a long time!!**

# Introduction

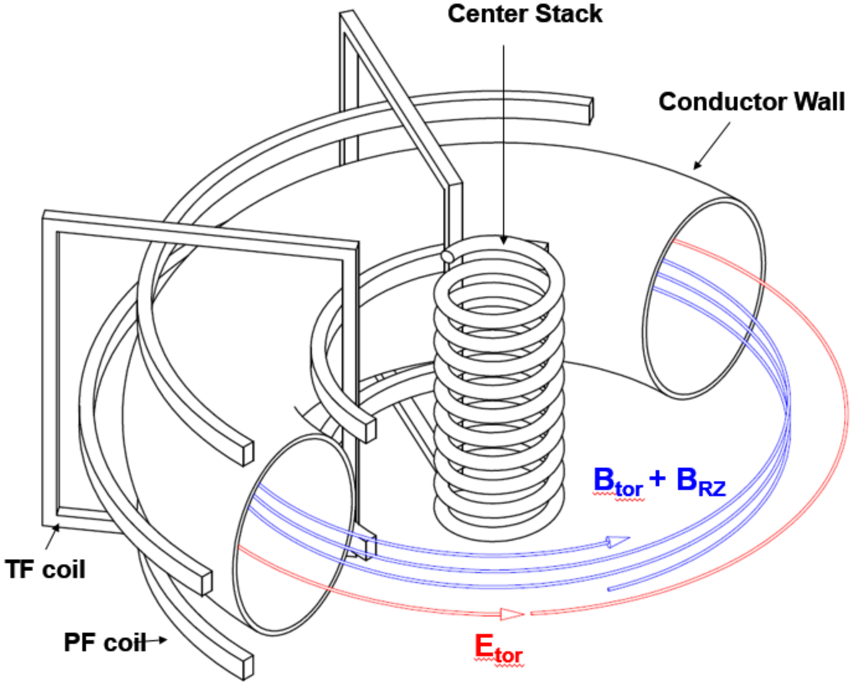
# Ohmic Breakdown in the Tokamak

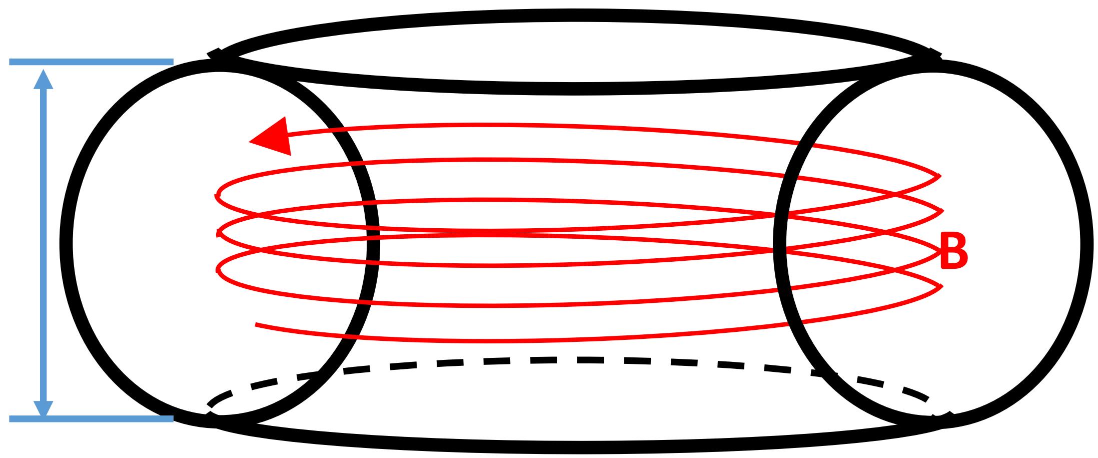



- **Toroidal electric fields** is induced by time-varying current of central solenoids (CS) to make electron avalanches in the tokamak.
- $|E_{tor}| \sim 1\ \mathrm{V/m}$ for usual tokamaks, $|E_{tor}| < 0.3\ \mathrm{V/m}$ for ITER due to engineering limits

# Stray magnetic fields

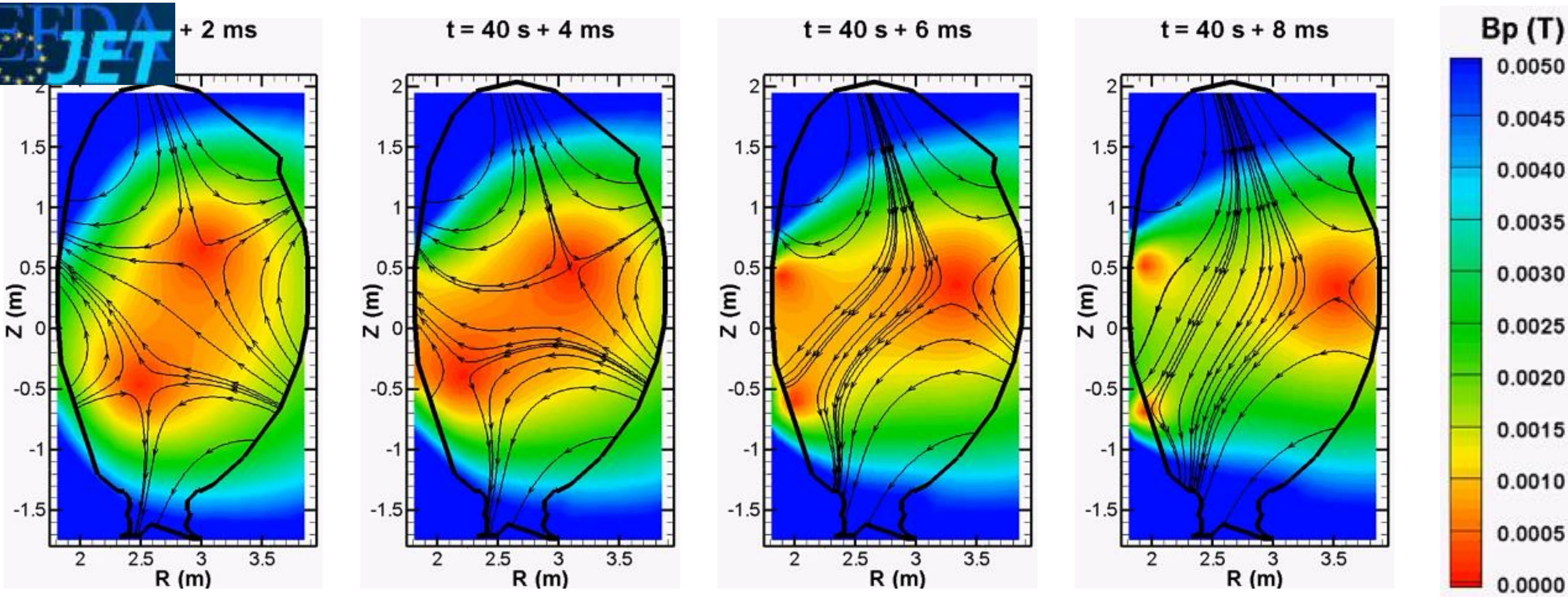



$$L \sim \left(\frac{B_{tor}}{B_z}\right) d$$

- **Stray magnetic fields are produced by CS currents and eddy currents on the wall**
- Since guiding centers of electrons tend to follow the magnetic field lines, electrons could be lost easily following stray magnetic fields.
- PF coil currents are adjusted to appropriately cancel the stray magnetic fields.

# Magnetic Configurations during ohmic breakdown




- **Time-varying** , **inhomogeneous** and **nonlinear** electromagnetic configurations are **inherently produced** in the tokamak which is totally different from any other discharge device.

# Motivation

# Distinct Characteristics of the Ohmic Breakdown

1. Low E (~ 1 V/m) by Faraday's induction
2. Long length ( L = 1000 ~ 10000 m)
3. Strong magnetic fields (~1 T)
4. Time-varying, inhomogeneous and nonlinear electromagnetic fields
5. Toroidal periodic & symmetric geometry

➡ **What's a picture of the ohmic breakdown physics under this unique situation?**

# No one have a clear picture of the ohmic breakdown!

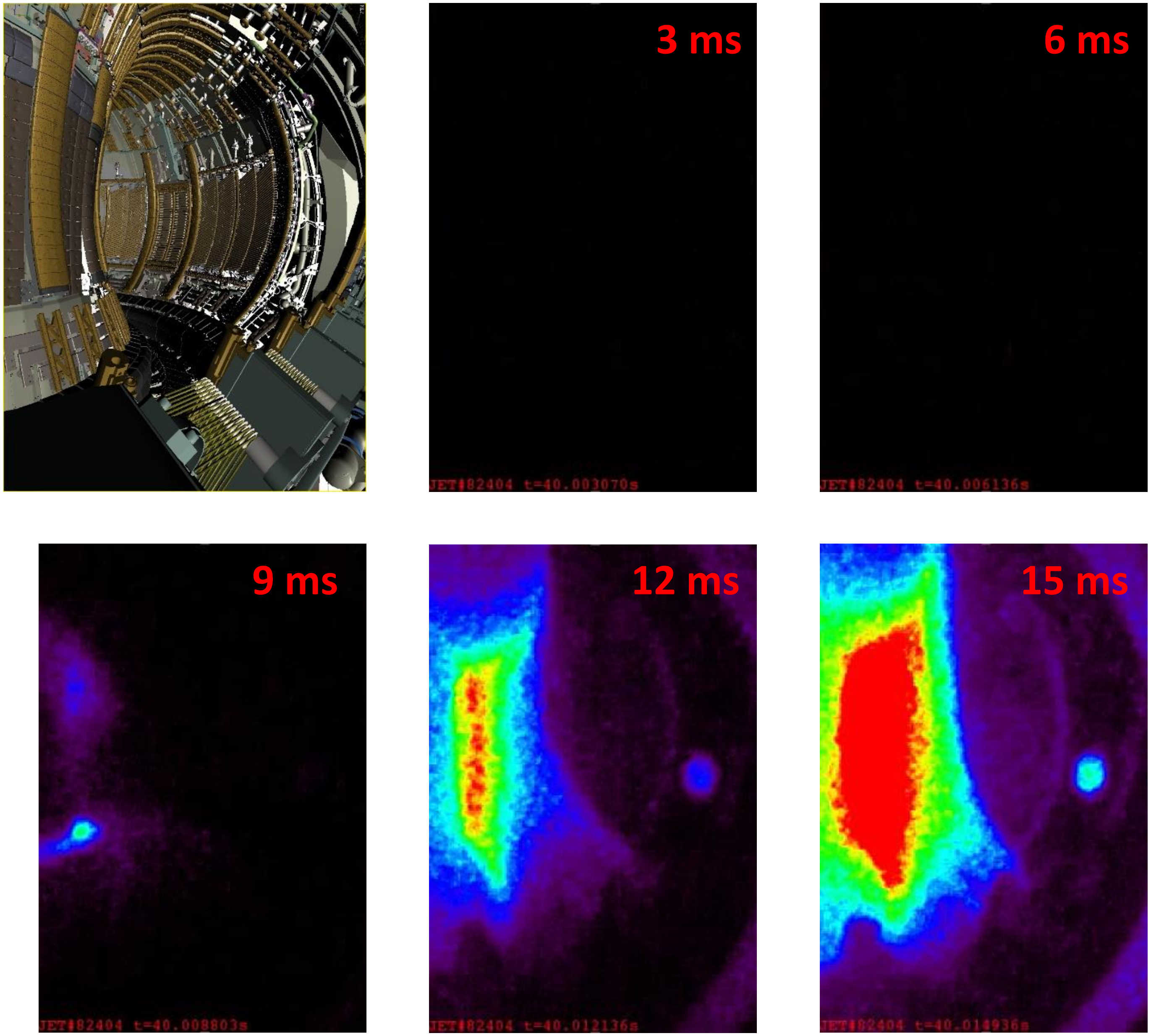
## Lack of observations !

- **Initial plasma** during the avalanche phase is **cold** **(10~100 eV)** and **rarefied** **($10^8$ ~$10^{15}$ $m^{-3}$)**

- Most diagnostics in the tokamak focus on **hot dense plasma**

➔ **Physics of the ohmic breakdown is not clearly revealed yet**

# JET Experimental Results (Fast Camera, KL8A)




**Black Box**

- What's going on here?

**Why? How?**

- Why inside?

- How is a channel like structure produced and maintained?

# Townsend avalanche & Paschen's law

- **First Townsend ionization coefficient $\alpha$**
  : Ionization growth rate

$$\alpha = Ap\exp(-Bp/E)$$

$$\frac{dn(x)}{dt} = \alpha n(x) \Rightarrow n(x) = n_0 \exp\left(\int \alpha\, dx\right)$$

- **Necessary condition for self-sustaining of avalanche**

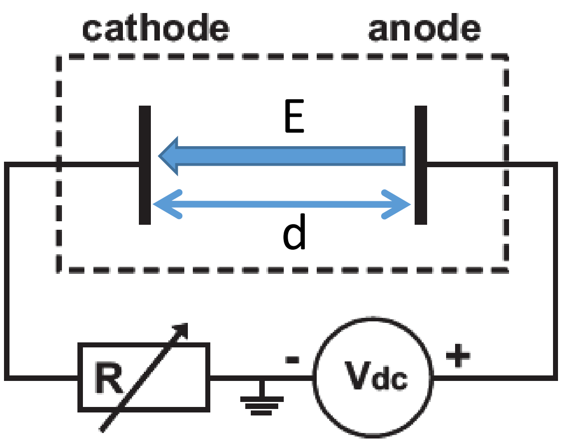
$$N_{e,sec} = \gamma\left(e^{\alpha d} - 1\right) \geq 1$$

- **Paschen's law**

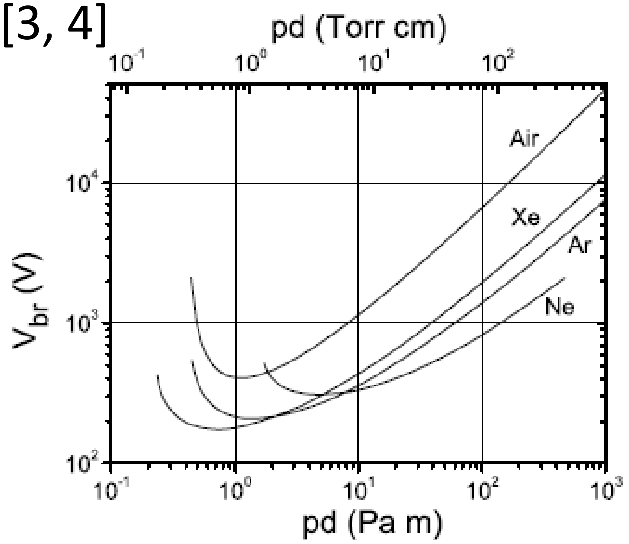
$$V = Ed \geq \frac{B(pd)}{\ln[A(pd)]}$$

⇒ **Breakdown is occurred by several generation of avalanches.**
**It can be determined by global parameter p, d and V,**
**because slab geometry is homogeneous system.**

[2]




[3, 4]





[2] Erik Wagenaars, "Plasma Breakdown of Low-Pressure Gas Discharges", Technische Universiteit Eindhoven, 2006 - Proefschrift

[3] Yu.B. Golubovskii, *et al*, J. Phys. D: Appl. Phys., 35(8):751–761, 2002.

[4] M.A. Folkardt and S.C. Haydon. : I. J. Phys. B: At. Molec. Phys., 6(1):214–226, 1973.

# Previous Study: Field Quality Approaches

## Conventional field quality analysis

- **Effective connection length [5]**

$$L_{\text{eff}} \cong 0.25\, a_{\text{eff}} B_T / B_p$$

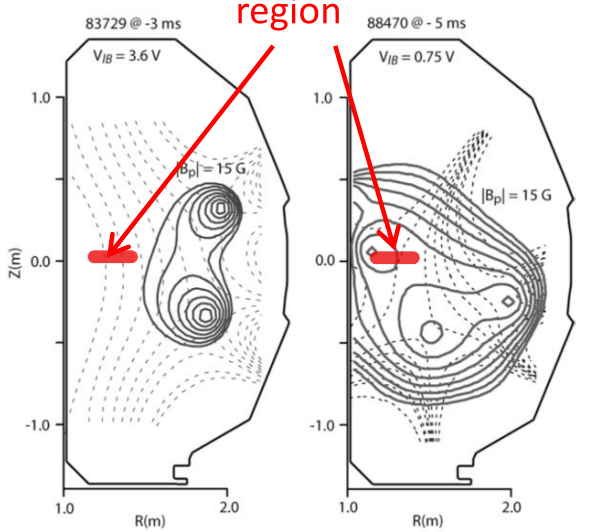
- **Empirical condition [6]**

$$E_T B_T / B_{\perp} > 1000\ \text{V/m}$$

- **Cons**

Breakdown region

[7]




**Breakdown occurs at unexpected region**

## Field-line-following analysis

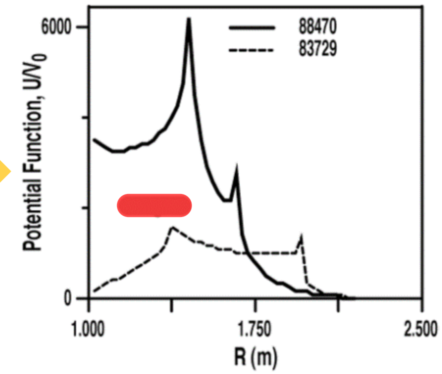
- **Estimation of 2D field map quality by field-line integration**

[7]

$$L = \int_{\vec{B}} dl$$

$$V = \int_{\vec{B}} \vec{E} \cdot d\vec{l}$$

$$M = \int_{\vec{B}} \alpha dl$$




- **Cons**

- Static analysis at a specific time
- Considering only external fields **(neglect fields produced by a plasma)**

➡ **No dynamic plasma evolution & response**


[5] R. Yoshino, *et al.*, Plasma Phys. Control. Fusion **39** 205 (1997)
[6] Tanga, A., in Tokamak Start-up (ed. U. Knoepfel), Plenum Press, New York 159 (1986)
[7] Lazarus E.A., *et al.,* Nucl. Fusion **38** 1083 (1998)

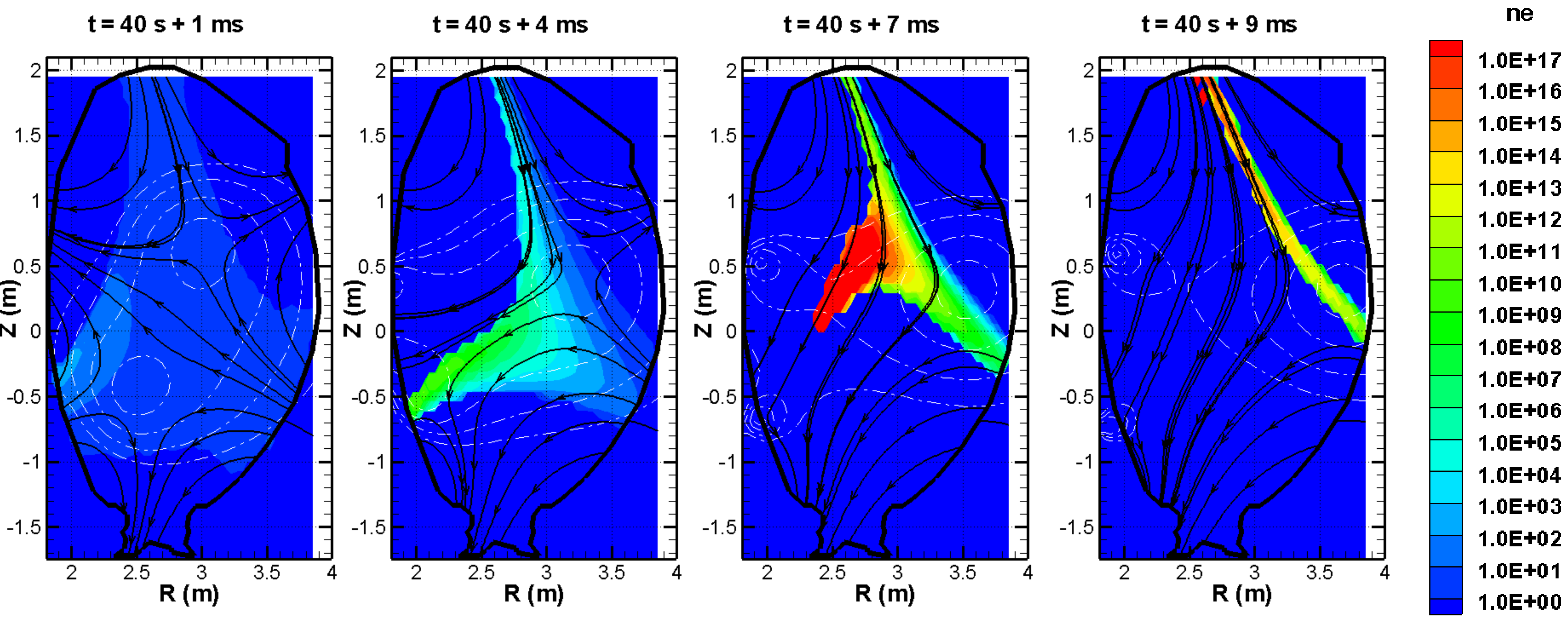
# Simple slab devcie v.s. Tokamak

| | Slab | Tokamak |
|---|---|---|
| E | > kV/m | ~ V/m |
| Pressure | ~ 100 Pa | ~ mPa |
| B | 0 | ~ 1 T |
| Track Length | < 1 m | > 1000 m |
| Fields characteristic | Homogeneous | Inhomogeneous |
| | Steady | Time-varying |
| Plasma response | negligible | ?? |

Same physics ??

➔ Dynamic evolution should be considered to understand the ohmic breakdown in the tokamak

# Electron density evolution of Mode D Scenario in JET




- Assume that the initial density of electron is 1 at everywhere
- Electron density evolve very dynamically with time-varying flux map (B-field and E-field).
- Multiplication of electron during 10 ms is **too large** (>$10^{17}$). (it's unreal value)

Electrons are fast
Ions are almost in rest

→ Breakdown occurs with only **one-pass electrons**, not by many generations.

# Mysterious results of Townsend avalanche theory for the tokamak

- **Too fast & large avalanche growth**

Townsend : Plasma is locally fully ionized in a only few ms

↕ Space charge

Experiment : Plasma still grows over than 10 ms in experiments

- **Transport**

Townsend : Electrons are swept away by external electric fields.

↕ Space charge, fast electrons

Experiment : Broad structure of a channel is produced and maintained

# Modeling

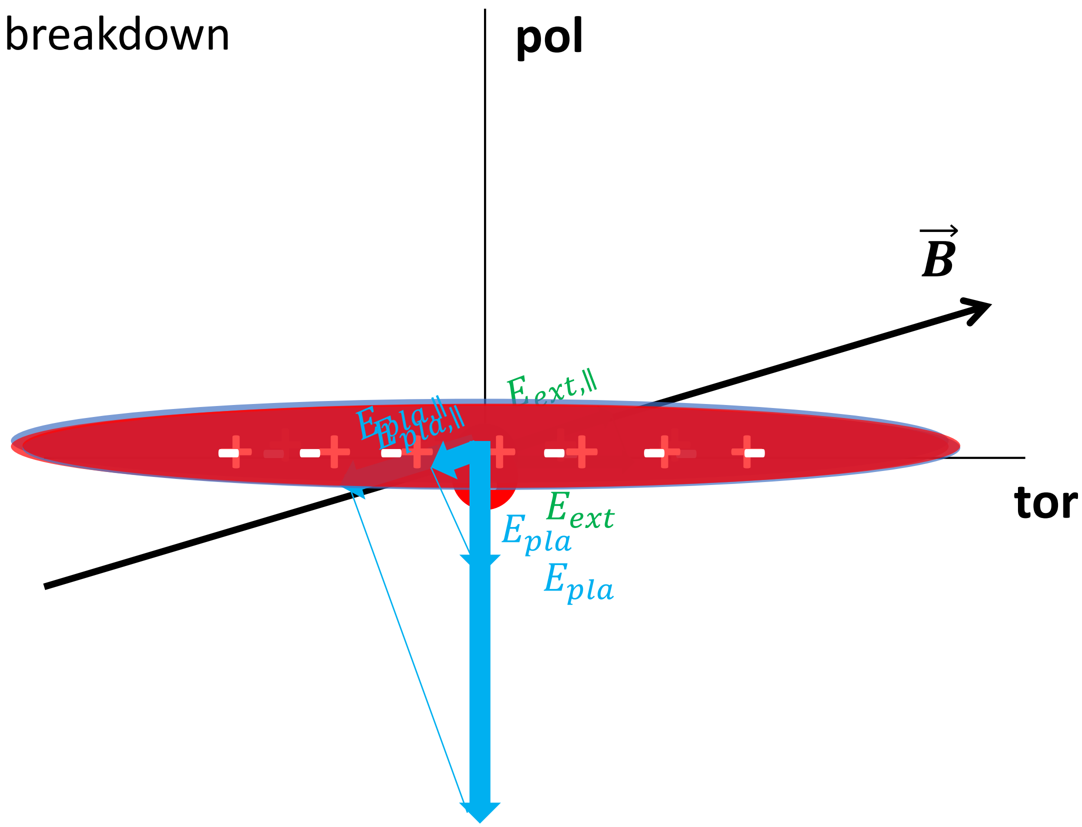
# Plasma Response: Effect of Space Charge

- **Electric Field** play a very important role in breakdown

  $E_{\parallel} \Rightarrow$ acceleration

  $E_{\perp} \Rightarrow$ drift motion

$$\boldsymbol{E} = -\nabla V - \frac{\partial \boldsymbol{A}}{\partial t} = \boldsymbol{E_{pla}} + \boldsymbol{E_{ext}}$$




**Low charge density** $|\boldsymbol{E_{pla,\parallel}}| \ll |\boldsymbol{E_{ext,\parallel}}|$ $\Rightarrow$ Electron and Ion move opposite direction.

**High charge density** $|\boldsymbol{E_{pla,\parallel}}| \sim |\boldsymbol{E_{ext,\parallel}}|$ $\Rightarrow$ Parallel heating reduced, Ambipolar like behavior

$|\boldsymbol{E_{pla,\perp}}| \gg |\boldsymbol{E_{ext,\perp}}|$ $\Rightarrow$ $\vec{E} \times \vec{B}$ drift motions

➡ **Electric field configuration can be modified by plasma response.**

# Effect of Space Charge: Plasma transport

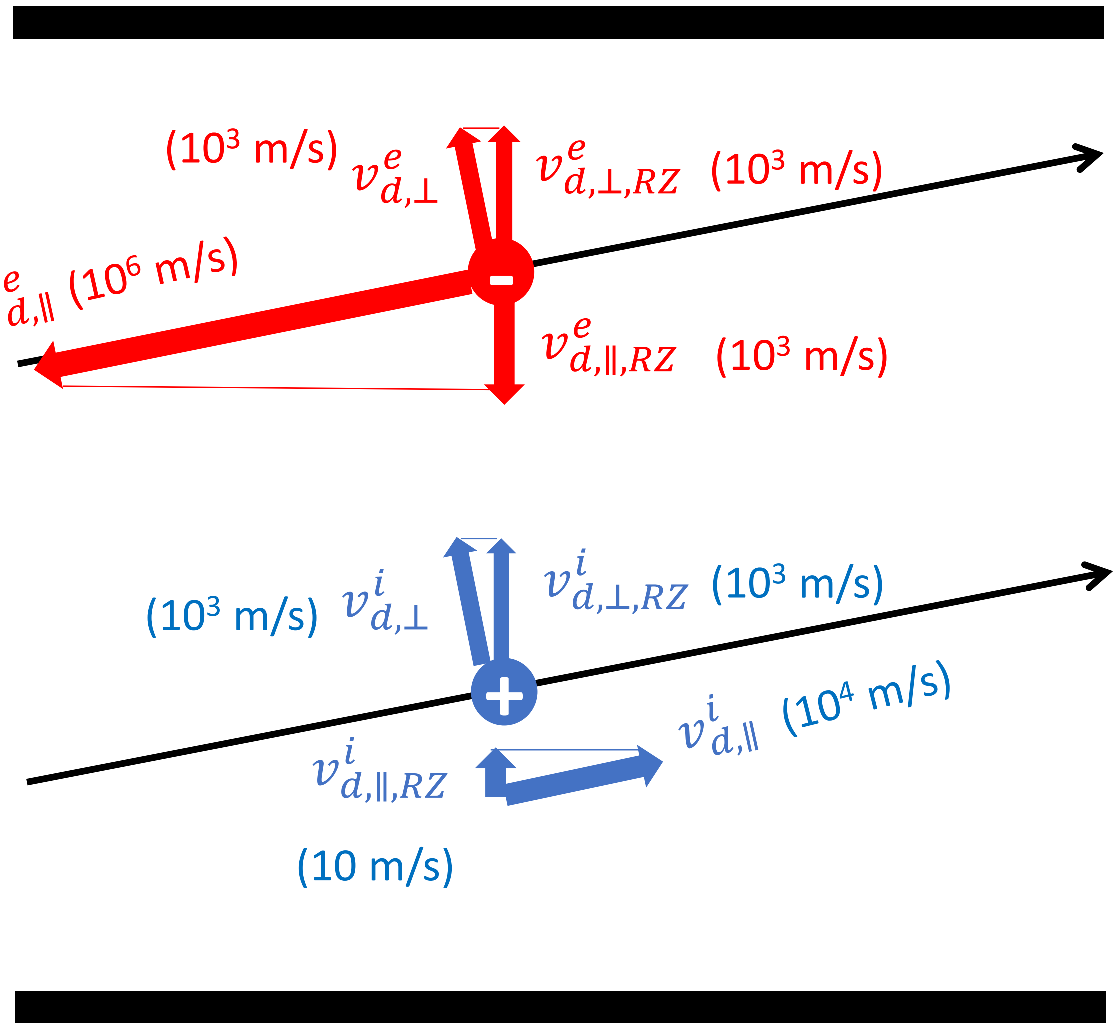
**Due to the toroidal symmetry, displacements in the RZ plane are important.**

$$|v^e_{d,\parallel}| \gg |v^e_{d,\perp}|$$

$$|v^e_{d,\parallel,RZ}| \sim |v^e_{d,\perp,RZ}|$$

$$|v^i_{d,\parallel}| \geq |v^i_{d,\perp}|$$

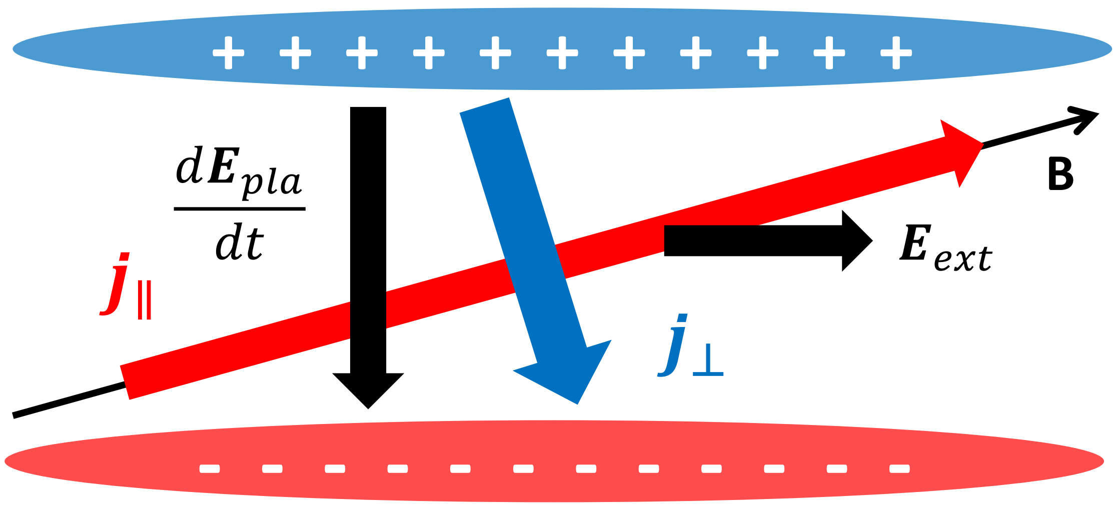
$$|v^i_{d,\parallel,RZ}| \ll |v^i_{d,\perp,RZ}|$$




**Perpendicular transport** could be dominant during the ohmic breakdown !!

➡ Townsend avalanche theory is not valid for this situation.

# Quasi-neutrality of initial plasma $\left(\frac{\partial \sigma}{\partial t} = ??\right)$




$$\mathrm{j}_{\parallel} = \mathrm{e}(\mathrm{n_i} v_{d,\parallel}^{i} - n_e v_{d,\parallel}^{e})$$

$$\boldsymbol{j}_{\perp} = \frac{(n_i M + n_e m)}{B^2}\frac{dE_{\perp}}{dt}$$

$$\nabla \times \boldsymbol{B} = \mu_0\left(\boldsymbol{j}_{\parallel} + \boldsymbol{j}_{\perp} + \epsilon_0 \frac{d\boldsymbol{E}}{dt}\right) \quad \rightarrow \quad \nabla \cdot (\boldsymbol{j}_{\parallel} + \boldsymbol{j}_{\perp} + \boldsymbol{j}_{\boldsymbol{d}}) = 0$$

If $\boldsymbol{n_i} \ll \frac{\boldsymbol{\epsilon_0 B^2}}{\boldsymbol{M}}$ , $(\boldsymbol{j}_{\perp} \ll \boldsymbol{j}_{\boldsymbol{d}})$ $\rightarrow$ $\nabla \cdot (\boldsymbol{j}_{\parallel} + \boldsymbol{j}_{\boldsymbol{d}}) = 0$ $\rightarrow$ $\frac{\partial \sigma}{\partial t} = -\nabla \cdot \boldsymbol{j}_{\parallel} \neq 0$

**Non quasi-neutral**

If $\boldsymbol{n_i} \gg \frac{\boldsymbol{\epsilon_0 B^2}}{\boldsymbol{M}}$ , $(\boldsymbol{j}_{\perp} \gg \boldsymbol{j}_{\boldsymbol{d}})$ $\rightarrow$ $\nabla \cdot (\boldsymbol{j}_{\parallel} + \boldsymbol{j}_{\perp}) = 0$ $\rightarrow$ $\frac{\partial \sigma}{\partial t} = -\nabla \cdot \boldsymbol{j} = 0$

($10^{15} \sim 10^{16}$ $\mathrm{m^{-3}}$)

**Quasi-neutral**

# My picture of the ohmic breakdown in the tokamak

Plasma Density

**Polarization currents**

Quasi-neutral

MHD like behavior

Closed flux surface

**Current force**
**+**
**Coulomb coll.**

$$n \sim \frac{\epsilon_0 B^2}{M_i}$$

($10^{15}$ ~ $10^{16}$ $m^{-3}$)

**Space charge**

Plasma oscillation

Ambipolar diffusion

ExB perp. drift

Collective behavior

**Electric force**
**+**
**(e-n), (i-n) coll.**

$$n \sim \left(\frac{B}{B_z}\right)^2 \left(\frac{\epsilon_0}{kT}\right) E_{ext}^2$$

($10^{12}$ ~ $10^{14}$ $m^{-3}$)

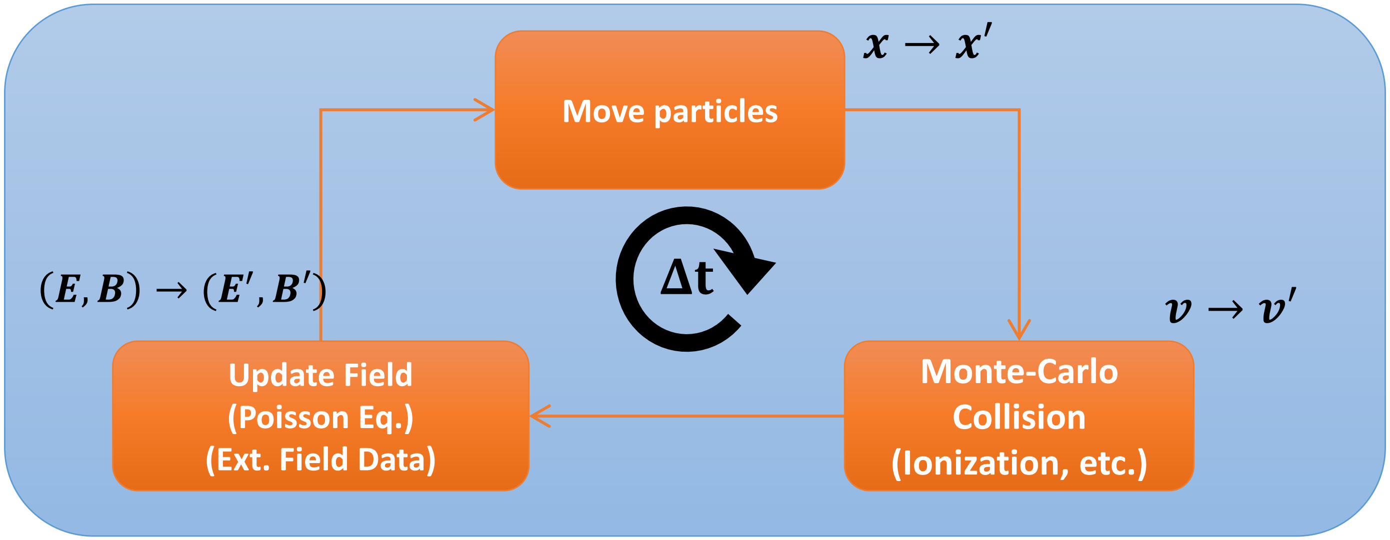
**External field**

Townsend avalanche

Charge separation

# Particle Simulation Development

## BREAK (Breakdown Evolution Analysis in tokamaK)




- **6 species (e, $H_2^+$, $H^+$, $H_3^+$, $H_{2(fast)}$, $H_{(fast)}$) are considered.**
- Guiding centers of the charged particle motions are calculated from **direct implicit method with D1 damping scheme** to reduce the computational cost.
- 26 collision reactions in the energy range of (0.01 – 1000) eV are treated by the **MCC (Monte Carlo Collision) scheme** to include atomic physics.
- **As a plasma response**, electric field generation due to the space charge is calculated from **Poisson equation** where the first-wall **is considered as a grounded conductor.**

# Simulation results

# Application to KSTAR Breakdown Scenario

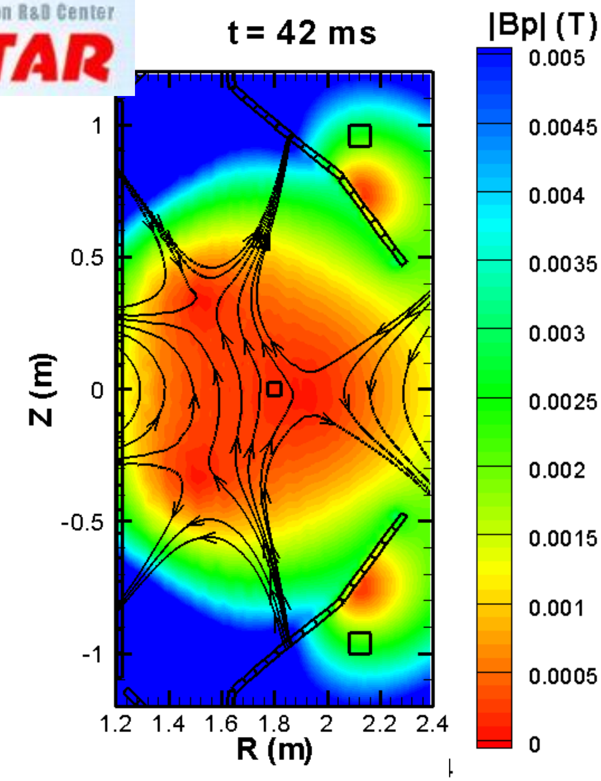
## Reference breakdown scenarios of *2010*

- **Breakdown scenarios** are designed by considering **eddy currents** as a ring model and **ferromagnetic incoloy 908 material effect** as a non-linear model **[8]**.

- Magnetic field configurations are changed rapidly during the breakdown phase. (30 - 60 ms)

| Initial Condition for Particle Simulation | $e^-$ | $H_2^+$ | Gas |
|---|---|---|---|
| Density ($\#/m^3$) | $2 \times 10^8$ | $2 \times 10^8$ | $4 \times 10^{17}$ |
| Temperature (eV) | 0.03 | 0.03 | 0.03 |
| Num. Super Particle (#) | $2 \times 10^6$ | $2\times 10^6$ | |

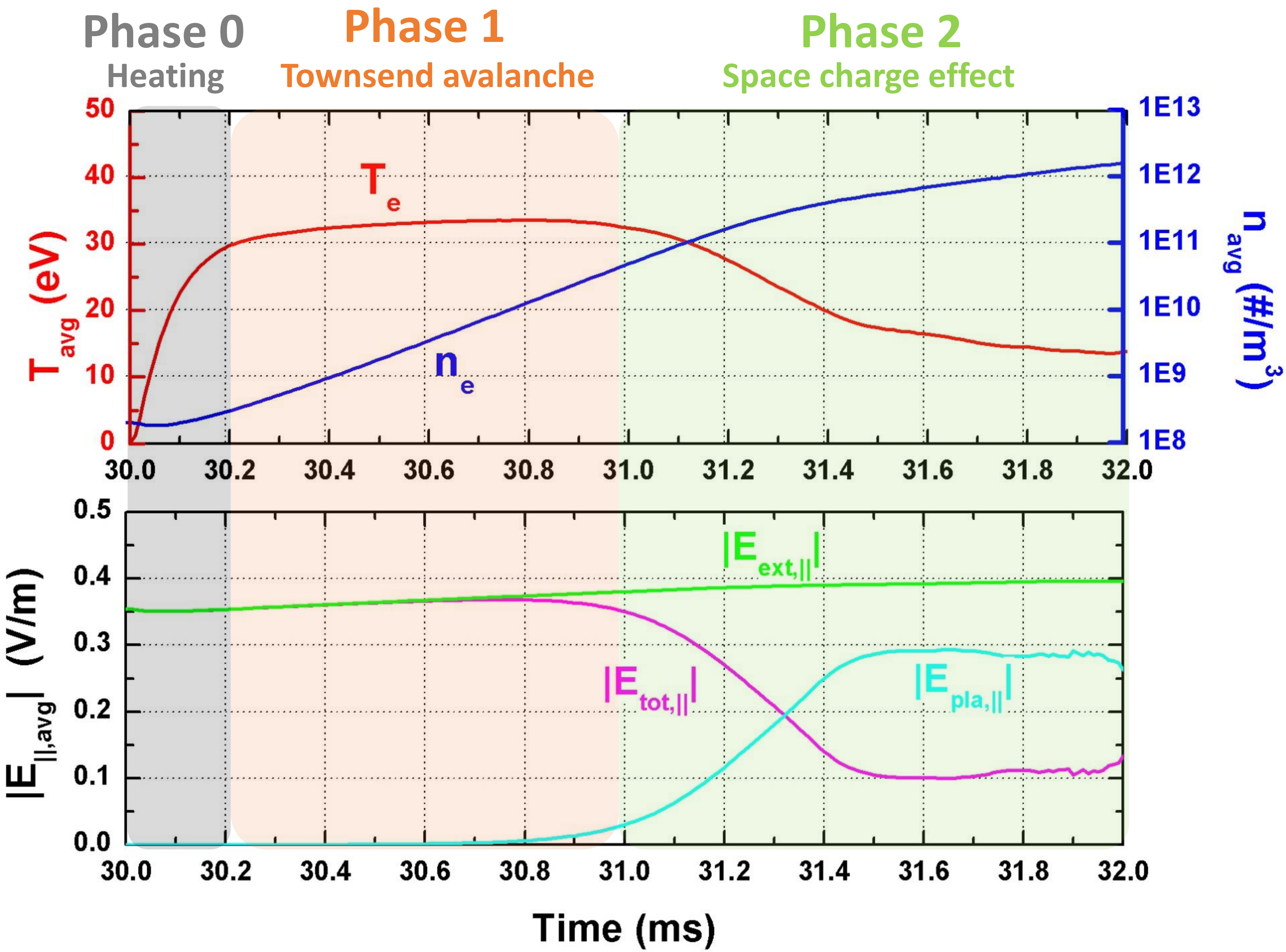
**p = 2 mPa** (Density and Temperature of Gas)



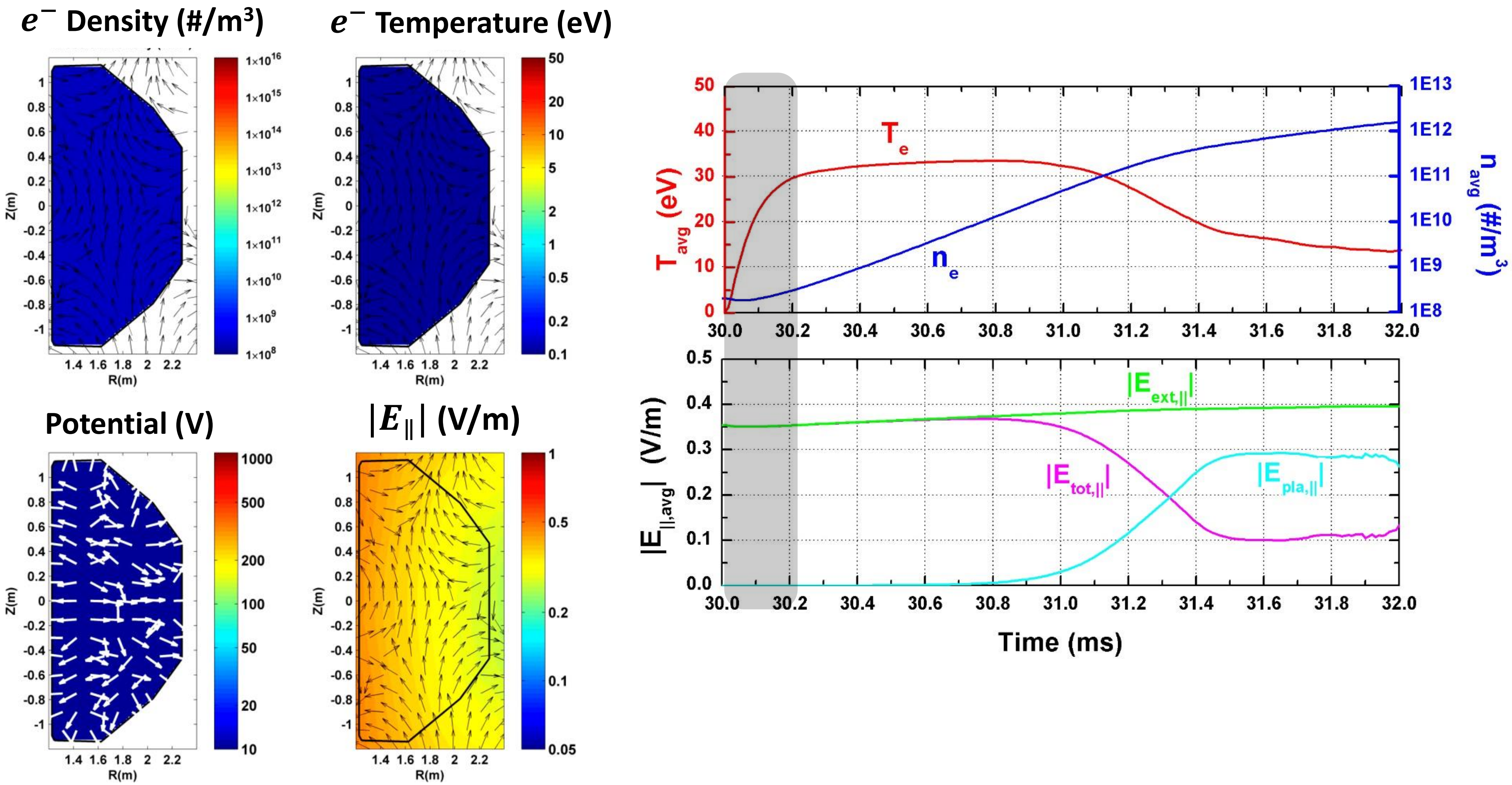

**KSTAR 2010 scenario magnetic field configuration**

[8] Jayhyun Kim, *et al.*, Nucl. Fusion **51** 083034 (2011)

# Breakdown Simulation of KSTAR 2010 Scenario

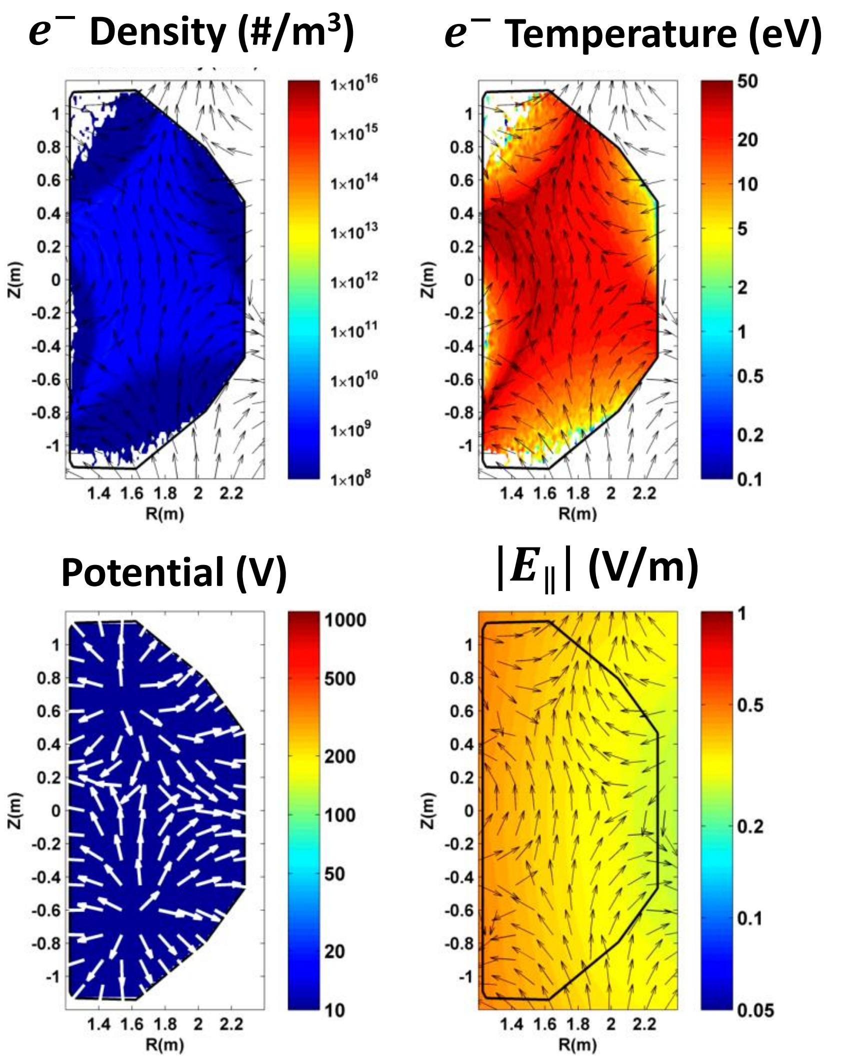


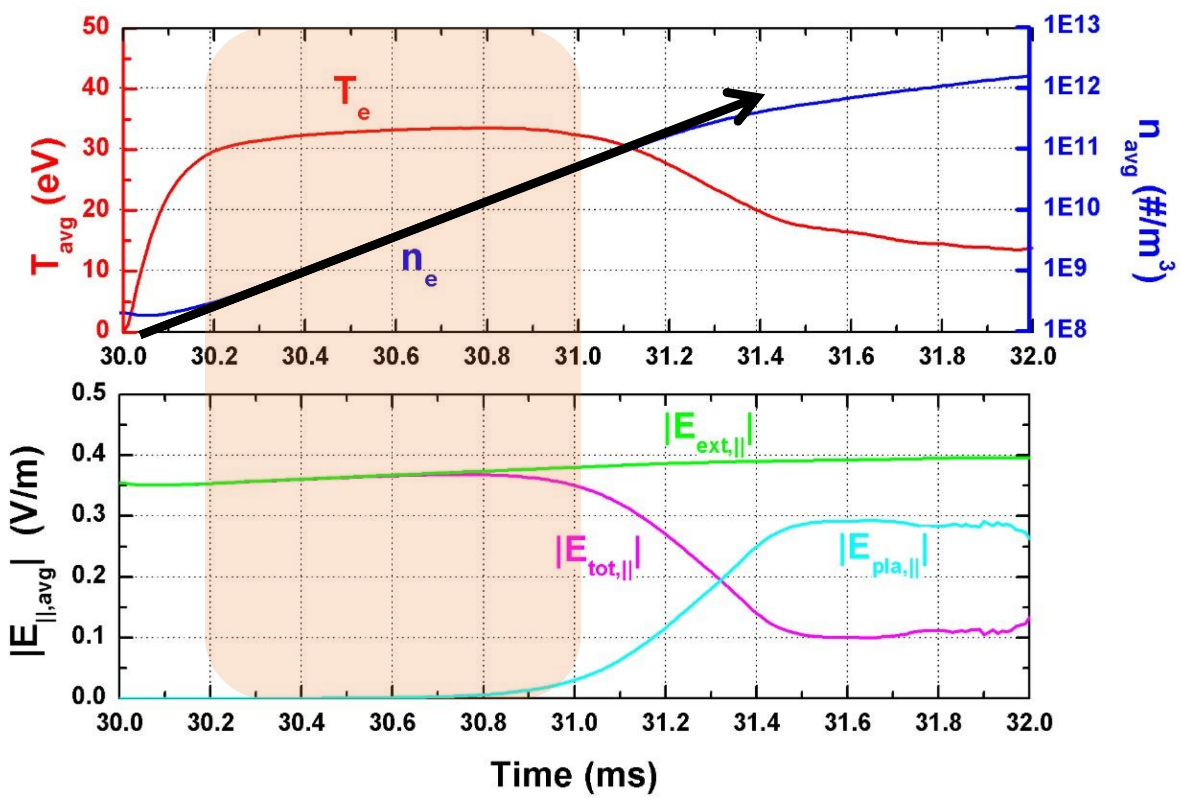
# Phase 0. Heating of Background Electrons




- ✓ Background electrons get energy from $E_{vac,\parallel}$ very rapidly.
- ✓ Electron temperature becomes saturated due to balance of the energy gain from the electric field and the energy loss by collisions at the end of the phase 0.

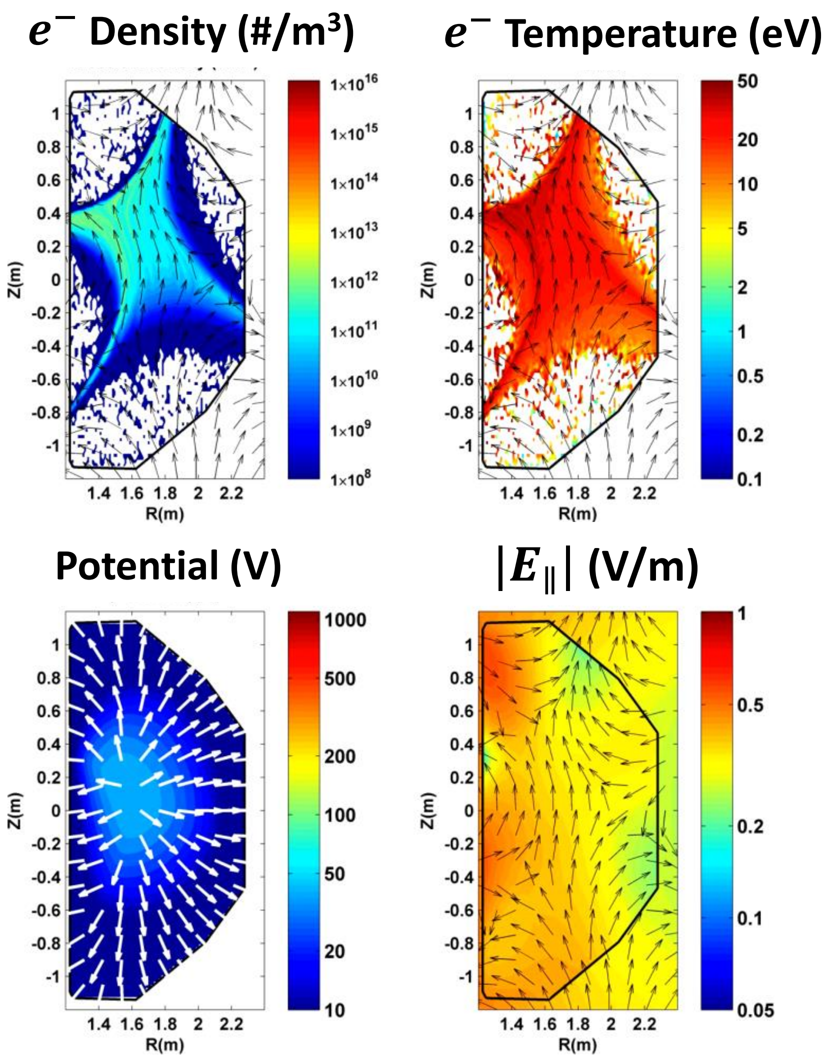
# Phase 1. Townsend Avalanche

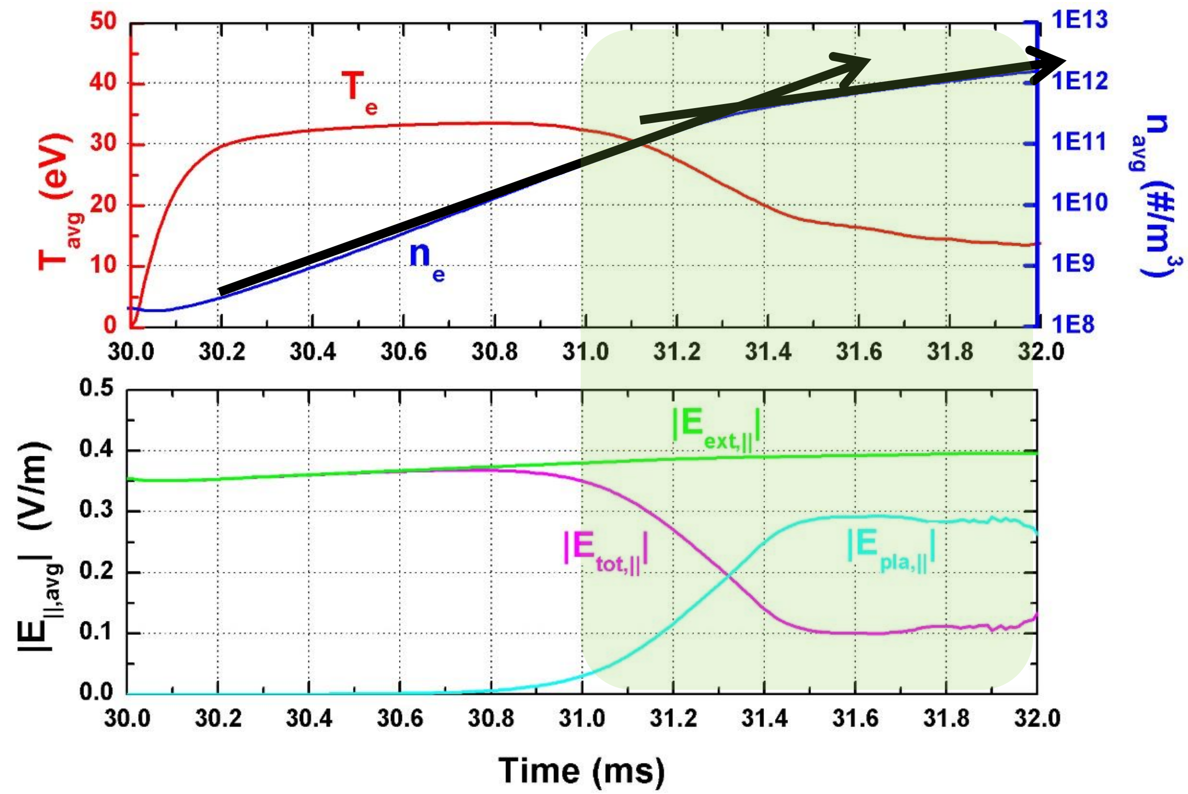


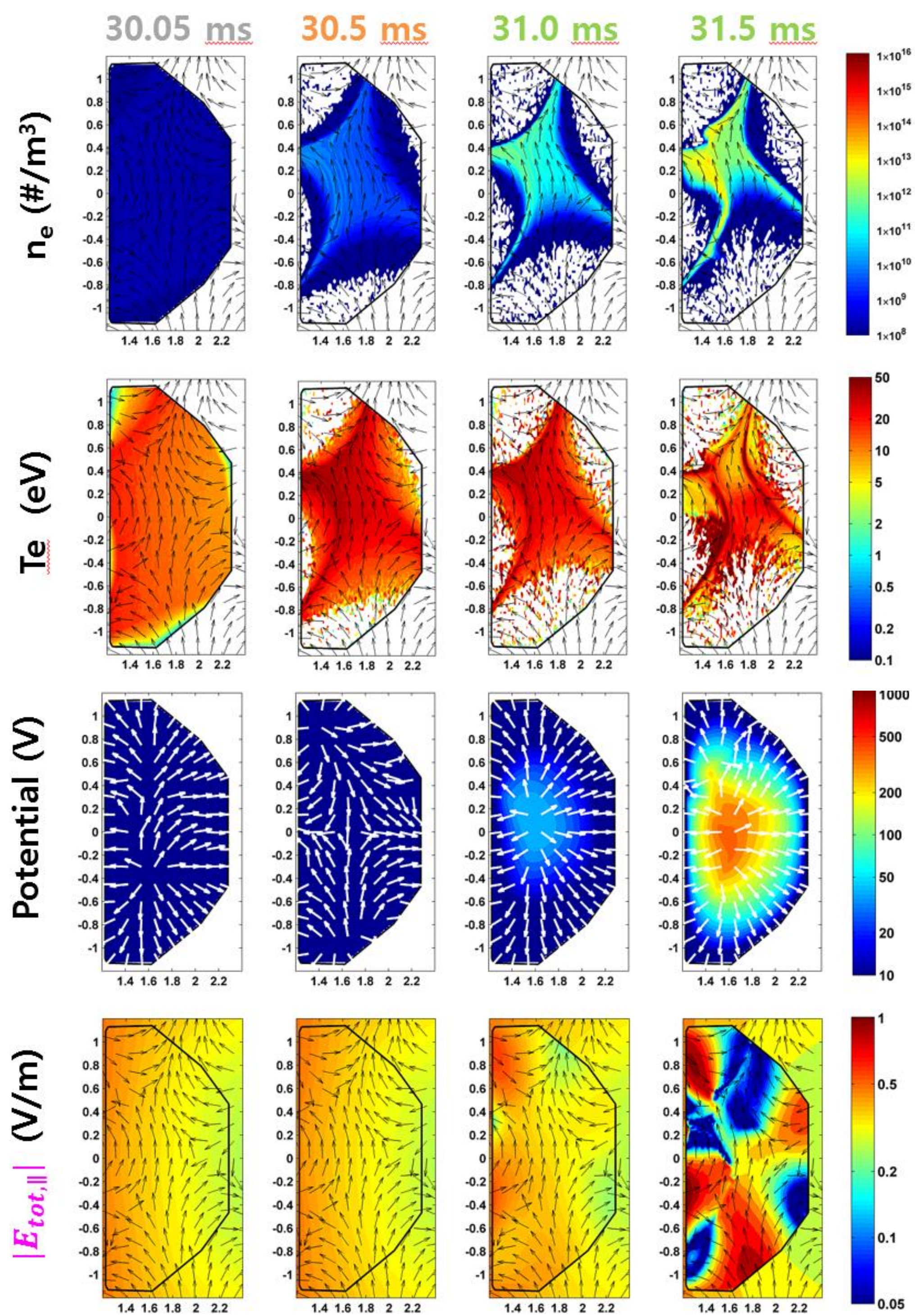

- ✓ External $\boldsymbol{E_{vac,\parallel}}$ **fields are dominant.**
- ✓ Electrons move along the magnetic field lines (Ions are almost in rest).
- ✓ **Electron avalanche** occurs with **constant exponential density growth rate according to Townsend avalanche theory** $\left(\frac{dn(\vec{x})}{d\vec{l}} = \alpha \mathrm{n}(\vec{\mathrm{x}})\ ;\ \ \alpha = \mathrm{A}p\exp(-\mathrm{B}p/E)\right)$.

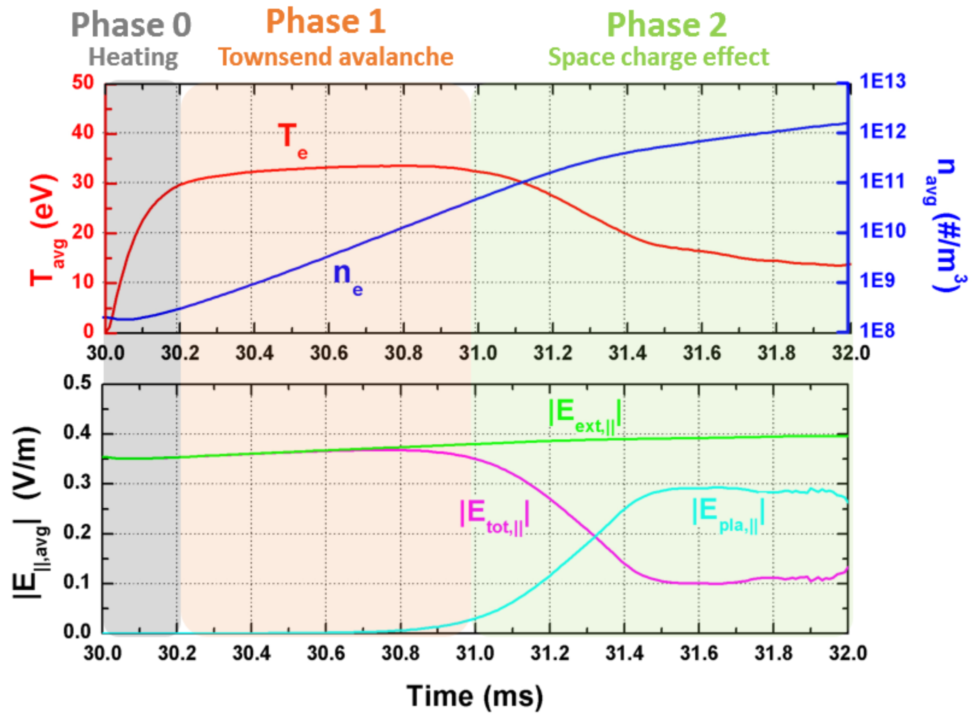
# Phase 2. Space Charge Effect



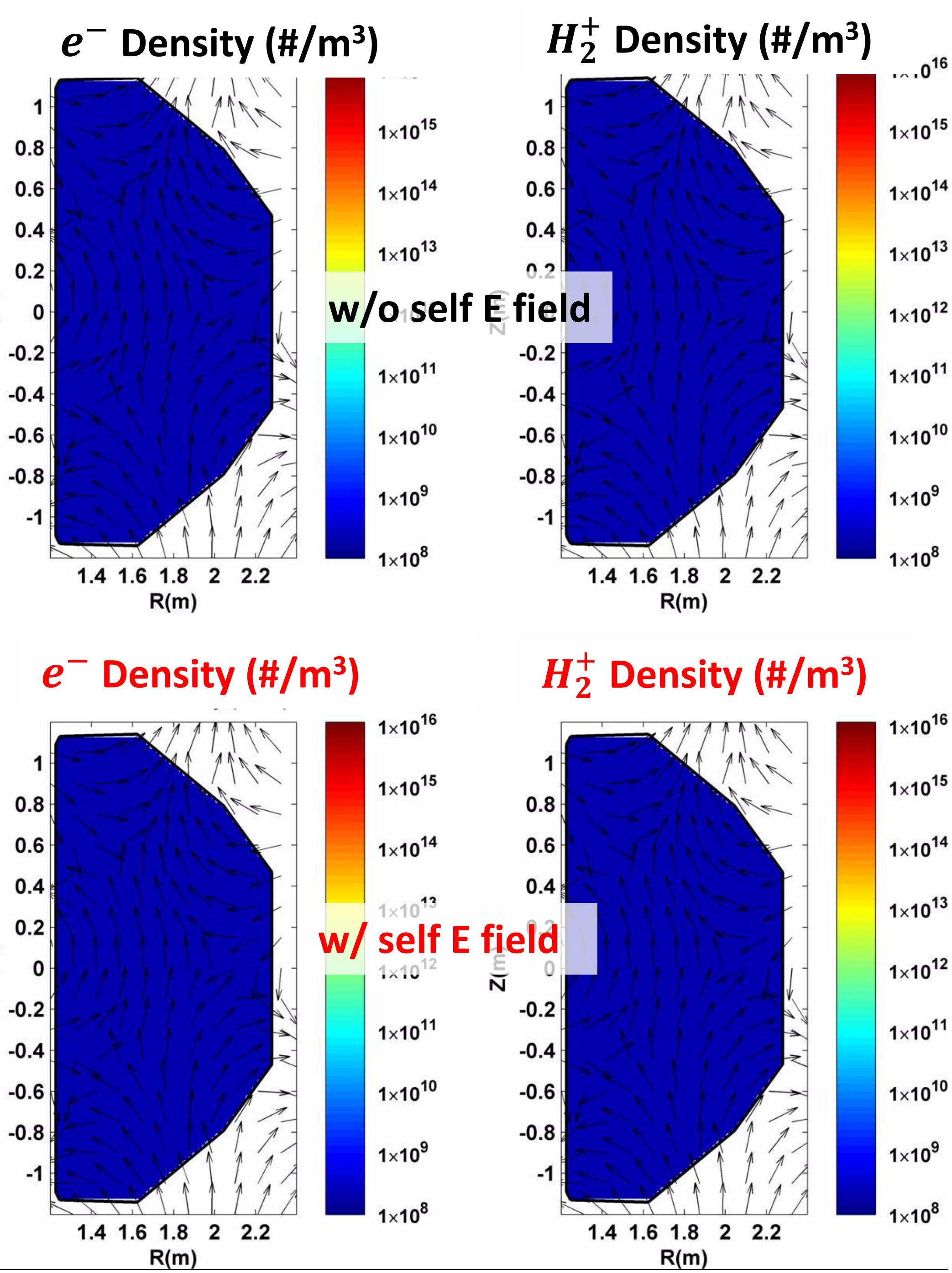




- ✓ **Plasma potential is built up** by positive space charge accumulated inside of the vessel.
- ✓ Electron energy is decreased due to **reduction of $|E_\parallel|$** so that the electron density growth rate is also reduced.
- ✓ **Perp. transport** of the plasma is enhanced by the **$\vec{E} \times \vec{B}$ drift motion.**


30.05 ms
30.5 ms
31.0 ms
31.5 ms
$n_e$ (#/m³)
Te (eV)
Potential (V)
$|E_{tot,||}|$ (V/m)


Phase 0
Heating
Phase 1
Townsend avalanche
Phase 2
Space charge effect
$T_e$
$n_e$
$T_{avg}$ (eV)
$n_{avg}$ (#/m³)
$|E_{ext,||}|$
$|E_{tot,||}|$
$|E_{pla,||}|$
$|E_{||,avg}|$ (V/m)
Time (ms)

# Discussion




## Role of space charge effect

✓ $|E_\parallel|$ **modification**

- Electron temperature is decreased due to **reduction of $|E_\parallel|$** so that the electron density growth rate is also reduced

✓ $\vec{E}_\perp \times \vec{B}$ **drift motion**

- New perpendicular transport is enhanced

$e^-$ : (⊥ transport) ~ ( ∥ transport)

$H_2^+$ : (⊥ transport) >> ( ∥ transport)

➡ **Plasma-wall interactions with $H_2^+$** such as **secondary electrons** come into play during breakdown phase

# Summary

- We establish **a toroidal symmetric plasma model** and develop **a particle simulation code** to have a **proper understanding** of the ohmic breakdown physics in the tokamak.

- In the modelling and the simulations, **crucial roles of the self-produced electric fields by the space charge** of the plasma are **newly observed**.

- In the parallel direction, the **avalanche growth rate is reduced** by $|\boldsymbol{E}_{tot,\parallel}|$ reduction due to the space charge effect.

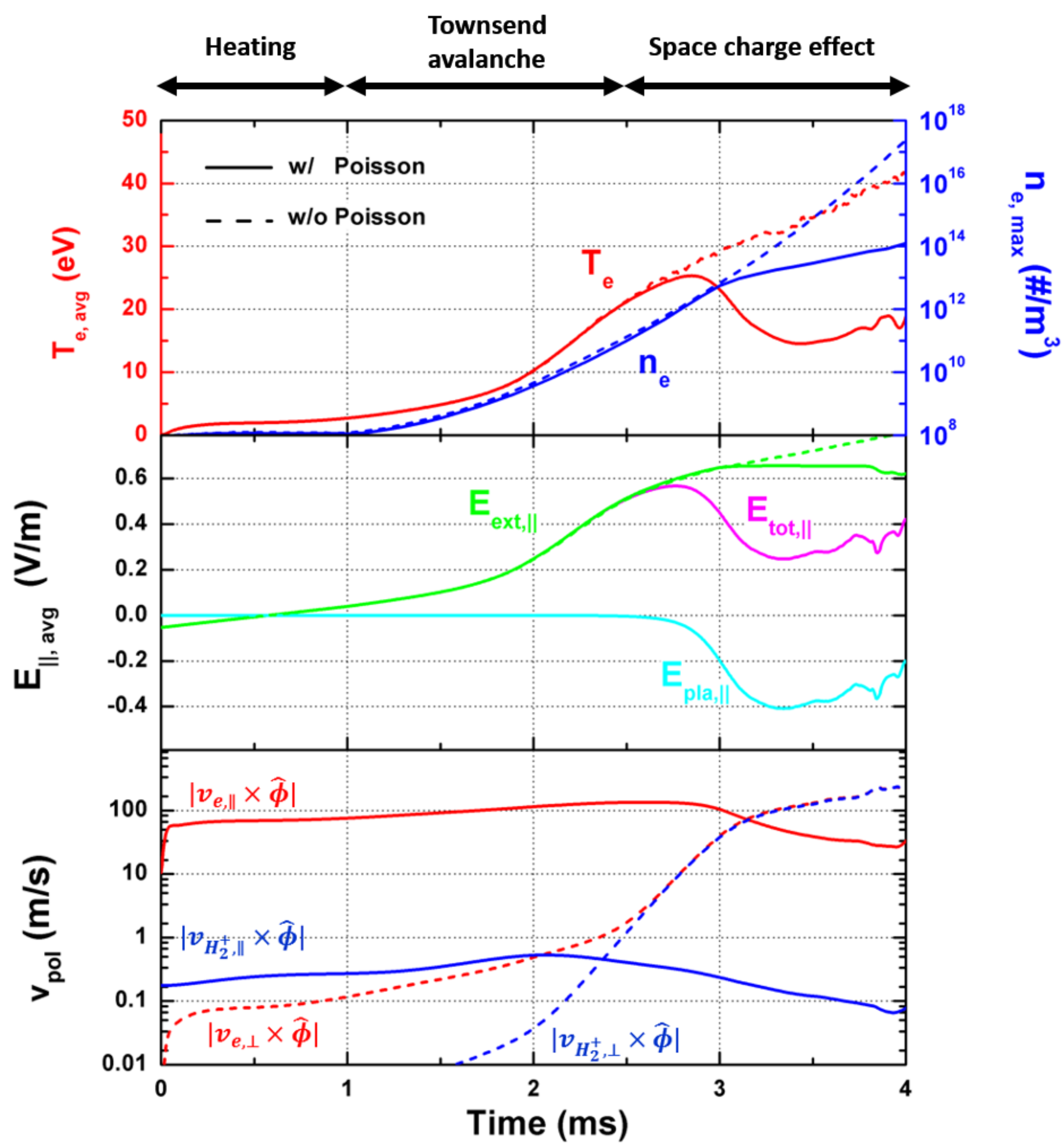
- In the perpendicular direction, $\mathbf{E}\times\mathbf{B}$ drift due to the self-produced perpendicular electric field results in **new perpendicular transport** especially for cold ions which can totally change the picture of the breakdown.

- **These space-charge effects newly observed in this research** could be important clues for a deeper understanding of unresolved issues of the ohmic breakdown in the tokamak.

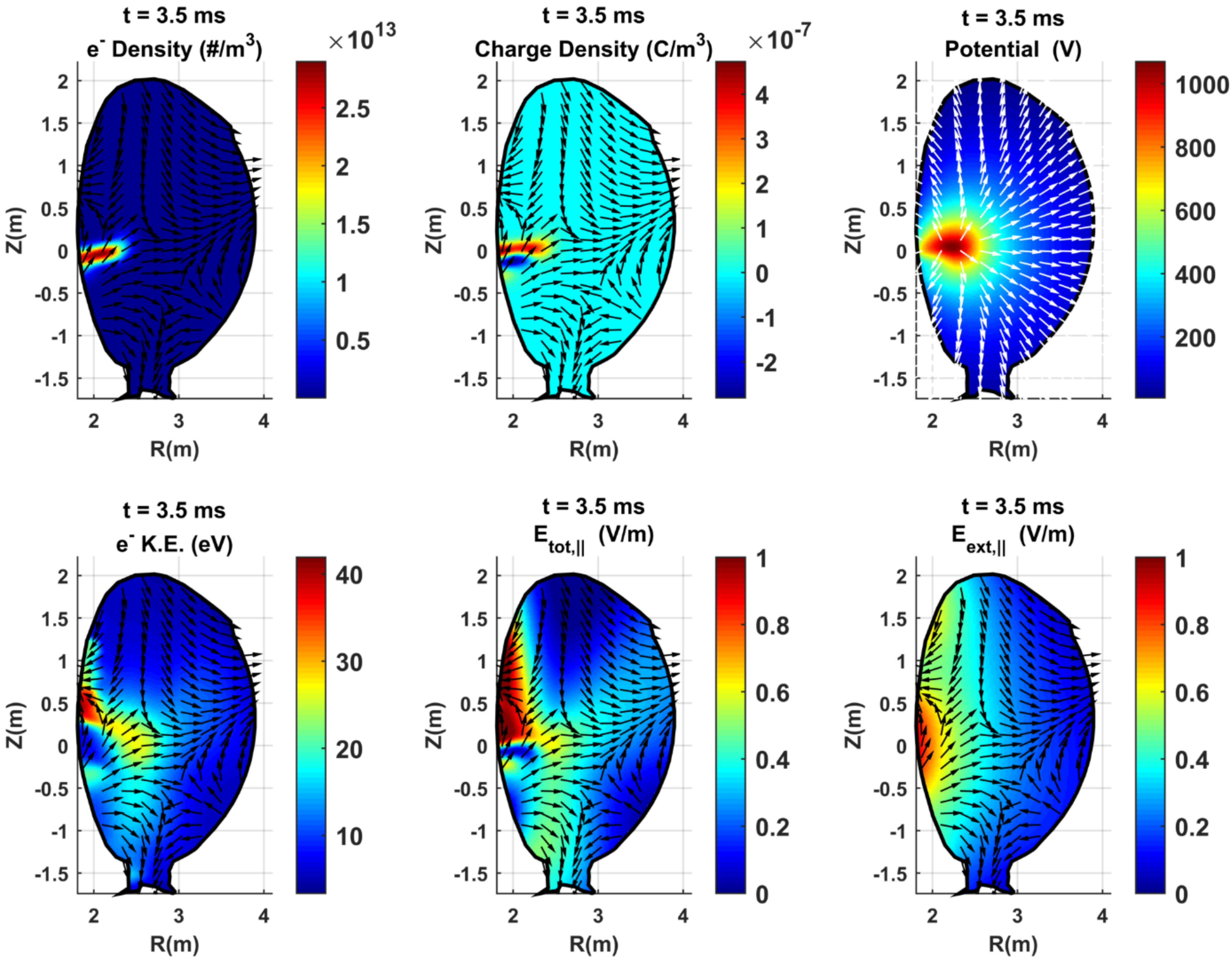
Thank you !!

# Backup

# Simulation results for #82404 shot of JET device



# Simulation results for #82404 shot of JET device



# Orderings

**R = 1.8 m, B = 1 T, Te = Ti = 20 eV, Tn = 400 K , Hydrogen Gas Pressure = 5 mPa**

| | Electron | Proton |
|---|---|---|
| Thermal velocity ($v_t$) | $2.7 \times 10^6\ m/s$ | $6.2 \times 10^4\ m/s$ |
| Drift velocity ($v_{\nabla B+curv}$) | $15\ m/s$ | $15\ m/s$ |
| Gyro-frequency ($\Omega$) | $1.8 \times 10^{11}\ Hz$ | $9.6 \times 10^7\ Hz$ |
| Gyro-radius ($\rho$) | $1.2 \times 10^{-5}\ m$ | $5.2 \times 10^{-4}\ m$ |

| Electron's scattering | Elastic | Ionization |
|---|---|---|
| Frequency ($\upsilon$) | $1.7 \times 10^5\ Hz$ | $2.7 \times 10^4\ Hz$ |
| Mean free path ($\lambda$) | 16 m | $100\ m$ |

i) $\Omega_e \gg \Omega_i \gg \upsilon$
ii) $\rho/L \ll 1$
iii) During 1 ms, $|\Delta x|_{max,col} = \rho\upsilon/1000 \ll L$ , $|\Delta x|_{max,drift} \ll L$

$\Rightarrow$ **Particle is almost attached to magnetic field,**
**particle motion could be treated sufficiently as a guiding center motion**

# Electron evolution model

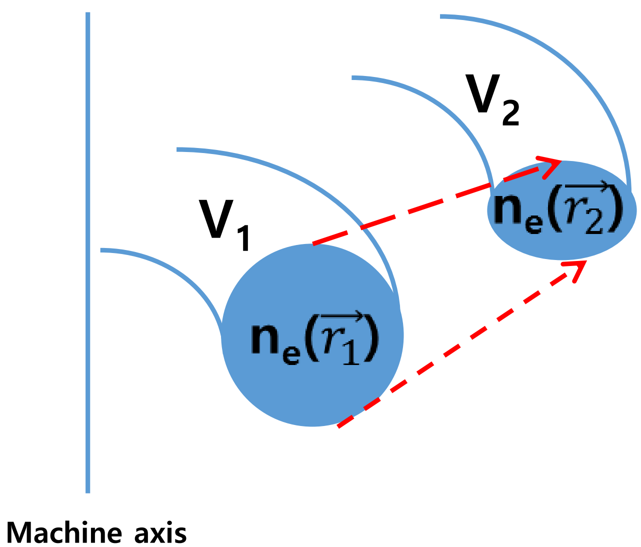
Empirical electron drift velocity by electric field : (Typically E/p = 80-800, $V_d$ ~ 0.55 – 2*$10^6$ m/s)

$$v_d \cong 6.9 * 10^4 \sqrt{(E/p)}$$ [m/s] (for 70<E/p<1500 [V/m/Pa] )

During dt, electrons follow the path $\vec{dl}$ parallel to magnetic field line with drift velocity $v_d$

$$\vec{dl} = \vec{v_d}(x,t) * dt$$

During some △t, 2D-axisymmetric electron cloud(toroidal ring) move to location 2 from location 1.
And electron density is multiplied by Townsend theory. (Neglect diffusion by coll.)




**For 1-D :**
(slab geometry)

$$\frac{dn(x)}{dt} = \alpha n(x) \Rightarrow n(x) = n_0 \exp(\int \alpha \, dx)$$

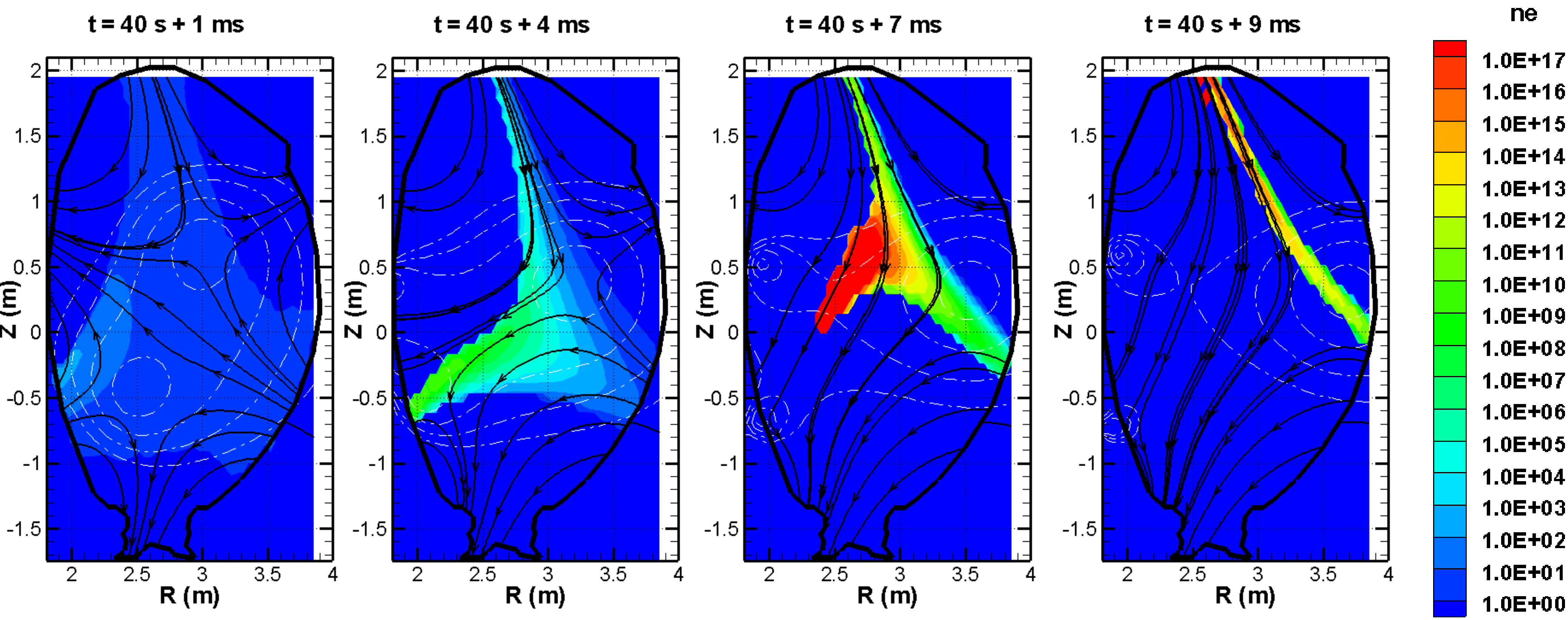
**For 3-D :**
(tokamak geometry)

$$n_e(\vec{r_2}) = n_e(\vec{r_1}) \times \exp\left(\int_{\vec{r_1}}^{\vec{r_2}} \alpha dl\right) \times \frac{V_2}{V_1}$$

**Townsend avlanche term (Integrate along B field)**

**Volume compression term**

# Electron density evolution of Mode D Scenario in JET




- Assume that the initial density of electron is 1 at everywhere
- Electron density evolve very dynamically with time-varying flux map (B-field and E-field).
- Multiplication of electron during 10 ms is **too large** ($>10^{17}$). (it's unreal value)

Electrons are fast
Ions are almost in rest

→

Breakdown occurs with only **a single avalanche**, not by many generations.

# Floating Potential Measurement on CASTOR tokamak [11]

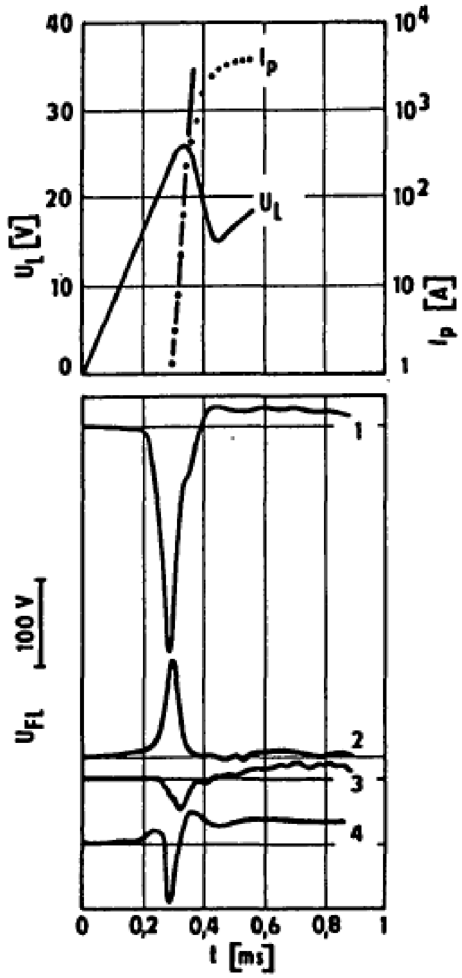



*FIG. 1. Temporal evolution of loop voltage $U_L$, plasma current $I_p$ and floating potential $U_{FL}$ of the probes for external magnetic fields: $B_H = 0.63$ mT, $B_V = -1.8$ mT. Numbers 1, 2, 3 and 4 denote the probe positions: $(R, z) = (R_0, a)$, $(R, z) = (R_0, -a)$, $(R, z) = (R_0 + a, 0)$ and $(R, z) = (R_0 - a, 0)$, respectively. (In the text, these are referred to as upper, lower, outer and inner probes.)*

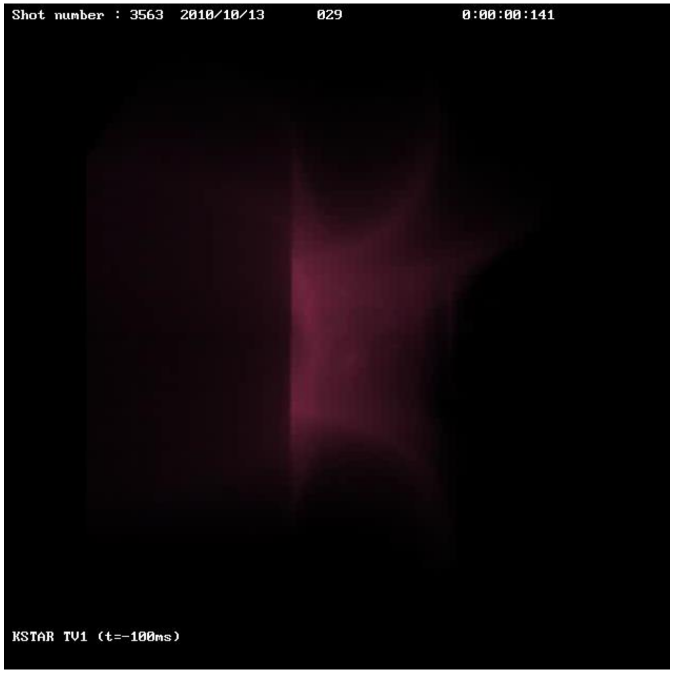
**The experiment was carried out on the CASTOR tokamak, which has a major radius R0 = 0.4 m, minor radius a = 85 mm**

**The maximum possible value of $\vec{E}_\perp$ is given by the condition that the projection of the electric field along the lines of force vanish ($v_D$ = 0):**

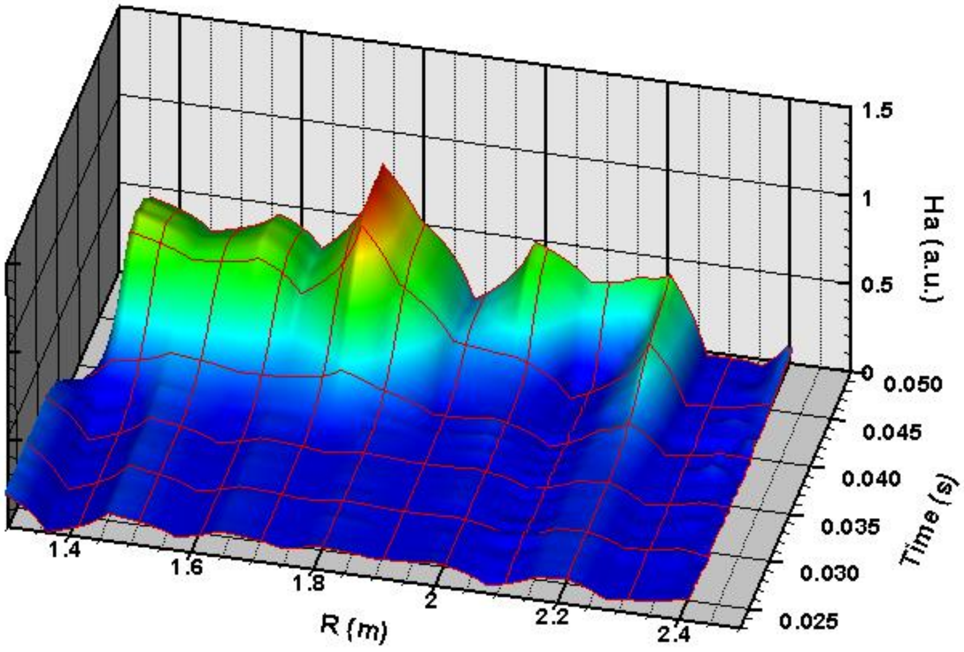
$$\vec{E}_\perp = -E \frac{B_T}{|B_\perp|^2} \cdot \vec{B}_\perp$$


[11] M. Valovic, Nucl. Fusion **27** 599 (1987)

# Measurement of 2010 reference scenario (#3563)




#3563 @ 40 ms

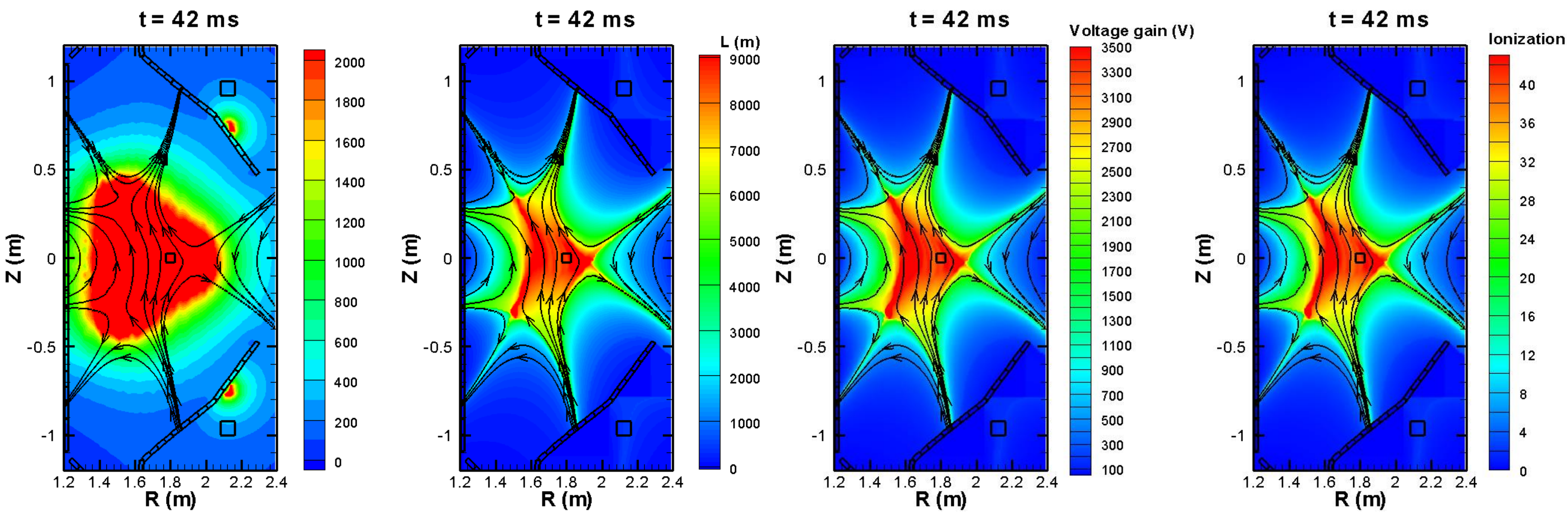



#3563 H-alpha signals

- Breakdown occurs widely from **40 ms**
- Most intense breakdown region matches with the field-line-following analysis

# Field Quality of KSTAR 2010 scenario

Pre-fill gas = **Hydrogen**, Pressure = **2 mPa** (1.5    $10^{-5}$ Torr), $B_T$ = **2 T** (at R = 1.8 m)




**Conventional condition** **Connection length** **Voltage gain** **Ionization number**

# Equation of Motion for Charged Particles [1]

- Equation of motion for charged particle

$$m\frac{d\boldsymbol{v}}{dt} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

- To the first order in m/q, the instantaneous acceleration $\frac{d\boldsymbol{v}}{dt}$ of the guiding center position **r** [2]

$$m\frac{d\boldsymbol{v}}{dt} = q[\boldsymbol{E}(\boldsymbol{r}) + \boldsymbol{v} \times \boldsymbol{B}(\boldsymbol{r})] - \mu\nabla B(\boldsymbol{r})$$

➡ This equation gives a solution whose **instantaneous value is not physically relevant**, **only the low frequency part** of the solution **has a physical meaning** (the **guiding center drifts**)

[1] F. Mottez, J.COMP.PHYSICS **227** (2008) 3260-2381
[2] T. Northrop, The Adiabatic Motion of Charged Particles, Interscience Publishers, 1963

# Implicit method under Cylindrical Coordinate

- Discretized equation of motion for charged particle

$$\boldsymbol{v}_{n+1/2} = \boldsymbol{v}_{n-1/2} + \Delta t\left[\overline{\boldsymbol{a}}_n - \frac{\mu}{m}\boldsymbol{\nabla}B_n(\boldsymbol{x}_n) + \frac{q}{m}\boldsymbol{u}_n \times \boldsymbol{B}_n(\boldsymbol{x}_n) + \boldsymbol{a}_{\boldsymbol{n}}^{\boldsymbol{fictious}}\right]$$

Centrifugal + Coriolis

$$\boldsymbol{x}_{n+1} = \boldsymbol{x}_{\boldsymbol{n}} + \mathbf{h}^{-1}\Delta t\, \boldsymbol{v}_{n+1/2}$$

metrics

Implicit parameters (D1)

$$\overline{\boldsymbol{a}}_n = \frac{1}{2}\left(\frac{q}{m}\boldsymbol{E}_{n+1} + \overline{\boldsymbol{a}}_{n-1}\right)$$

$$\boldsymbol{u}_n = \frac{1}{2}\left(\boldsymbol{v}_{n+1/2} + \overline{\boldsymbol{v}}_{n-1/2}\right)$$

$$\overline{\boldsymbol{v}}_{n-1/2} = \frac{1}{2}\left(\boldsymbol{v}_{n+1/2} + \overline{\boldsymbol{v}}_{n-3/2}\right)$$

- Substituting implicit parameters with D1 scheme

$$\boldsymbol{v}_{n+1/2} = \boldsymbol{v}_{n-1/2} + \frac{\Delta t}{2}\overline{\boldsymbol{a}}_{n-1} - \frac{\mu\Delta t}{m}\nabla B_n + \frac{q\Delta t}{4m}\overline{\boldsymbol{v}}_{n-3/2} \times \boldsymbol{B}_n(x_n) + \Delta t\boldsymbol{a}_{\boldsymbol{n}}^{\boldsymbol{fictious}} + \frac{\Delta t}{2}\frac{q}{m}\boldsymbol{E}_{n+1} - \boldsymbol{v}_{n+1/2} \times \boldsymbol{\Theta}_n(\boldsymbol{x}_n)$$

$t_{level} \leq n$ $\qquad$ $t_{level} = (n+1)$

where $\boldsymbol{\Theta}_n(\boldsymbol{x}_n) = \frac{3q\Delta t}{4m}\boldsymbol{B}_n(\boldsymbol{x}_n)$

# Prediction & Correction terms ( Case 1 )

- **Let** $\boldsymbol{v}_{n+1/2} = \widetilde{\boldsymbol{v}}_{n+1/2} + \boldsymbol{\delta v}_{n+1/2}$

For prediction ( $\equiv \widetilde{w}$)

$$\widetilde{\boldsymbol{v}}_{n+1/2} + \delta\boldsymbol{v}_{n+1/2} = \boldsymbol{v}_{n-1/2} + \frac{\Delta t}{2}\overline{\boldsymbol{a}}_{n-1} - \frac{\mu \Delta t}{m}\boldsymbol{\nabla} B_n + \frac{q\Delta t}{4m}\overline{\boldsymbol{v}}_{n-3/2} \times \boldsymbol{B}_n(x_n) + \Delta t \boldsymbol{a}_{\boldsymbol{n}}^{fictious} - \widetilde{\boldsymbol{v}}_{n+1/2} \times \boldsymbol{\Theta}_n(x_n)$$

$$+ \frac{\Delta t}{2}\frac{q}{m}\boldsymbol{E}_{n+1} - \delta\boldsymbol{v}_{n+1/2} \times \boldsymbol{\Theta}_n(\boldsymbol{x}_n)$$

For correction ($\equiv \delta$w)

- **The equation of motion is divided into two parts**

$$\widetilde{\boldsymbol{v}}_{n+1/2} = \widetilde{w} - \widetilde{\boldsymbol{v}}_{n+1/2} \times \boldsymbol{\Theta}_n(\boldsymbol{x}_n)$$

$$\delta\boldsymbol{v}_{n+1/2} = \delta w - \delta\boldsymbol{v}_{n+1/2} \times \boldsymbol{\Theta}_n(\boldsymbol{x}_n)$$

Predicted velocity $\widetilde{\boldsymbol{v}}_{n+1/2} = \frac{1}{1+\boldsymbol{\Theta}^2}(\mathbb{I} - \boldsymbol{\Theta} \times \mathbb{I} + \boldsymbol{\Theta\Theta}) \cdot w_n$

Correction velocity $\delta\boldsymbol{v}_{n+1/2} = \frac{1}{1+\boldsymbol{\Theta}^2}(\mathbb{I} - \boldsymbol{\Theta} \times \mathbb{I} + \boldsymbol{\Theta\Theta}) \cdot \delta w$

Note that

$\boldsymbol{E}_{n+1} = \boldsymbol{E}_{n+1}(\boldsymbol{x}_{n+1})$

Pre-Push $\widetilde{\boldsymbol{x}}_{n+1} = \boldsymbol{x}_{\boldsymbol{n}} + \boldsymbol{h}^{-1}\Delta t\, \widetilde{\boldsymbol{v}}_{n+1/2}$

Final-Push $\boldsymbol{x}_{n+1} = \widetilde{\boldsymbol{x}}_{n+1} + \boldsymbol{h}^{-1}\Delta t\, \delta\boldsymbol{v}_{n+1/2}$

➡ $\boldsymbol{x}_{n+1} = \boldsymbol{x}_{\boldsymbol{n}} + \boldsymbol{h}^{-1}\Delta t\, \boldsymbol{v}_{n+1/2}$

# Implicit Field Equation (Case 1)

- **How to guess $\boldsymbol{E}_{\boldsymbol{n+1}}(\boldsymbol{x}_{\boldsymbol{n+1}})$ ?**

**Gauss's law**

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}_{n+1}=\frac{\rho}{\epsilon_0}=\frac{1}{\epsilon_0}(\tilde{\rho}+\delta\rho)$$

**Continuity equation**

$$\delta n_s=-\boldsymbol{\nabla}\cdot(\tilde{n}_s\delta\boldsymbol{x}_s)=-\boldsymbol{\nabla}\cdot(\tilde{n}_s\delta\boldsymbol{v}_s\Delta t)$$

**Correction velocity**

$$\delta\boldsymbol{v}_{s,n+1/2}=\frac{1}{1+\boldsymbol{\Theta}_s^2}(\mathbb{I}-\boldsymbol{\Theta}_s\times\mathbb{I}+\boldsymbol{\Theta}_s\boldsymbol{\Theta}_s)\cdot\frac{\Delta t}{2}\frac{q_s}{m}\boldsymbol{E}_{n+1}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}_{n+1}=\frac{1}{\epsilon_0}\sum_s(q_s\tilde{n}_s+q_s\delta n_s)=\frac{\tilde{\rho}}{\epsilon_0}-\boldsymbol{\nabla}\cdot\left[\sum_s\frac{1}{2}\frac{\tilde{n}_sq_s^2}{\epsilon_0 m_s}\frac{\Delta t^2}{1+\Theta_s^2}(\mathbb{I}-\boldsymbol{\Theta}\times\mathbb{I}+\boldsymbol{\Theta}\boldsymbol{\Theta})\cdot\boldsymbol{E}_{\boldsymbol{n+1}}\right]$$

- **Implicit field equation**

$$\boldsymbol{\nabla}(\mathbb{I}+\overleftrightarrow{\chi})\cdot\boldsymbol{E}_{n+1}=\frac{\tilde{\rho}}{\epsilon_0}$$

where $\overleftrightarrow{\chi}\equiv\sum_s\frac{1}{2}\frac{\tilde{n}_sq_s^2}{\epsilon_0 m_s}\frac{\Delta t^2}{1+\Theta_s^2}(\mathbb{I}-\boldsymbol{\Theta}\times\mathbb{I}+\boldsymbol{\Theta}\boldsymbol{\Theta})$

$$\boldsymbol{\nabla}(\mathbb{I}+\overleftrightarrow{\chi})\cdot\boldsymbol{\nabla}\Phi_{n+1}=-\frac{\tilde{\rho}}{\epsilon_0}$$

assuming $\boldsymbol{E}=-\boldsymbol{\nabla}\Phi$