Theory of Alfven-Sound Frequency Gaps and Discovery of Alfven-Sound Eigenmodes in Tokamaks

C. Z. Cheng, PPPL

Co-authors:

G. J. Kramer, PPPL

M. Podesta, PPPL

R. Nazikian, PPPL

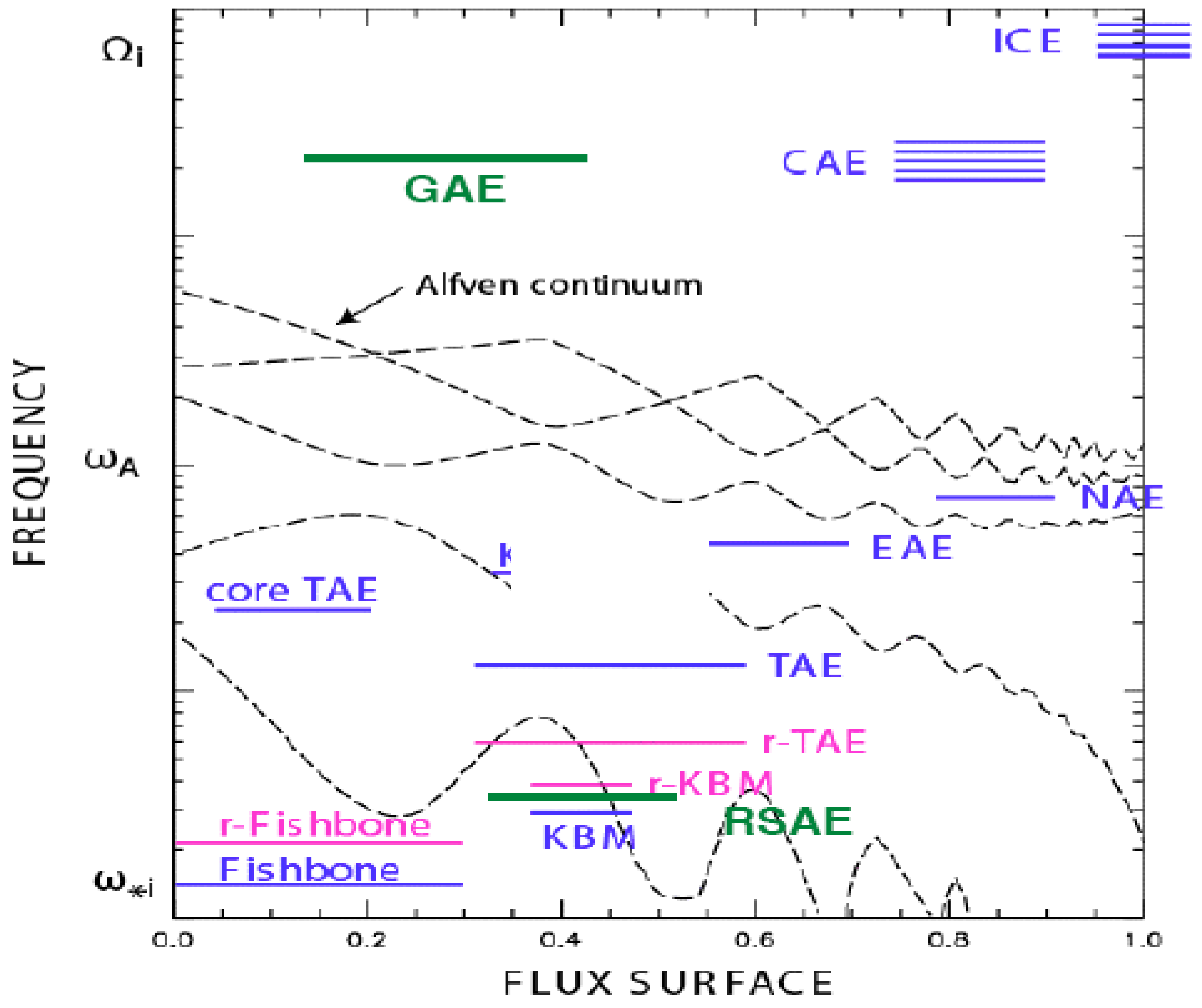
Outline

- **Introduction**
  - Alfven frequency gaps and Eigenmodes (TAE, EAE, NAE, RSAE, etc.)
- **Theory of Alfven-Sound Frequency Gaps**
  - MHD continuous spectrum of Alfven wave and slow mode
  - Coupling of Alfven wave and slow mode breaks up continuous spectrum to form Alfven-Sound (AS) gaps; Gap width increases with plasma pressure
- **Discovery of Alfven-Sound Eigenmodes (ASE)**
  - Existence of many types of ASEs: BAAE is one type of ASEs

    5 types of ASEs for $\beta(0)$=5% ; **6 types of ASEs** for $\beta(0)$=25%
  - ASEs can have dominant **shear Alfven wave** nature, or dominant **compressional slow mode** nature, or mixture of **shear and compressional** components
- **Summary - Application of ASEs**

Introduction:

Zoo of Alfven Eigenmodes in Toroidal Plasmas

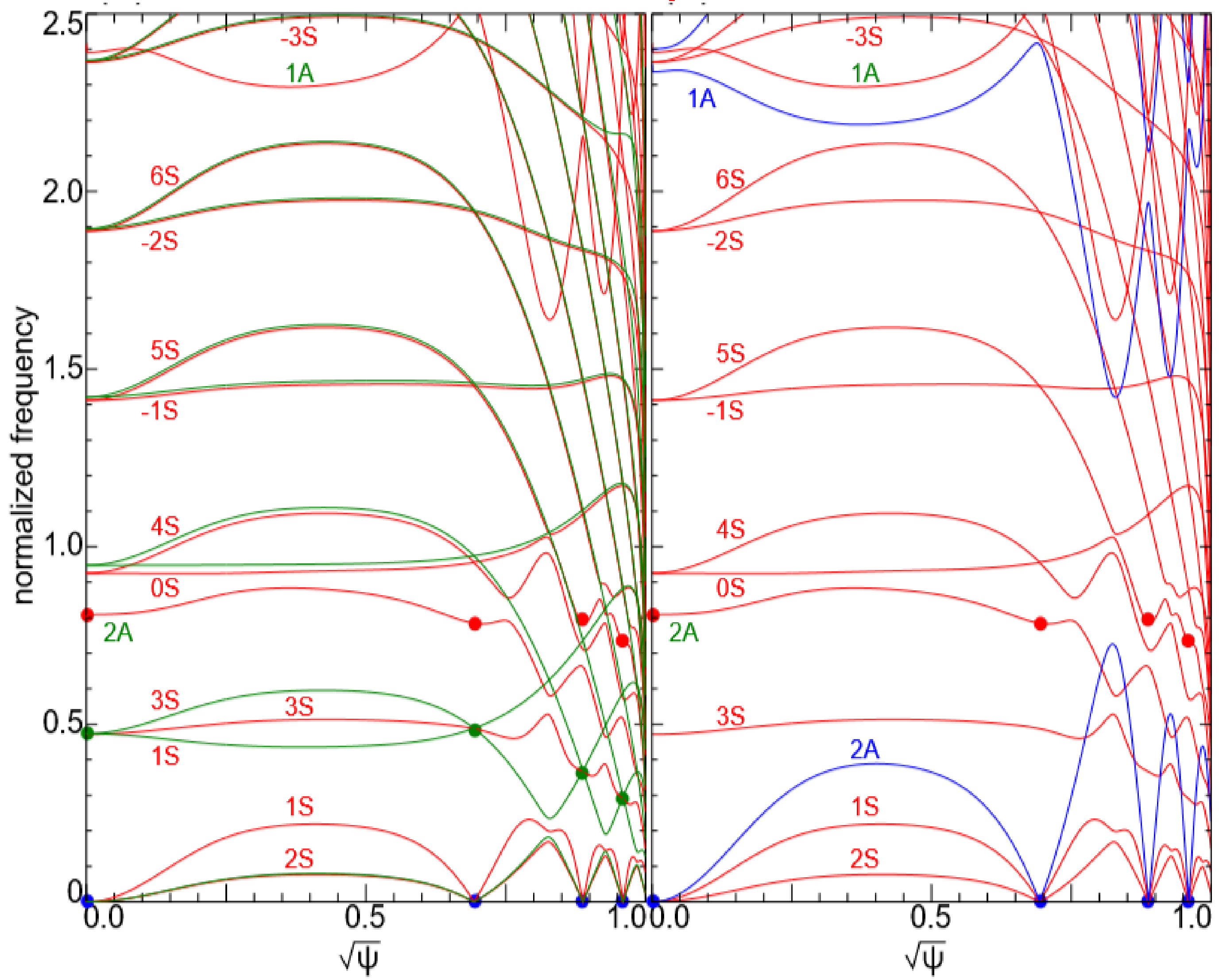


Heidbrink, Phys. Pl. 9 (2002) 2113

Coupling physics between Alfven wave and slow mode was not well studied !

Comparison: Full MHD Solution and Uncoupled Alfven & Slow Mode Continuous Spectrum

$n = 2$, $\beta(0) = 5\%$



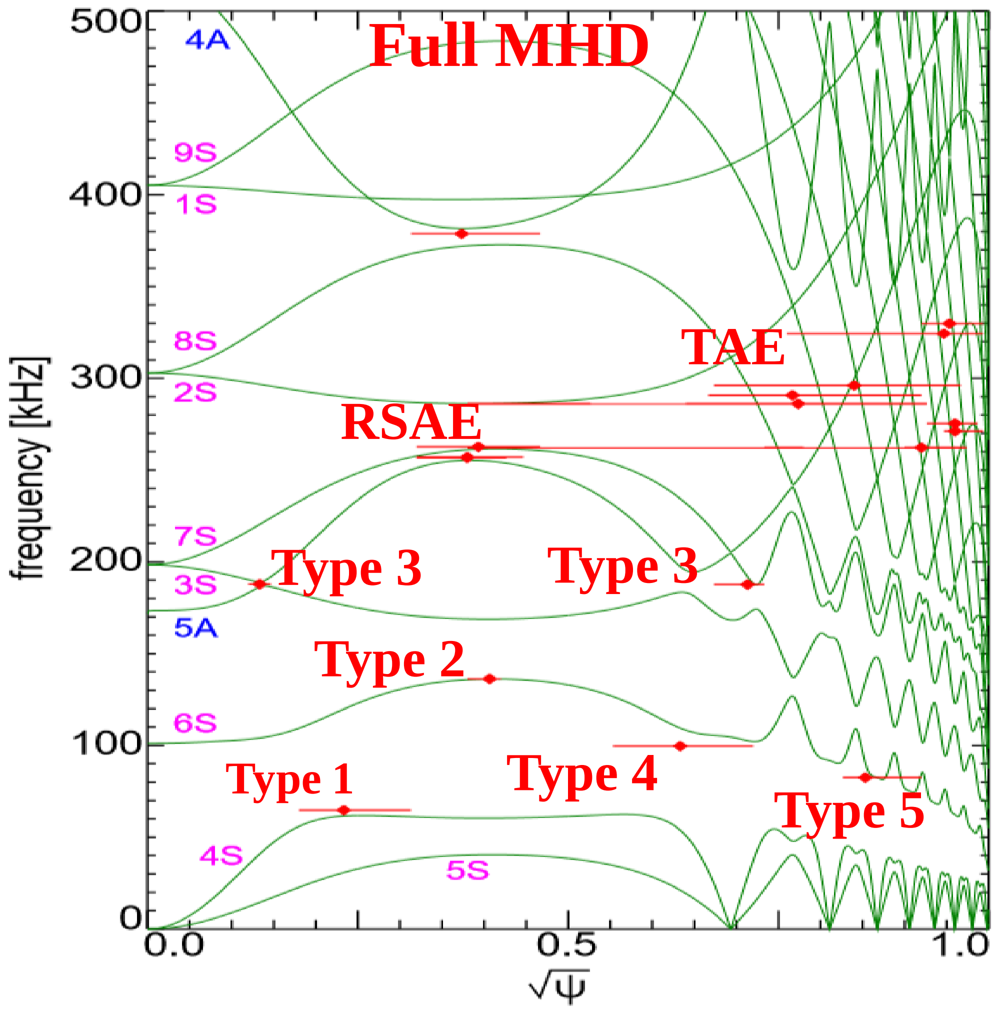
Reverse shear, circular shape tokamak equilibrium with R/a = 3; q(0) = 1; q_{min} = 0.92; q(1)=2.42

- **Alfven wave - slow mode coupling breaks up continuous spectrum below TAE gap to form Alfven-Sound (AS) gaps**
- **Several types of Alfven-Sound Eigenmodes (ASE) exist with frequency in AS gaps**

- **Red curves:** Full MHD results
- **Green curves:** uncoupled slow mode
- **Blue curves:** uncoupled Alfven wave

Alfven-Slow Mode Coupling: Alfven-Sound Gaps & Alfven-Sound Eigenmodes

n **= 5 AS gaps & ASEs for $\beta(0)$=5% Circular Tokamak**



$q(0) = 1$; $q_{min} = 0.92$; $q(1) = 2.42$; R/a=3

- **Coupling of Alfven wave and slow mode breaks up continuous spectrum below TAE gap to form Alfven-Sound (AS) gaps**
- **Several types of Alfven-Sound Eigenmodes (ASE) exist with frequency in AS gaps**

MHD Continuous Spectrum

Coupling of Alfven wave and slow mode (Cheng-Chance (1986)):

Alfven wave

$$\boxed{\mathbf{B}\cdot\nabla\left(\frac{|\nabla\psi|^2}{B^2}\mathbf{B}\cdot\nabla\xi_s\right)+\frac{\rho\omega^2|\nabla\psi|^2}{B^2}\xi_s}+\gamma_s P K_s\Delta=0$$

Slow mode

$$\boxed{\mathbf{B}\cdot\nabla\left(\frac{\gamma_s P}{\rho\omega^2 B^2}\mathbf{B}\cdot\nabla\Delta\right)+\frac{B^2+\gamma_s P}{B^2}\Delta}+K_s\xi_s=0$$

coupling

Magnetic field: $\mathbf{B}=\nabla(\zeta-q\theta)\times\nabla\psi$; **Safety factor:** q

Shear displacement: $\xi_s=\boldsymbol{\xi}\cdot\mathbf{B}\times\nabla\psi/|\nabla\psi|^2$

Plasma compression: $\Delta=\nabla\cdot\boldsymbol{\xi}$

Geodesic magnetic curvature: $K_s=2\boldsymbol{\kappa}\cdot\mathbf{B}\times\nabla\psi/B^2$

Continuous Spectrum by Lagrangian Functional

Cheng-Chance (1986):

$$\delta L = \int_0^{2\pi} d\theta \left\{ \rho\omega^2 \mathcal{J} \left(\frac{|\nabla\psi|^2}{\mathcal{J}B^2} |Y_1|^2 + B^2|Z|^2 \right) - \left[\frac{|\nabla\psi|^2}{\mathcal{J}B^2} |\frac{\partial Y_1}{\partial\theta}|^2 + \left(\frac{\mathcal{J}\gamma_s P B^2}{\gamma_s P + B^2} \right) |K_s Y_1 - \frac{1}{\mathcal{J}} \frac{\partial Z}{\partial\theta}|^2 \right] \right\} = 0$$

$$Z = \gamma_s P(\partial Y_2/\partial\theta)/(\mathcal{J}\omega^2\rho B^2)$$

$$Y_1 = \hat{\xi}_s exp[in(\zeta - q\theta)]$$

Jacobian: $\mathcal{J} = (\nabla\psi \times \nabla\theta \cdot \nabla\zeta)^{-1}$

$$Y_2 = \hat{\Delta} exp[in(\zeta - q\theta)]$$

Toroidal mode number: ***n***

$$\xi(\psi,\theta,\zeta) = \hat{\xi}(\psi,\theta)e^{-in\zeta}$$

Coupling of Slow Mode *m*–1 and *m*+1 Harmonics with Alfven Wave *m* harmonic

$$\boxed{\omega^2-\omega_A^2(m-nq)^2-\omega_{Kp}^2 = \omega_{AS}^4\left[\frac{(m-1-nq)^2}{(\omega^2-\omega_S^2(m-1-nq)^2}+\frac{(m+1-nq)^2}{(\omega^2-\omega_S^2(m+1-nq)^2)}\right]}$$

$$\omega_A^2=\frac{\int_0^{2\pi}d\theta|\nabla\psi|^2/(\mathcal{J}B^2)}{\int_0^{2\pi}d\theta\mathcal{J}\rho|\nabla\psi|^2/B^2} \quad ; \quad \omega_S^2=\frac{\int_0^{2\pi}d\theta\gamma_s PB^2/[\mathcal{J}(\gamma_s P+B^2)]}{\int_0^{2\pi}d\theta\mathcal{J}\rho B^2}$$

$$\omega_{Kp}^2=\frac{\int_0^{2\pi}d\theta\mathcal{J}\gamma_s PB^2K_s^2/(\gamma_s P+B^2)}{\int_0^{2\pi}d\theta\mathcal{J}\rho|\nabla\psi|^2/B^2} \quad ; \quad \omega_{AS}^4=\frac{\left(\int_0^{2\pi}d\theta\gamma_s PB^2K_s sin\theta/(\gamma_s P+B^2)\right)^2}{\left(\int_0^{2\pi}d\theta\mathcal{J}\rho B^2\right)\left(\int_0^{2\pi}d\theta\mathcal{J}\rho|\nabla\psi|^2/B^2\right)}$$

Large aspect ratio limit:

$$\boxed{V_A^2 = B^2/\rho \quad ; \quad C_S^2=\gamma_s PB^2/\rho(\gamma_s P+B^2)}$$

$$\boxed{\omega_A^2 \simeq (V_A/qR)^2 \quad ; \quad \omega_S^2 \simeq (C_S/qR)^2 \quad ; \quad \omega_{Kp}^2\simeq 2C_S^2/R^2 \quad ; \quad \omega_{AS}^4\simeq C_S^4/q^2R^4}$$

Coupling of Slow Mode m–1 and m+1 Harmonics with Alfven Wave m harmonic

Dispersion relation:

$$\boxed{\begin{aligned}(\omega^2-\omega_A^2\eta^2-\omega_{Kp}^2)[(\omega^2-\omega_S^2-\omega_S^2\eta^2)^2-4\eta^2\omega_S^4]\\ =2\omega_{AS}^4[(1+\eta^2)\omega^2-\omega_S^2(1-\eta^2)^2]\end{aligned}}$$

$\eta=m-nq$, and consider $m-1/2\le nq\le m+1/2$, so that $\eta^2\le 1/4$.

$$\boxed{(\omega^2-\omega_S^2)[(\omega^2-\omega_A^2\eta^2-\omega_{Kp}^2)(\omega^2-\omega_S^2)-2\omega_{AS}^4]=0}$$

Three solutions:

Alfven m harmonic: $\boxed{\omega^2\simeq\omega_A^2(m-nq)^2+\omega_{Kp}^2+\omega_S^2}$

One of slow mode $m\pm1$ harmonic: $\boxed{\omega^2\simeq\omega_S^2-(\omega_S^4+2\omega_{AS}^4)/[\omega_A^2(m-nq)^2+\omega_{Kp}^2+\omega_S^2]}$

Other slow mode $m\pm1$ harmonic: $\boxed{\omega^2=\omega_S^2}$

Coupling of Slow Mode *m*–1 and *m*+1 Harmonics with Alfven Wave *m* harmonic

Large aspect ratio limit, and for $m-1/2 \leq nq \leq m+1/2$

Alfven *m* harmonic:

$$\boxed{\omega^2 \simeq (V_A/qR)^2(m-nq)^2 + (1+2q^2)(C_S/qR)^2.}$$

One of slow mode *m* ±1 harmonic:

$$\boxed{\omega^2 \simeq (C_S/qR)^2\left[1-\frac{(1+2q^2)}{(V_A/C_S)^2(m-nq)^2+(1+2q^2)}\right]}$$

Other slow mode *m* ±1 harmonic: $\boxed{\omega^2 \simeq (C_S/qR)^2}$

Coupling of Alfven Wave *m*–1 and *m*+1 Harmonics with Slow Mode *m* harmonic

$$\omega^2 - \omega_S^2(m-nq)^2 = \frac{\omega_{AS}^4(m-nq)^2}{\omega^2 - \omega_A^2(m-1-nq)^2 - \omega_{Kp}^2} + \frac{\omega_{AS}^4(m-nq)^2}{\omega^2 - \omega_A^2(m+1-nq)^2 - \omega_{Kp}^2}.$$

Slow mode *m*-harmonic:

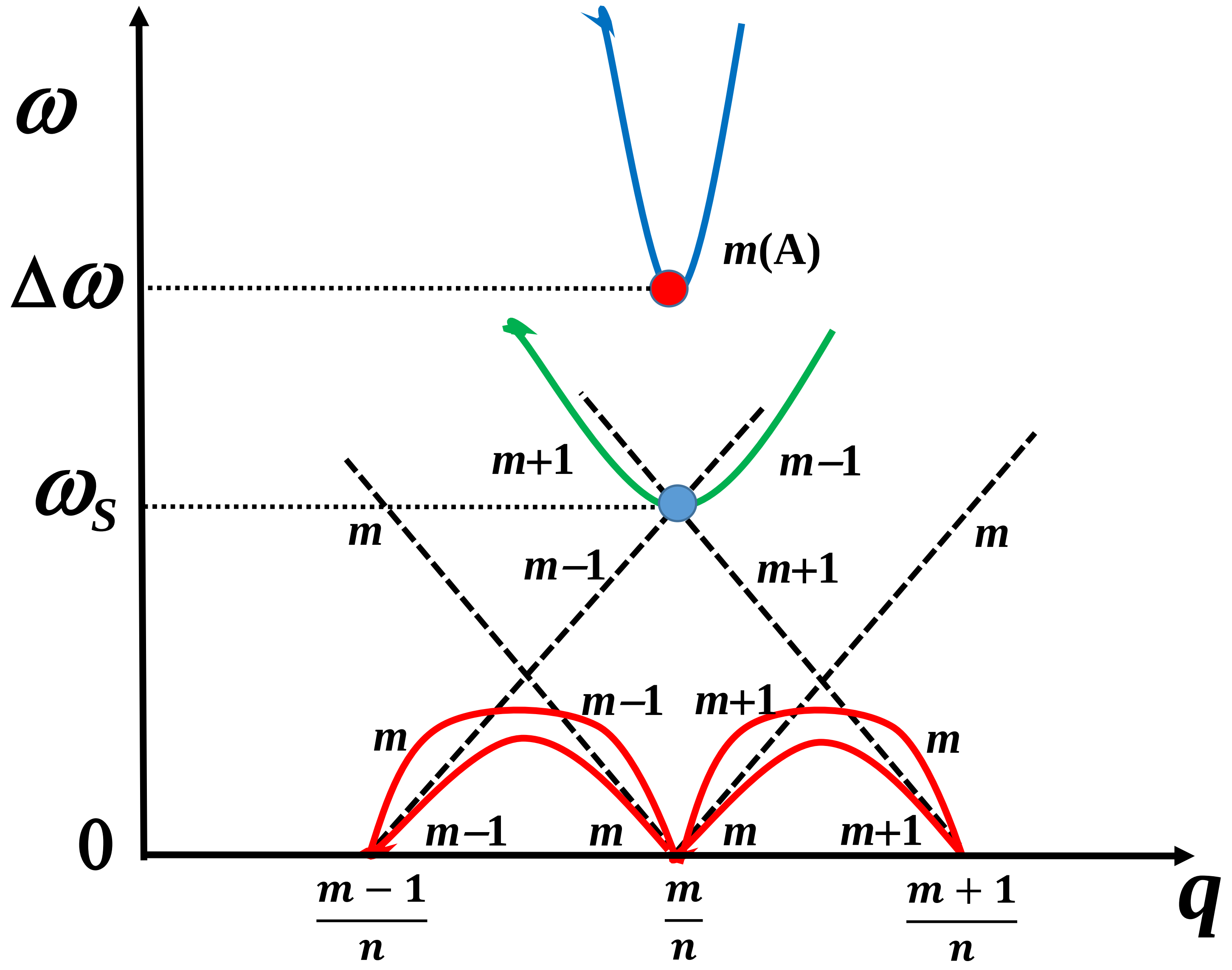
$$\omega^2 \simeq \omega_S^2(m-nq)^2\left[1 - 2\omega_{AS}^4/\omega_S^2(\omega_A^2+\omega_{Kp}^2)\right]$$

Large aspect ratio limit:

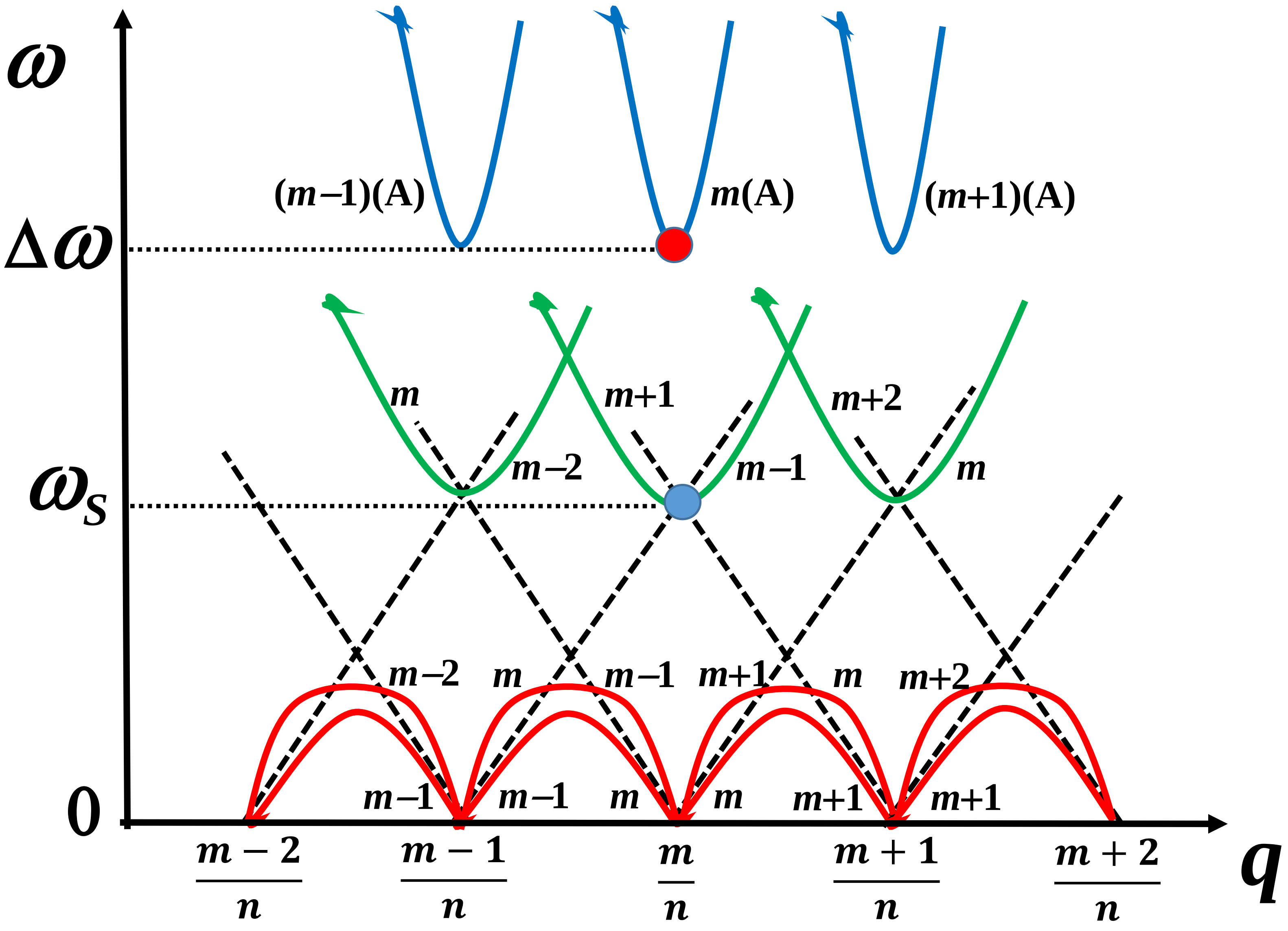
$$\omega^2 \simeq \omega_S^2(m-nq)^2/\left[1+2q^2\gamma_s P/(B^2+\gamma_s P)\right]$$

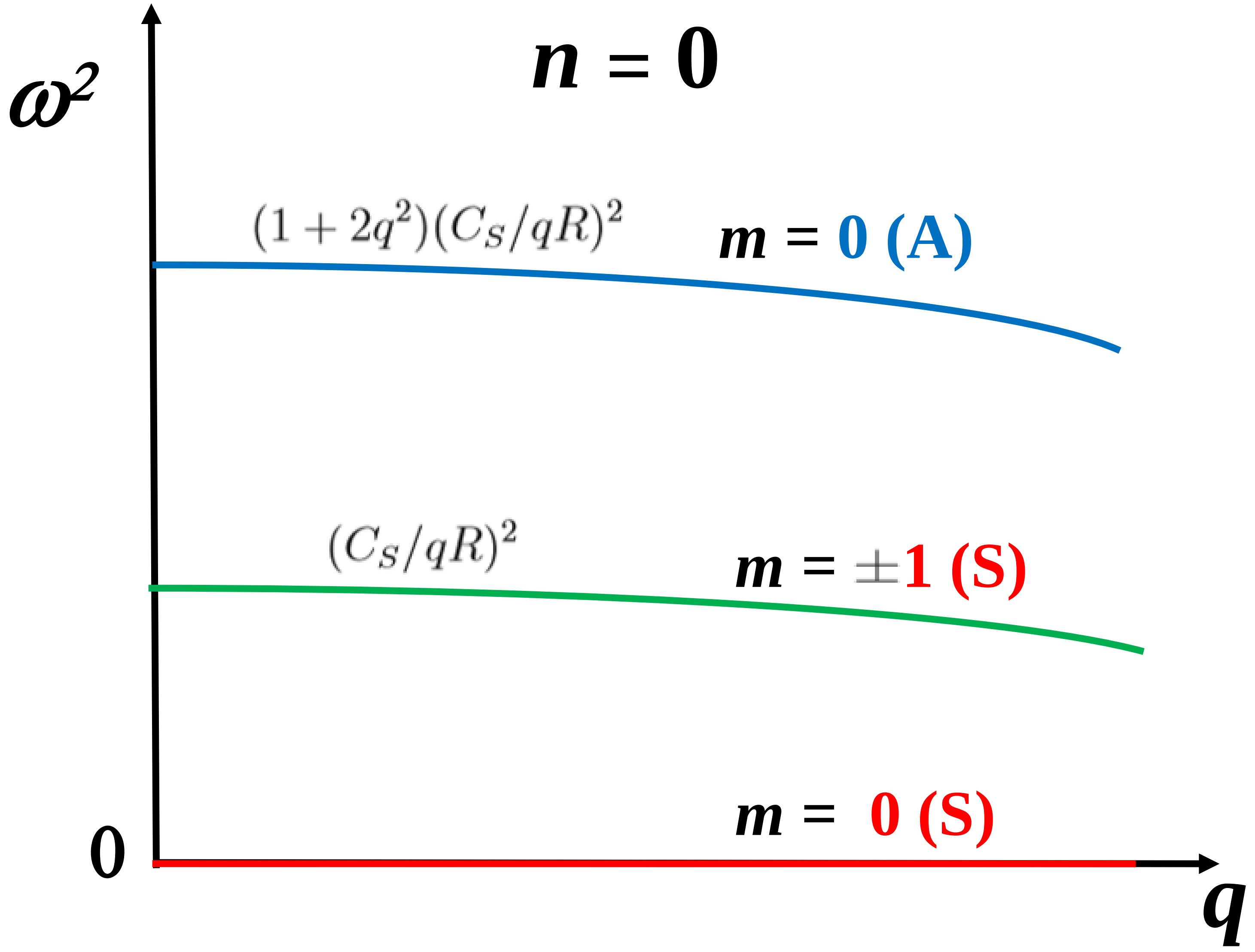
Alfven *m*±1 harmonics:

$$\omega^2 \simeq \omega_A^2(m-nq\pm 1)^2 + \omega_{Kp}^2$$



ω
$\Delta\omega$
ω_s
0
$m(A)$
$m+1$
$m-1$
m
$m-1$
$m+1$
m
$m-1$
$m+1$
m
m
$m-1$
m
m
$m+1$
$\frac{m-1}{n}$
$\frac{m}{n}$
$\frac{m+1}{n}$
q


ω
Δω
ω_S
0
q
(m–1)(A)
m(A)
(m+1)(A)
m
m+1
m+2
m–2
m–1
m
m–2
m
m–1
m+1
m
m+2
m–1
m–1
m
m
m+1
m+1
(m − 2)/n
(m − 1)/n
m/n
(m + 1)/n
(m + 2)/n


ω^2
$n = 0$
$(1+2q^2)(C_S/qR)^2$
m = 0 (A)
$(C_S/qR)^2$
$m = \pm 1$ (S)
m = 0 (S)
0
q

Circular Tokamak

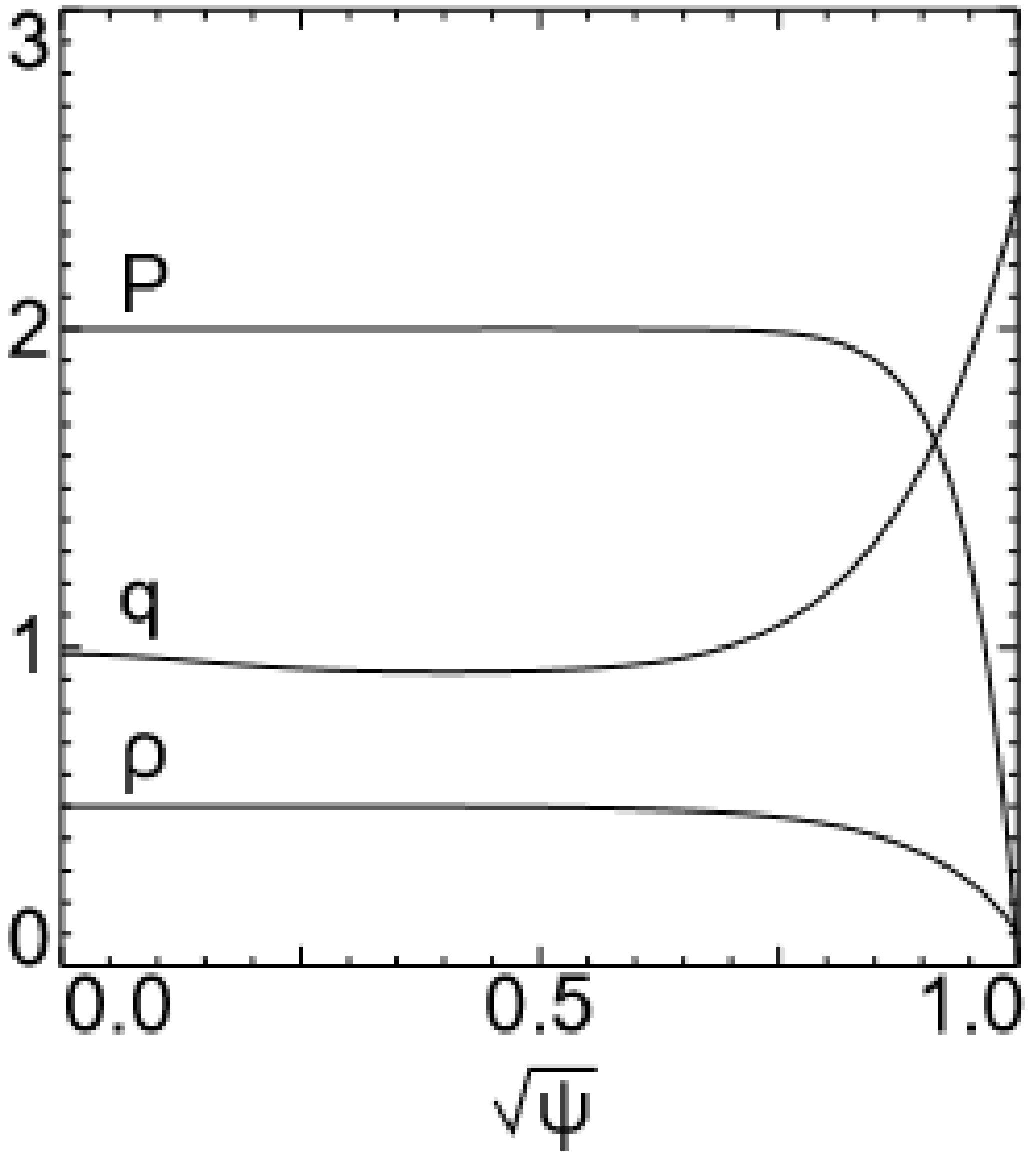
Reverse shear,

Circular shape tokamak equilibrium with

$R/a = 3$

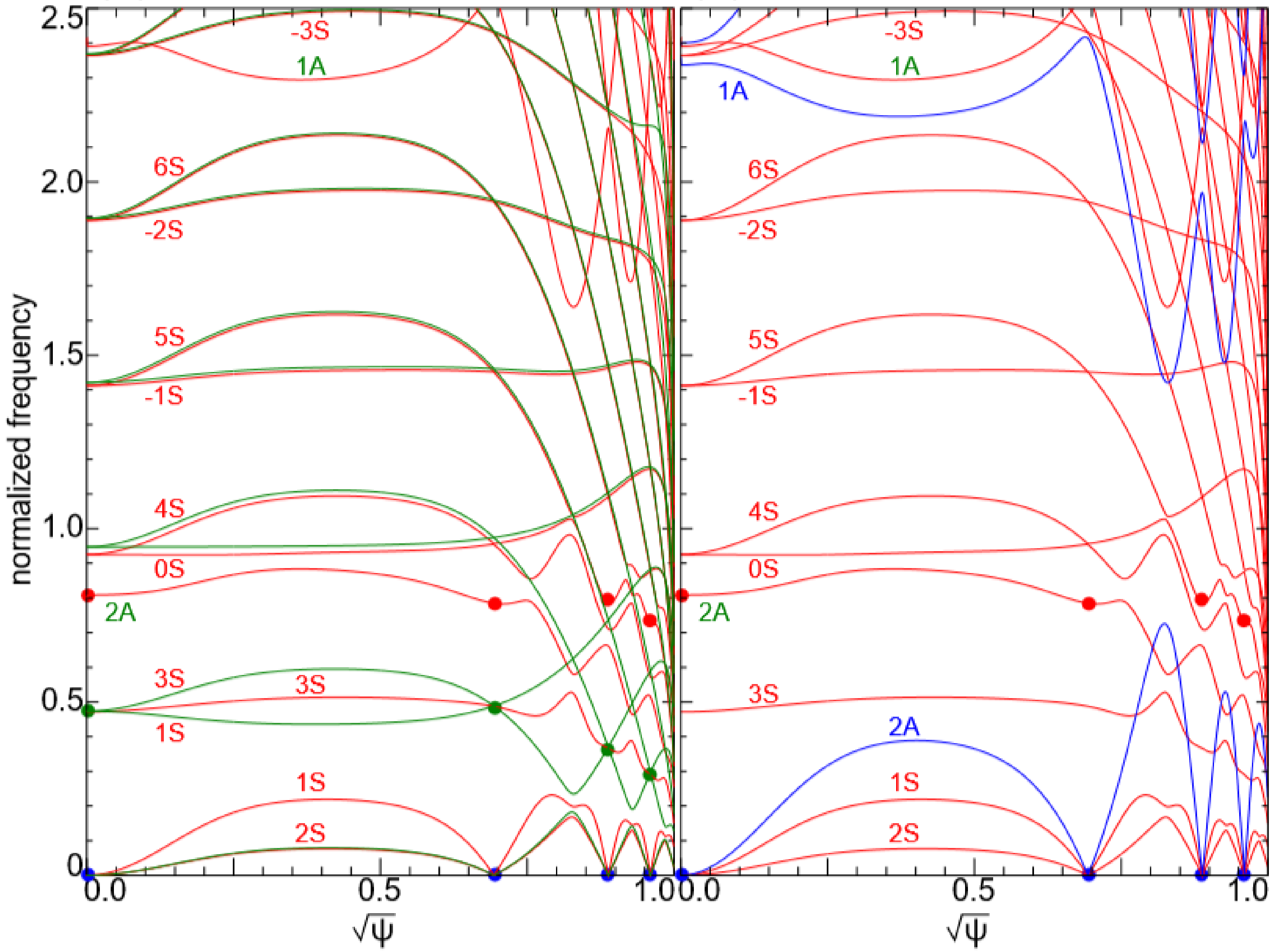
$q(0) = 1$; $q_{min} = 0.92$;

$q(1)=2.42$



Comparison: Full MHD Solution and Uncoupled Alfven & Slow Mode Continuous Spectrum

$n = 2$, $\beta(0) = 5\%$



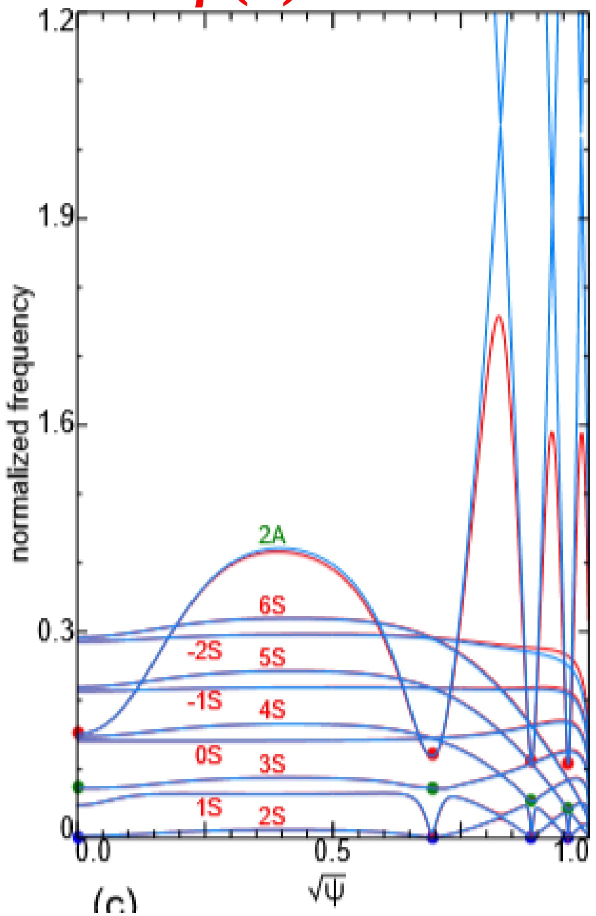
Reverse shear, circular shape tokamak equilibrium with
R/a = 3
$q(0) = 1$; $q_{min} = 0.92$
$q(1)=2.42$

- **Red curves:**
  Full MHD results
- **Green curves:**
  Uncoupled slow mode
- **Blue curves:**
  uncoupled Alfven wave

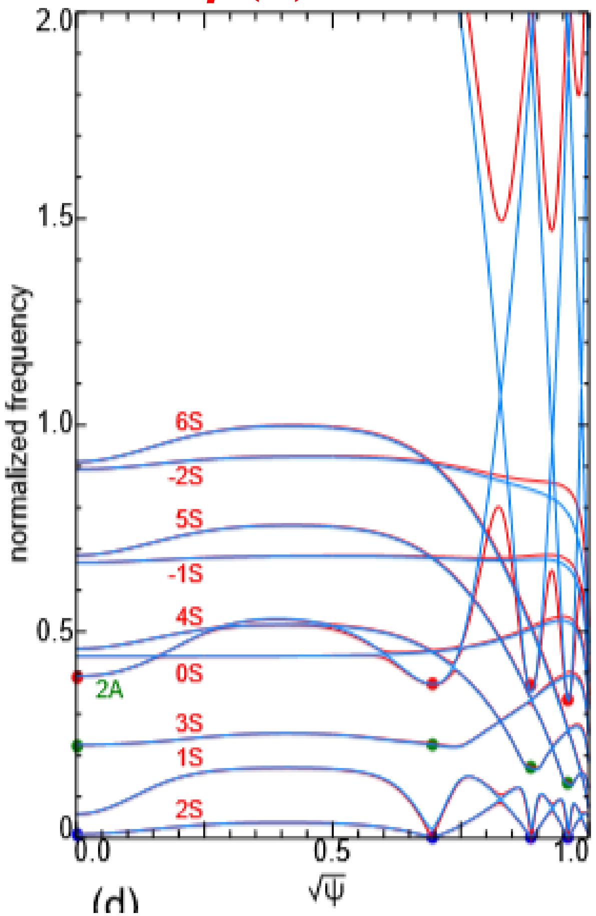
n = 2 Continuous Spectrum:

Comparison of 3-Harmonic Coupling Theory and Full MHD Solution

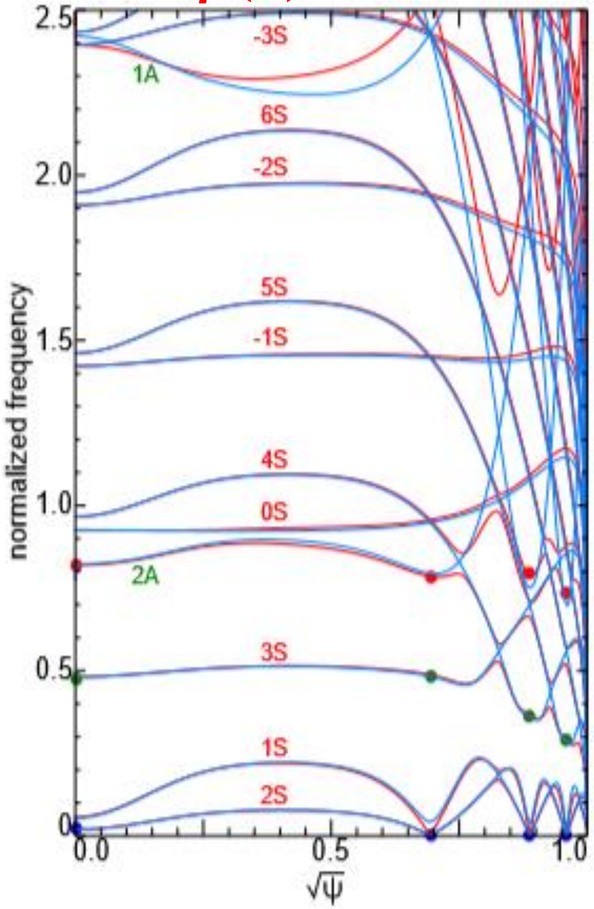
β(0)=0.1%



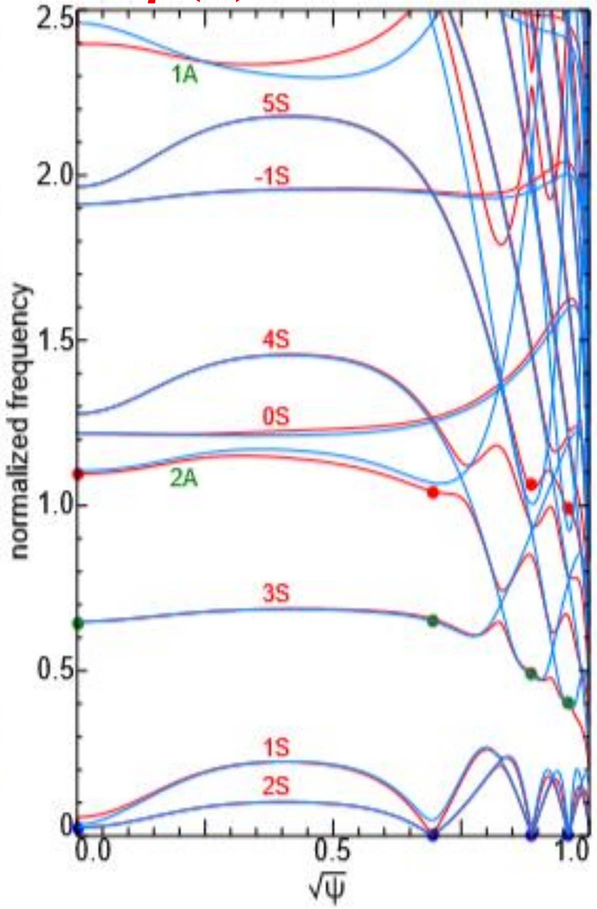
β(0)=1%



β(0)=5%

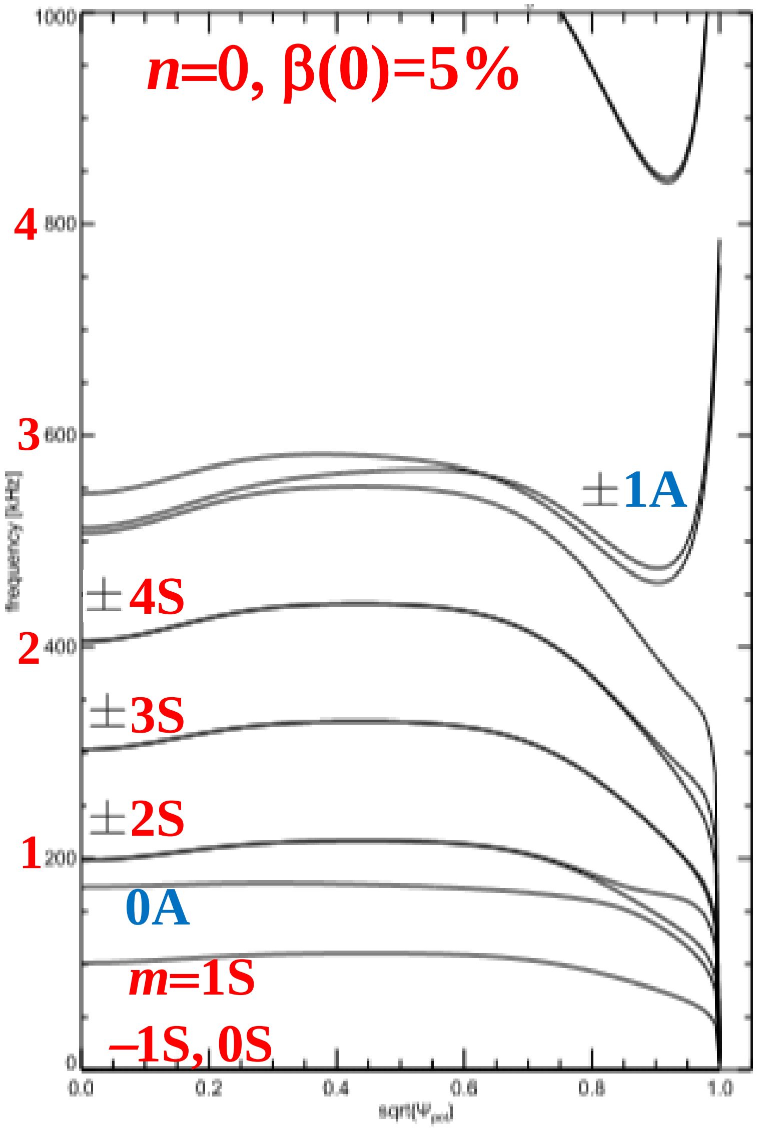


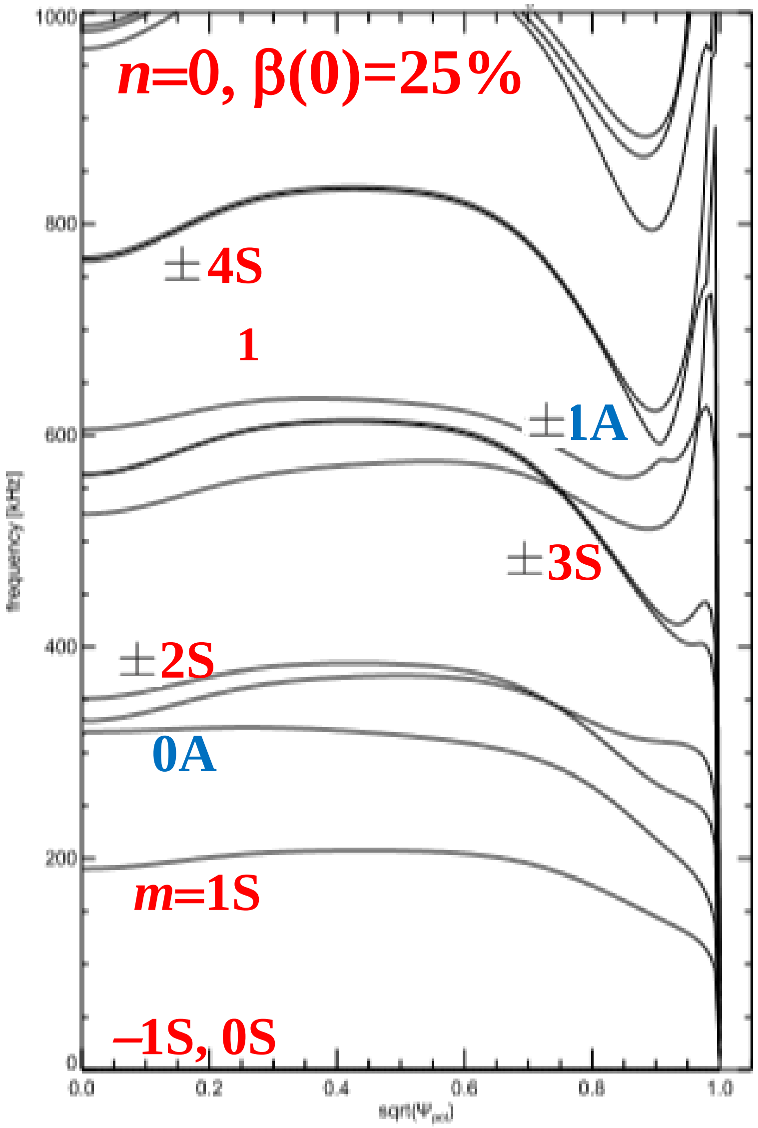
β(0)=10%



Blue curves: theory results ; **Red curves:** NOVA full MHD results

n = 0 MHD Continuous Spectrum
Coupling between Alfven Wave & Slow Mode

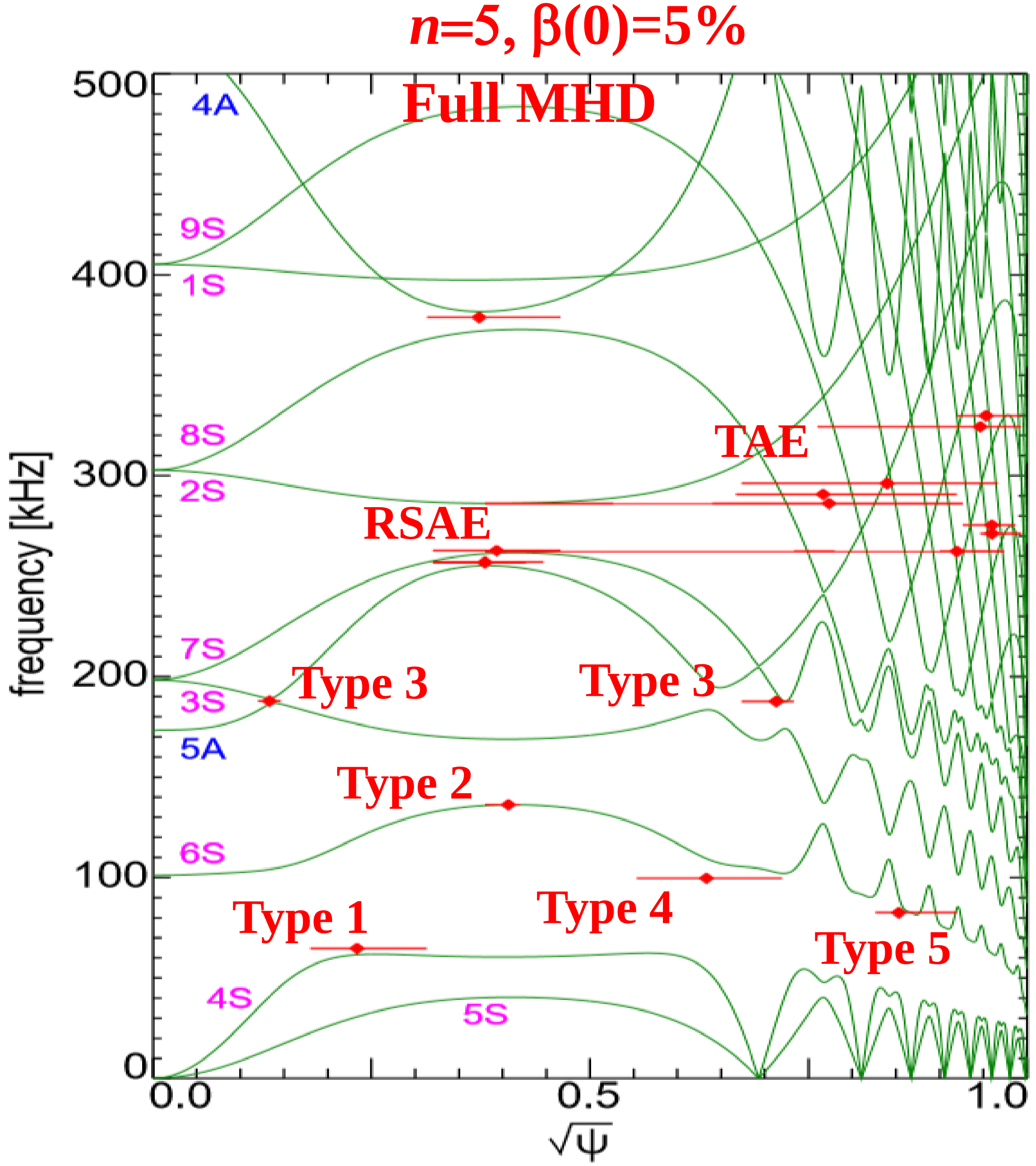




- **Reverse shear, circular shape equilibrium with**
  R/a = 3; q(0) = 1;
  q_{min} = 0.92; q(1)=2.42
- **normalized frequency**
  =214 kHz

- ***n*=0 MHD results**
- **Coupling of Alfven *m*=0 harmonic with slow mode *m*=±1 harmonics creates Alfven-Sound gaps**
- **Can GAM eigenmodes exist in the *n*=0 AS gaps ?**

n = 5 Eigenmodes for β(0)=5% Tokamak

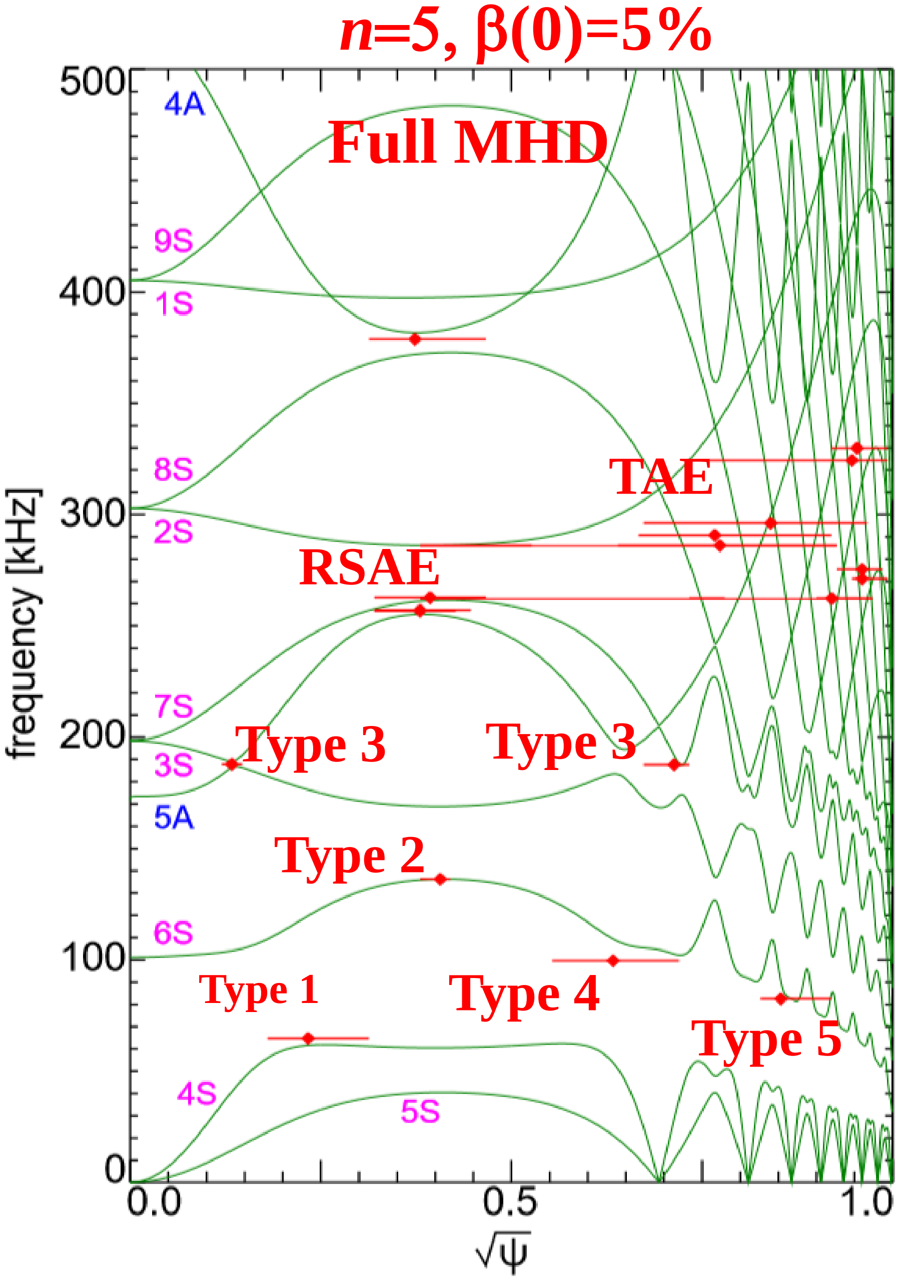


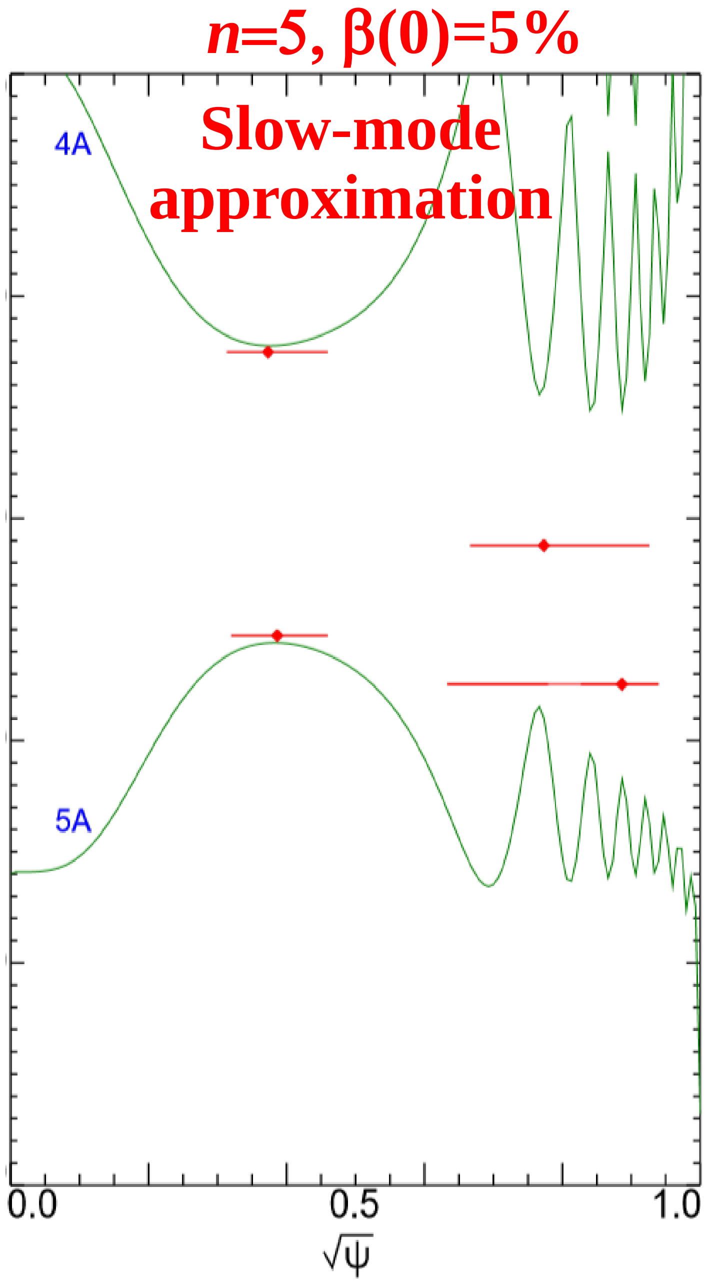
$q(0) = 1$; $q_{min} = 0.92$; $q(1) = 2.42$

5 types of ASEs are found:

- **Type 1:** Frequency above $m=nq-1= 4$ slow mode continuum in core region;
- **Type 2:** Frequency above $m=nq+1= 6$ slow mode continuum in core region ;
- **Type 3:** Frequencies in AS gaps due to coupling of Alfven $m=5$ harmonic with slow mode m-2 = 3 and m+2= 7 harmonics in core region
- **Type 4 (BAAE):** Frequency below $m=nq+1= 6$ slow mode continuum; mode localized around $q=m/n$
- **Type 5:** Frequency in lowest AS gap, mode localized in outer radial region

$n = 5$ Eigenmodes for β(0)=5% Circular Tokamak





- **Horizontal bars** indicate eigenmode frequencies & half-width of radial structure

normalized frequency

$$B_0/2\pi\sqrt{\rho(0)}q(1)R = 214.445 \text{ kHz}$$

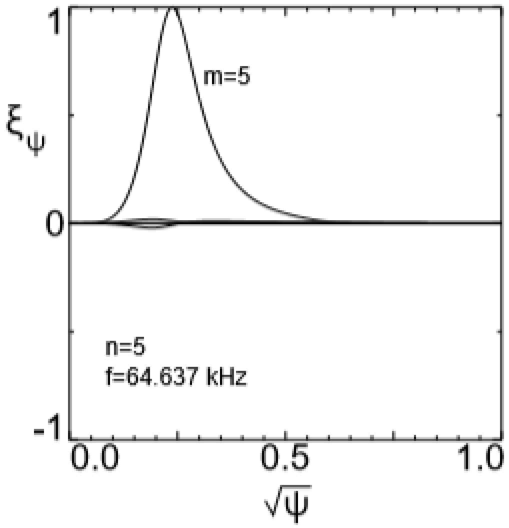
- **Slow-mode approximation**
  - no Alfven-Sound coupling physics
  - no ASE solutions
  - less number of TAEs & RSAEs

Five Types of ASEs for β(0)=5% Case

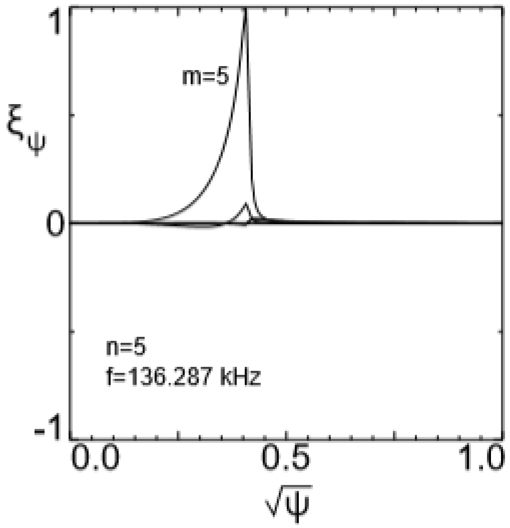
- **Type 1:** Frequency above ($m=nq-1$) slow mode continuum in core region; for **n=5, *q*=1, *m*=4**
- **Type 2:** Frequency above ($m=nq+1$) slow mode continuum in core region ; for **n=5, *q*=1, *m*=6**
- **Type 3:** Frequencies in AS gaps due to coupling of Alfven m-harmonic and slow mode (m-2) and (m+2) harmonics in core region
- **Type 4 (BAAE):** Frequency below ($m=nq+1$) slow mode continuum; mode localized around $q=m/n$; for ***n*=5, *q*=1, *m*=6**
- **Type 5:** Frequency in lowest AS gap, mode localized in outer radial region

n = 5 Alfven-Sound Eigenmodes for β(0)=5%

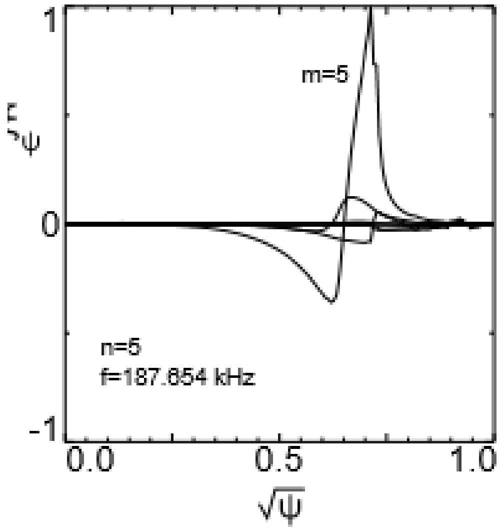
Type 1



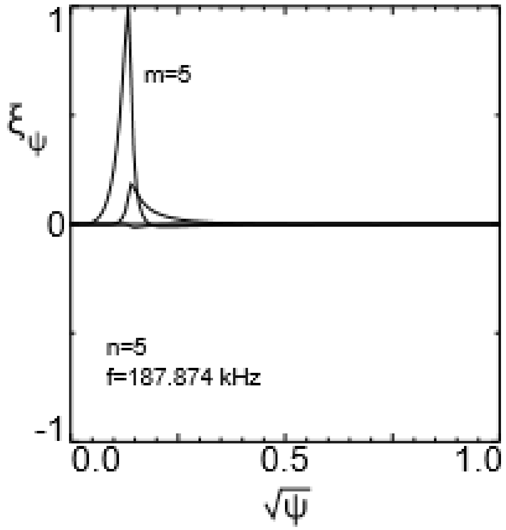
Type 2



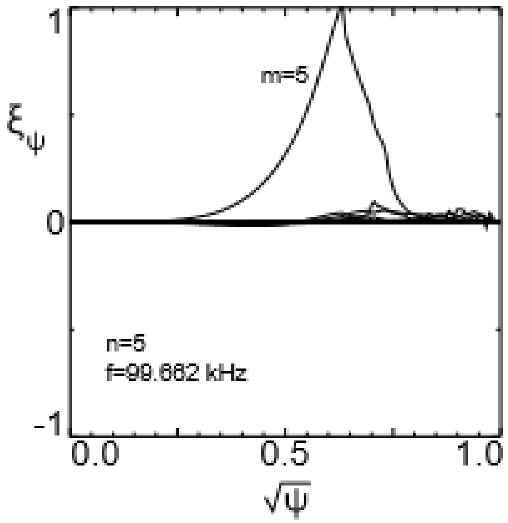
Type 3



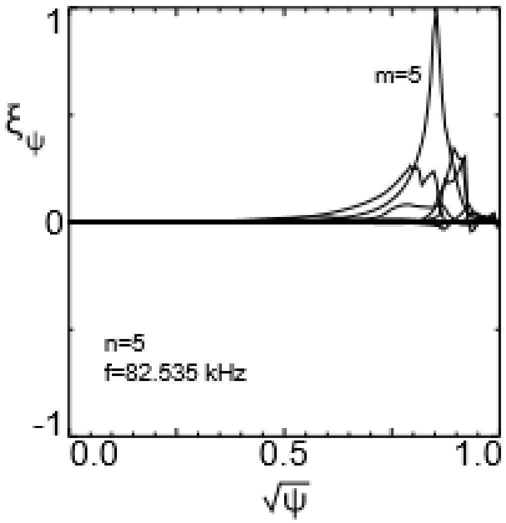
Type 3



Type 4

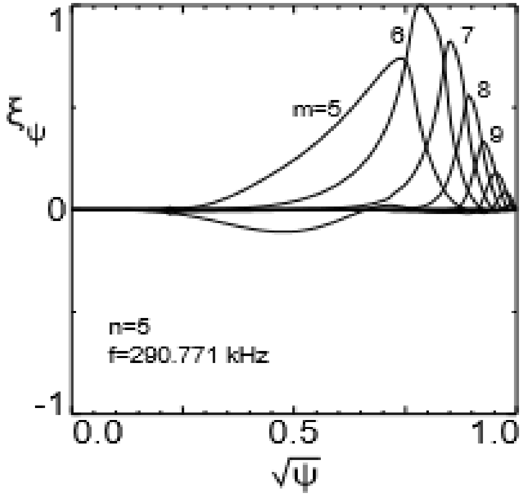


Type 5

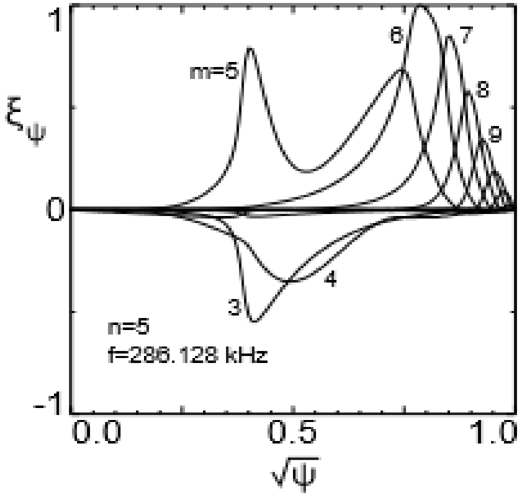


n = 5 Eigenmodes for β(0)=5%

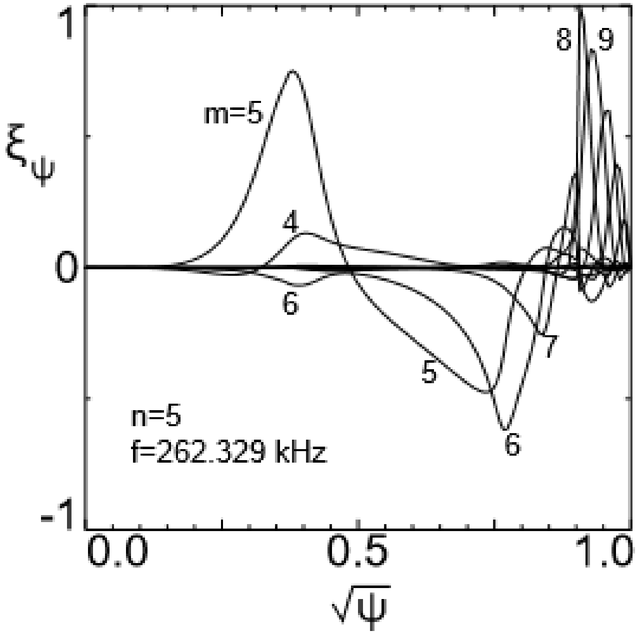
TAE



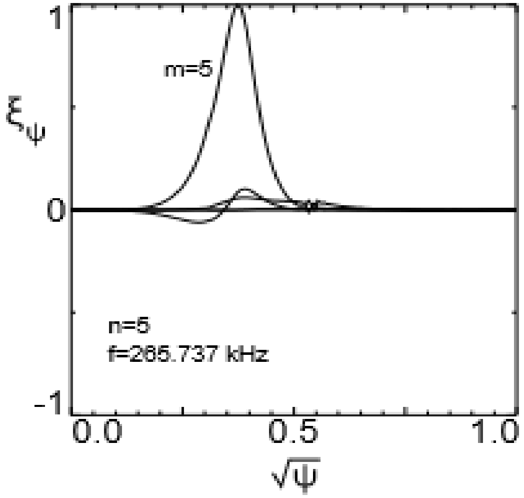
RSAE-Sound-TAE



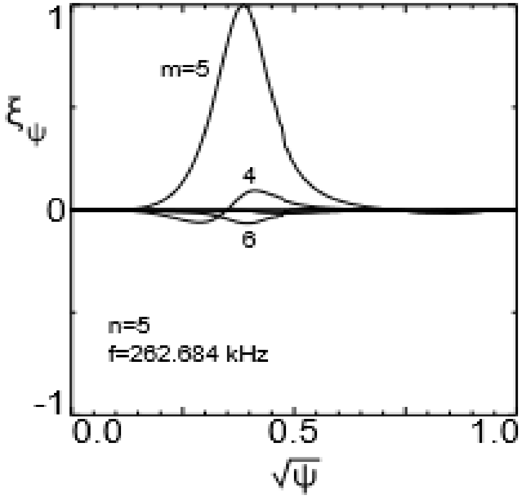
RSAE-TAE-Sound



RSAE

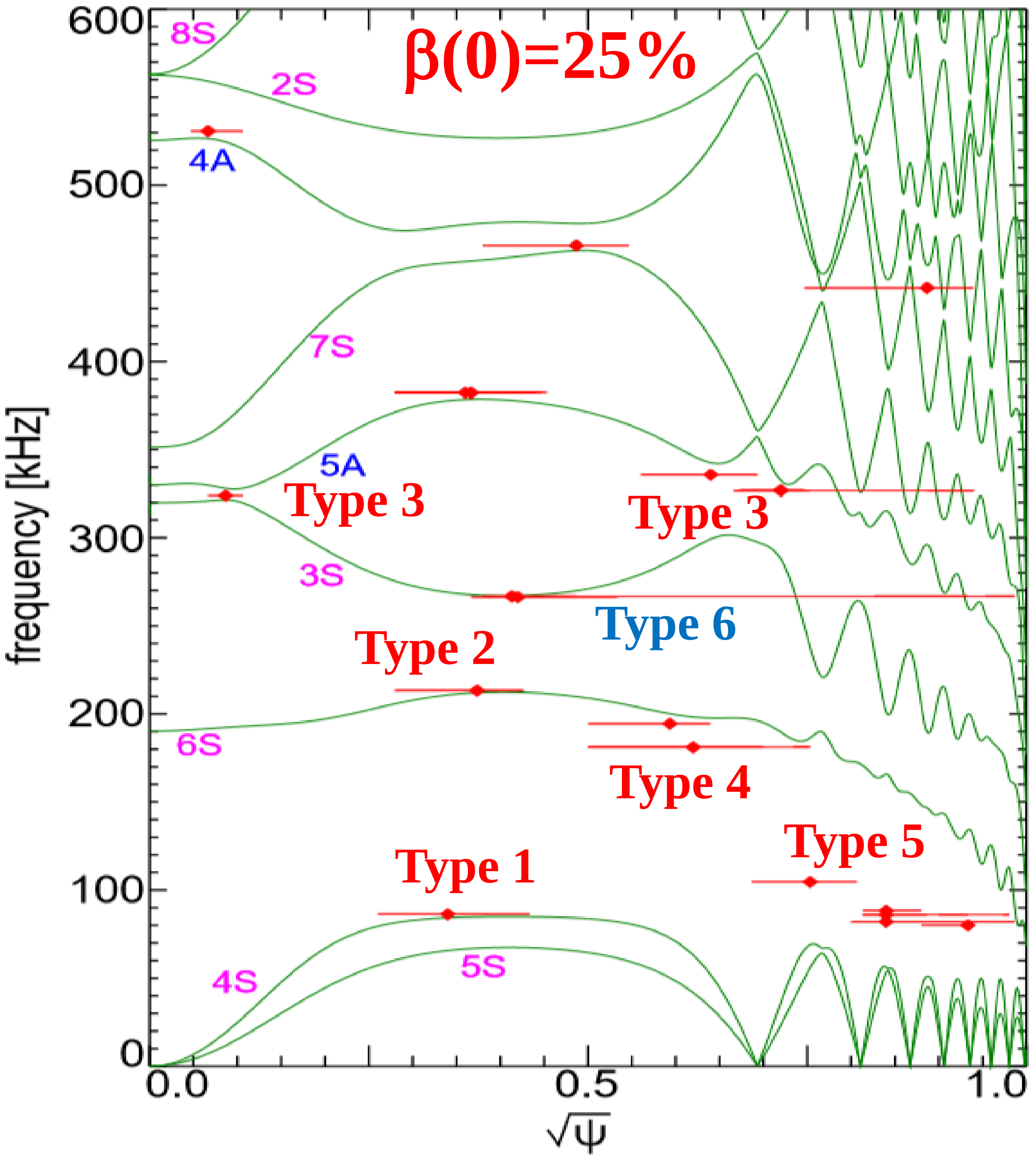


RSAE-Sound



- **More TAEs, RSAEs than model without considering Alfven-Sound coupling.**
- **New modes with mixed features of TAE, RSAE, slow mode**

n = 5 Eigenmodes for β(0)=25% Circular Tokamak



- **Horizontal bars indicate eigenmode frequencies & half-width of radial structure**

normalized frequency

$B_0/2\pi\sqrt{\rho(0)}q(1)R = 214.445\ \text{kHz}$

- **6 types of ASEs**

  - more ASE Type than β(0)=5% case

  - more ASE solutions per Type

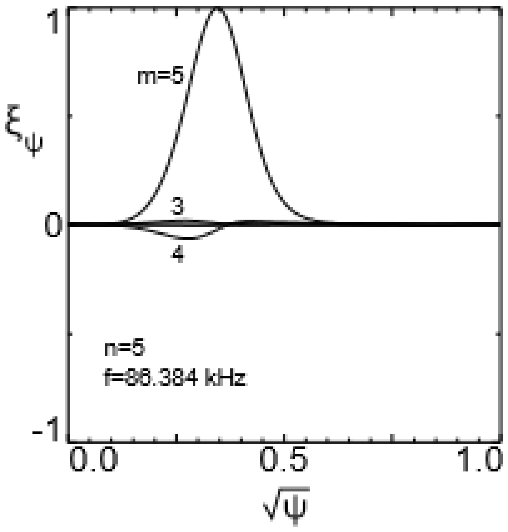
  - less number of TAEs & RSAEs

Six Types of ASEs for β(0)=25% Case

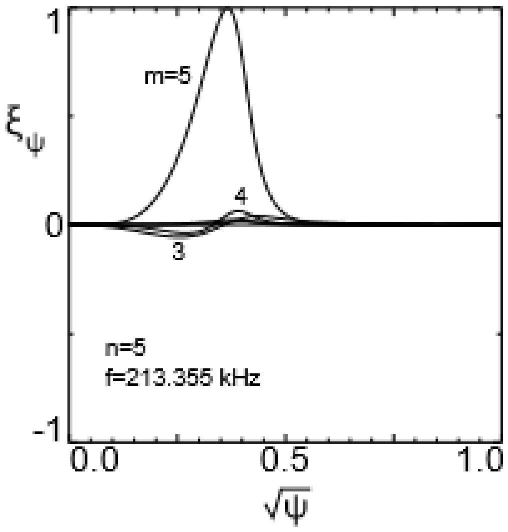
- **Type 1:** Frequency above (m=nq-1) slow mode continuum in core region; for **n=5, *q*=1, *m*=4**
- **Type 2:** Frequency above (m=nq+1) slow mode continuum in core region; for **n=5, *q*=1, *m*=6**
- **Type 3:** Frequencies in AS gaps due to coupling of Alfven m-harmonic and slow mode (m-2) and (m+2) harmonics in core region
- **Type 4 (BAAE):** Frequency below (m=nq+1) slow mode continuum; mode localized around q=m/n ; for ***n*=5, *q*=1, *m*=6**
- **Type 5:** Frequency in lowest AS gap, mode localized in outer radial region
- **Type 6:** Frequency below (m=nq–2) slow mode continuum in core region; for **n=5, *q*=1, *m*=3**

n = 5 Alfven-Sound Eigenmodes for β(0)=25%

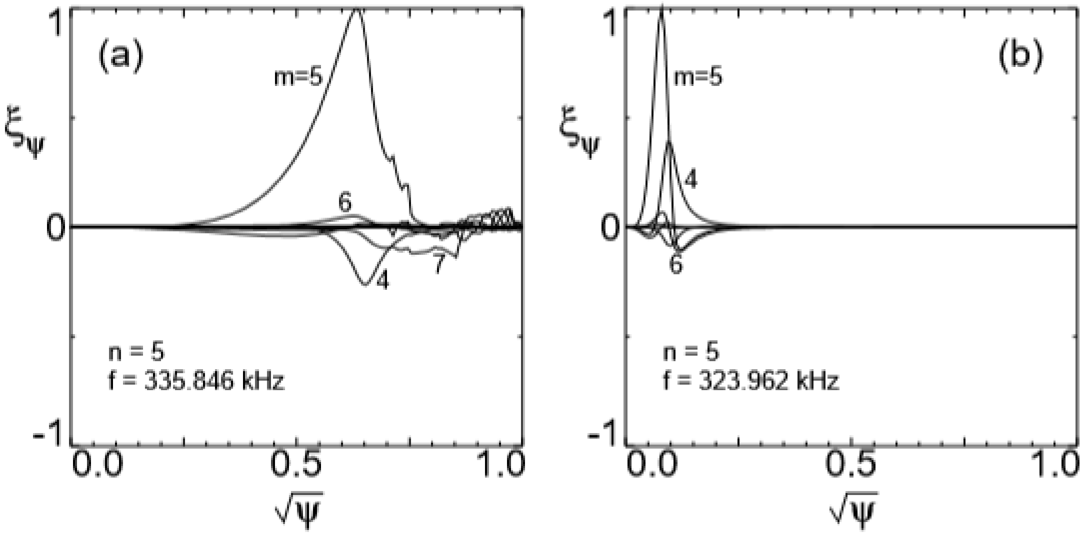
Type 1



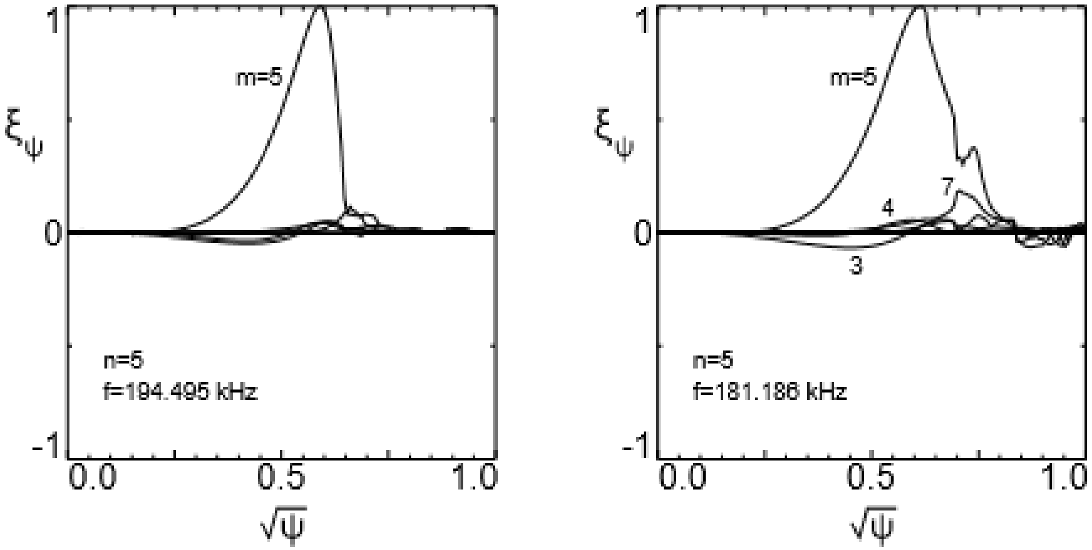
Type 2



Type 3

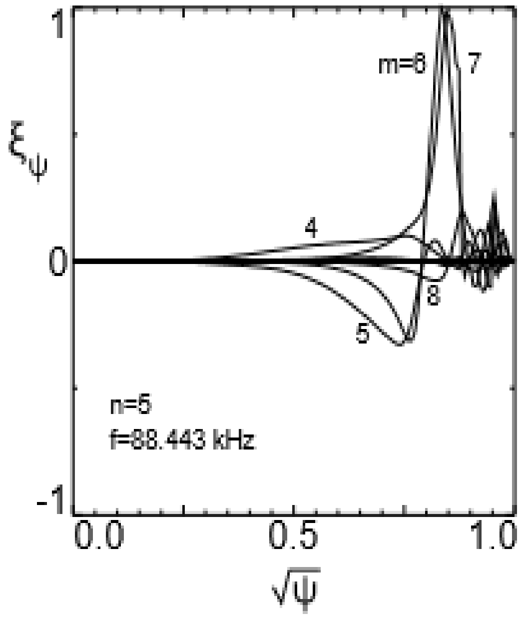


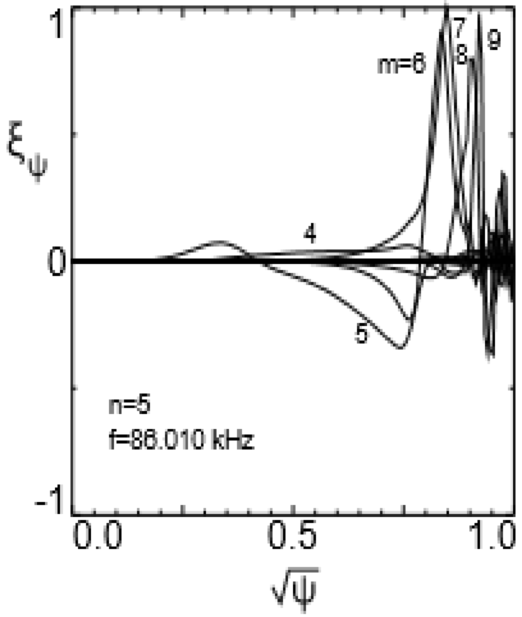
Type 4



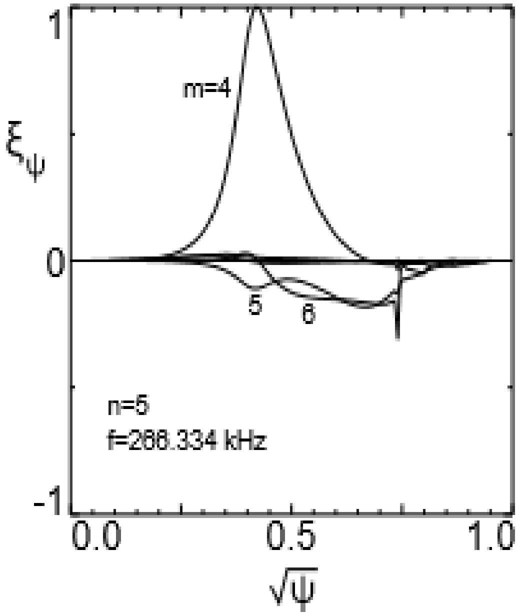
n = 5 Alfven-Sound Eigenmodes for β(0)=25%

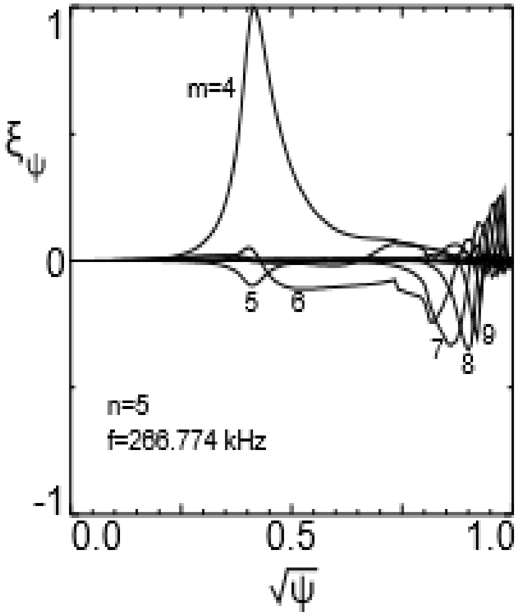
Type 5 Alfven-Sound



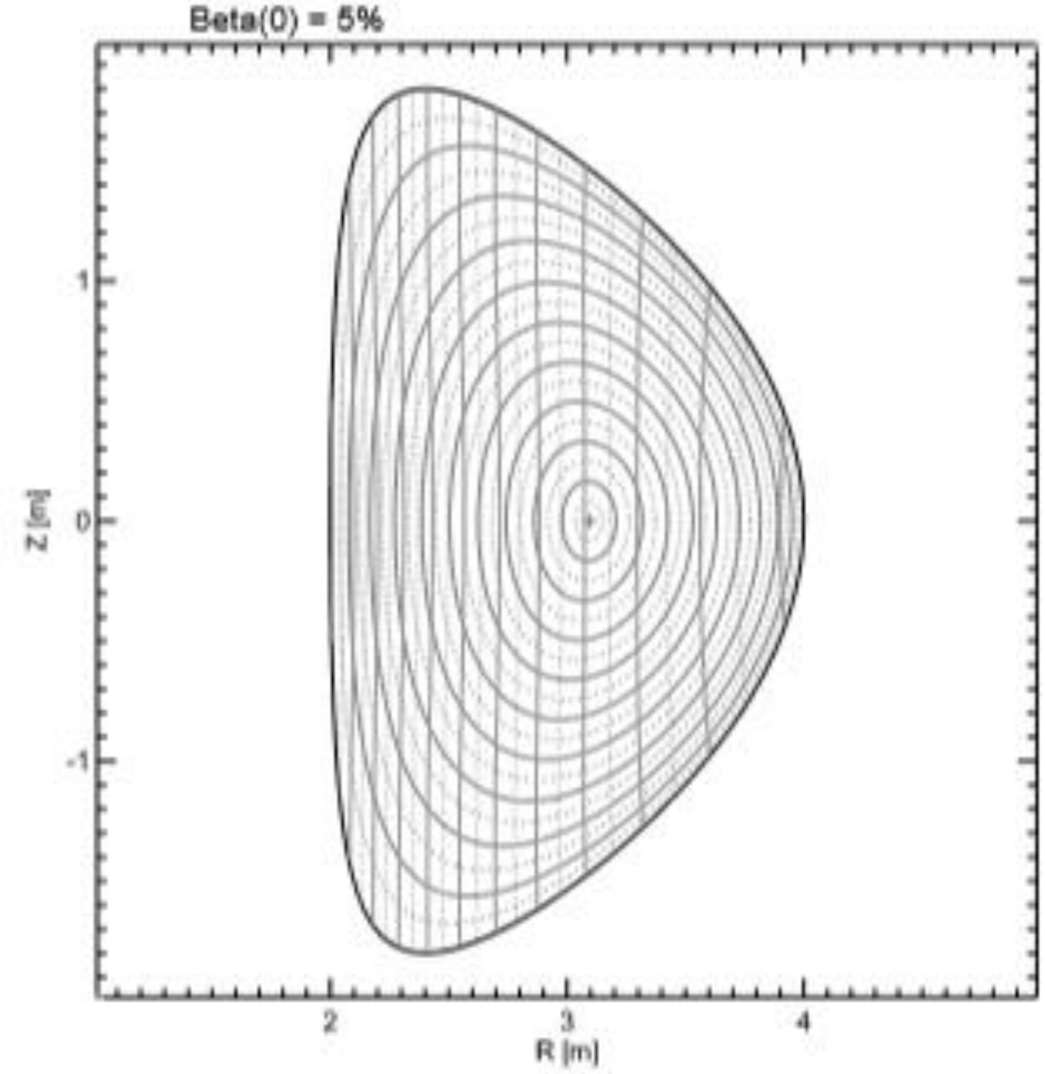


Type 6 Alfven-Sound

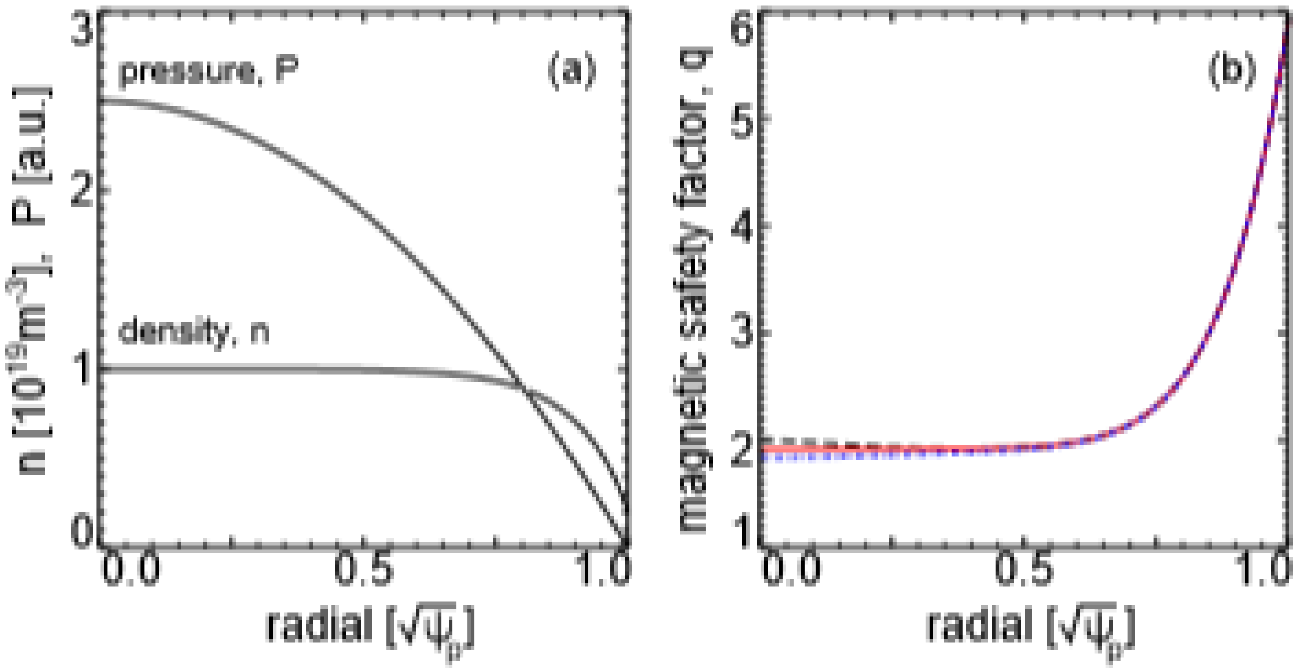




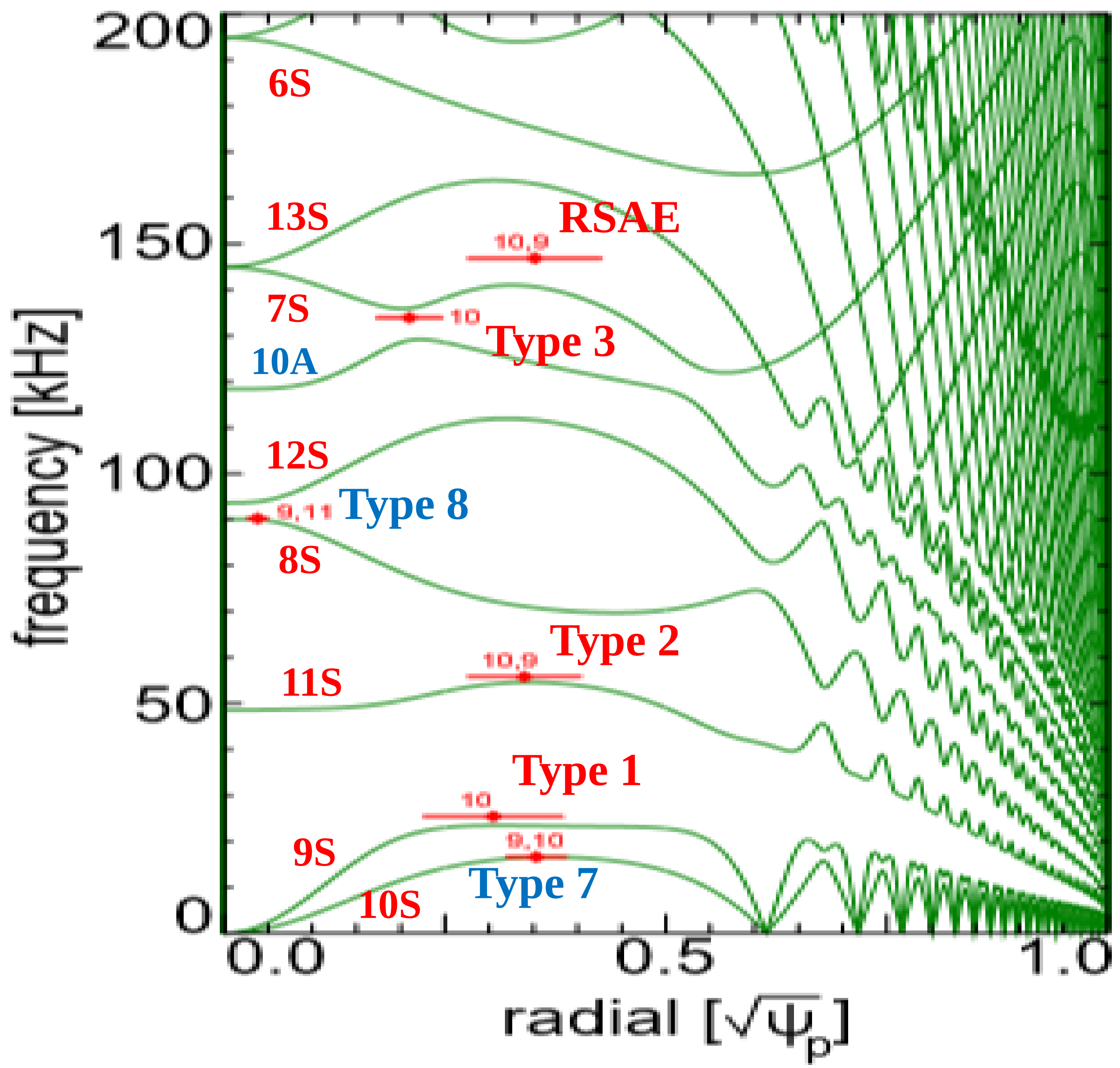
$\beta(0)$=5% Triangular Tokamak



Reverse shear, triangular shape equilibrium with
R/a = 3
q(0) = 2; q_{min} = 1.92 ; q(1)=6



n = 5 Eigenmodes for β(0)=5% Triangular Tokamak



- **Horizontal bars** indicate eigenmode frequencies & half-width of radial structure

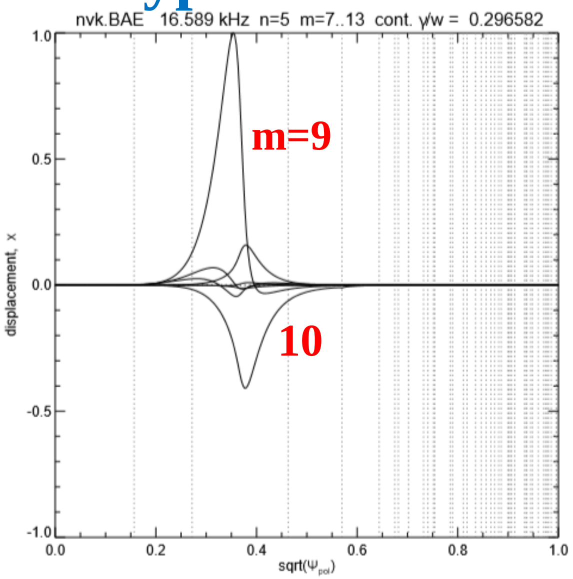
normalized frequency

$B_0/2\pi\sqrt{\rho(0)}q(1)R = 214.445$ kHz

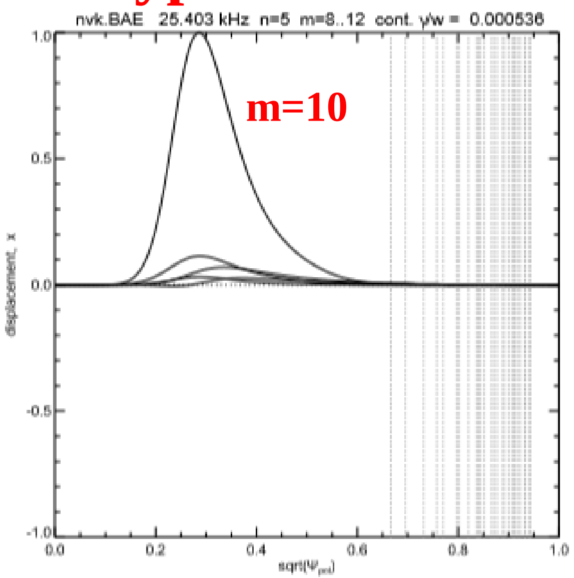
- **new types of ASEs**
  - 5 types of ASEs
  - 2 new types of ASEs
  - no BAAE

n = 5 Eigenmodes for β(0)=5% Triangular Tokamak

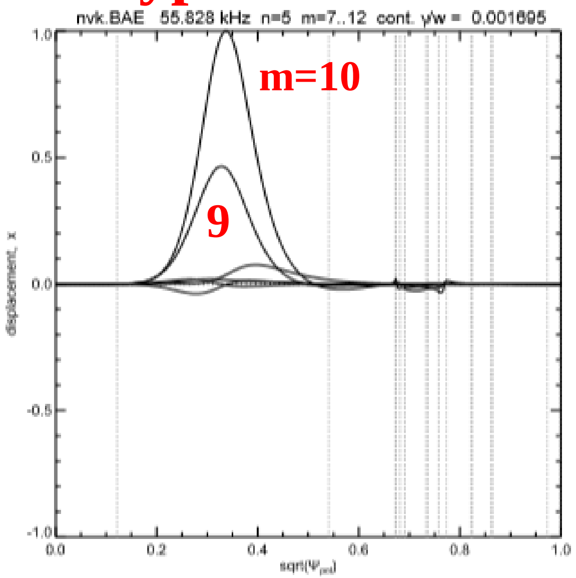
Type 7



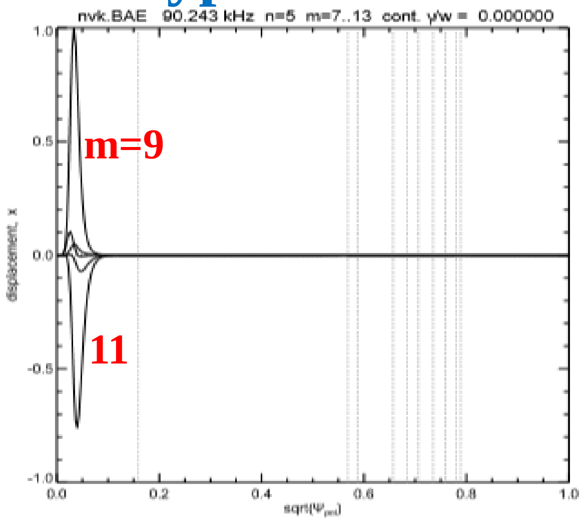
Type 1



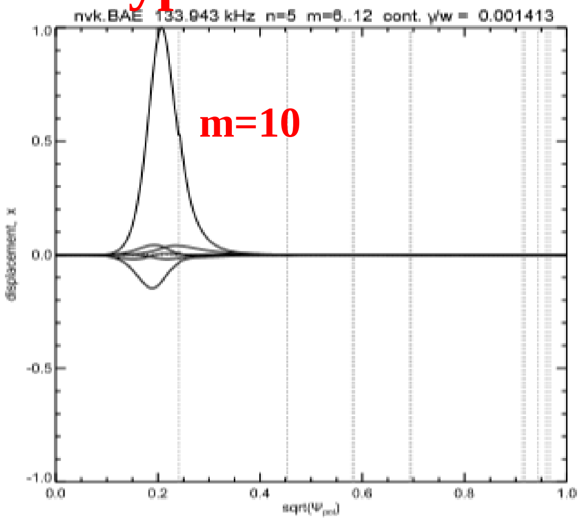
Type 2



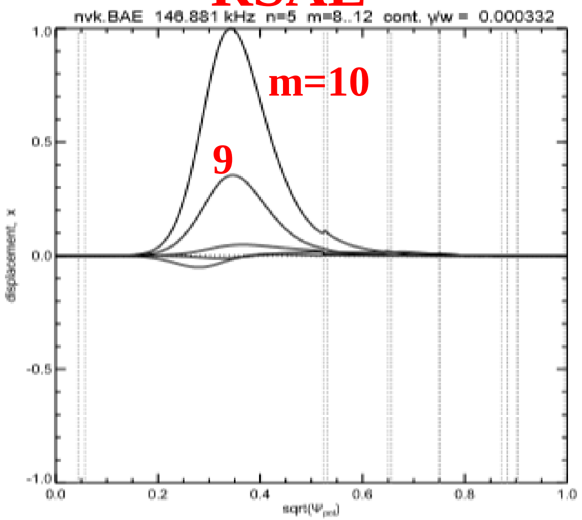
Type 8



Type 3



RSAE



Summary

- **Theory of coupling between Alfven wave and slow magnetosonic wave**
  - Formation of Alfven-Sound (AS) gaps
- **Discovery of Alfven-Sound Eigenmodes (ASE)**
  - ASEs are ubiquitous in tokamak plasma for different q, P, ρ profiles, plasma β and toroidal mode number n
  - More than 5 types of ASEs are found in circular tokamak; BAAE is one type of ASEs
- **Slow-mode approximation does not have slow mode physics and thus has no AS gaps and ASEs**
- **Application of ASE concept**
  - Explanation of BAEs observed in high-β tokamak experiments
  - ASEs resonance with lower energy fast particles than TAEs or RSAEs
  - Removal of Helium ash (slowing-down α population) from fusion reactor core