Near-Resonant Heat Flux Reduction in Gyrokinetic Dimits-Shift Analysis and Quasilinear Model Building

Ping-Yu Li[1], P. W. Terry[1], M.J. Pueschel[2,3] and G.G. Whelan[1]

[1]University of Wisconsin – Madison

[2]Dutch Institute for Fundamental Energy Research

[3]Eindhoven University of Technology

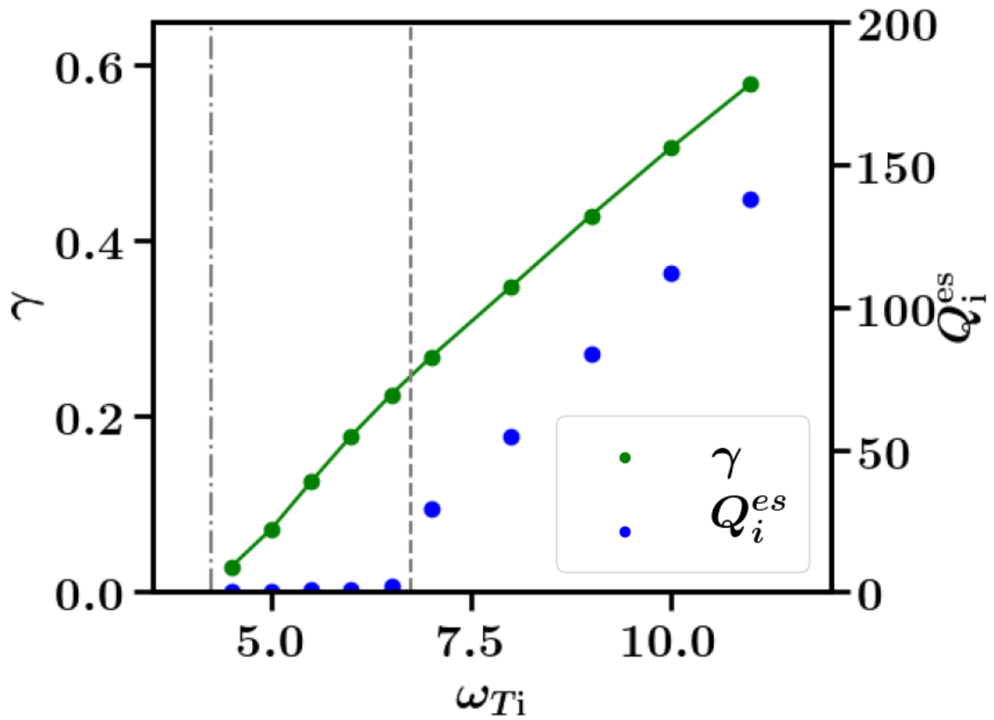
Outline

- The Critical Gradient Upshift in Gyrokinetics
- ITG Fluid Model with Threshold Physics
  - The Modified Horton-Holland Model and Eigenmode Decomposition
  - Triplet Correlation Time
- Analysis of the Resonance Effect in Gyrokinetics
  - Gyrokinetic Observations and Numerical Experiments
- Triplet Correlation Time in Quasilinear Model Building
  - ITG and Grad-n TEM
- Conclusions

The Critical Gradient Upshift in Gyrokinetics

What is the Dimits shift?

Cyclone Base Case Gyrokinetic Simulation



γ : Growth rate of the most unstable mode

Q_i^{es} : Ion heat flux

$\omega_{T_i} = R/L_{T_i}$: Normalized ion temperature gradient

- What is it?

  The difference in onsets of γ and Q_i^{es}

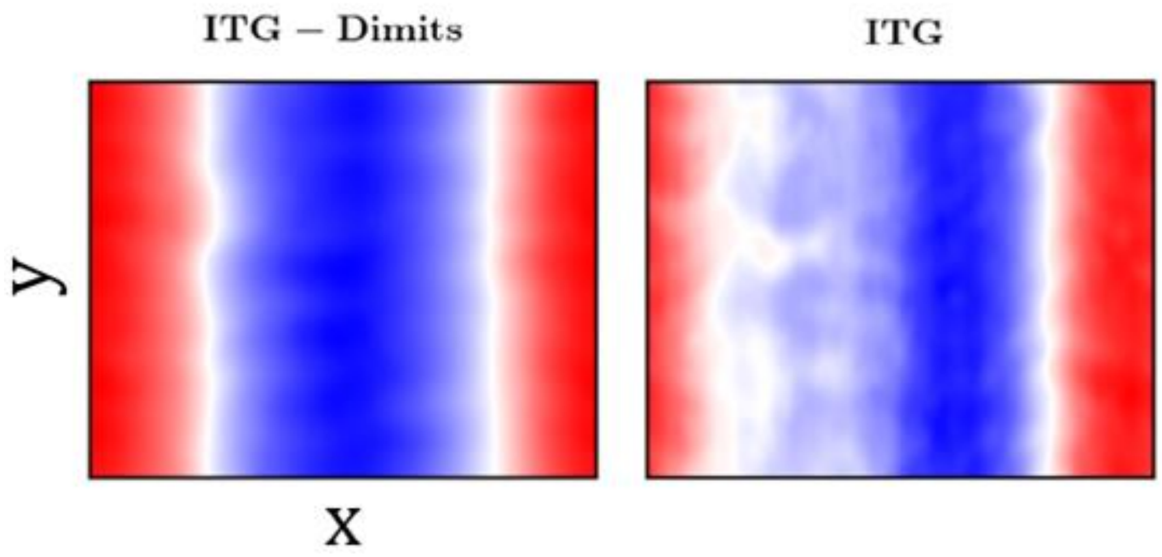
- Why do we care?

  Low but finite heat flux within the Dimits regime

  → Predicting fluxes near criticality correctly is important for transport modeling

Gyrokinetic Observations

Snapshot for the field line averaged $\Phi(x, y)$



- Both with clear band structure

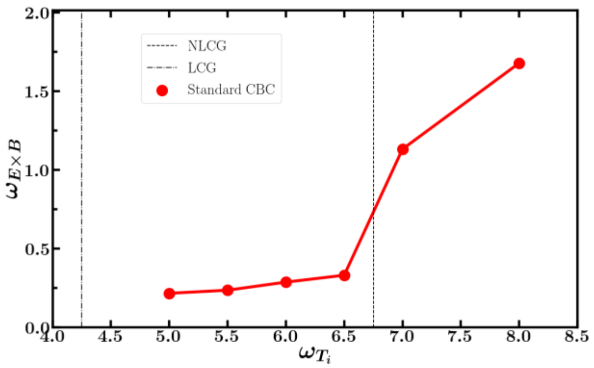
  → Strong zonal flow

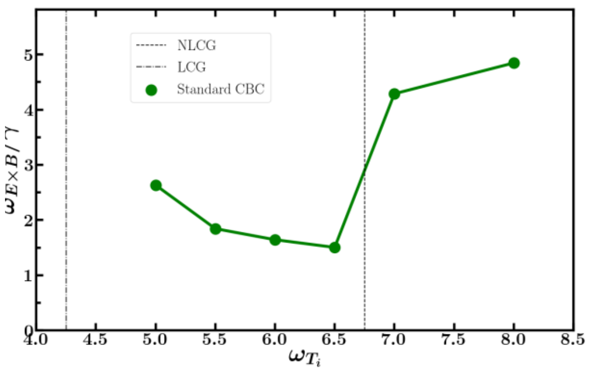
- More turbulent behavior above the Dimits regime

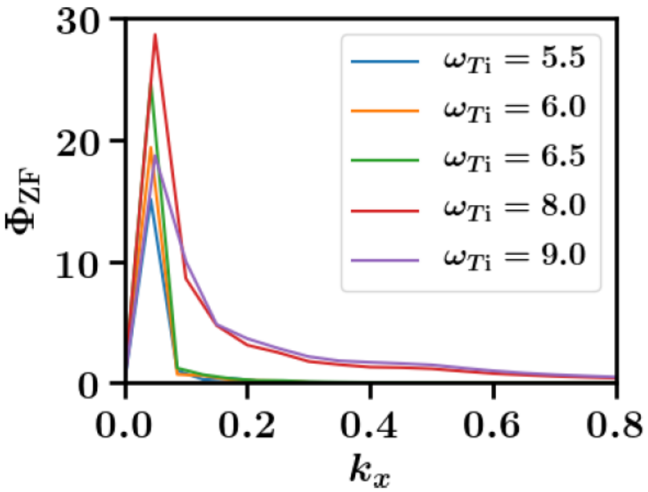
  → Stronger non-zonal catalyzed interactions

In the Dimits regime (left), and above(right)

Gyrokinetic Observations







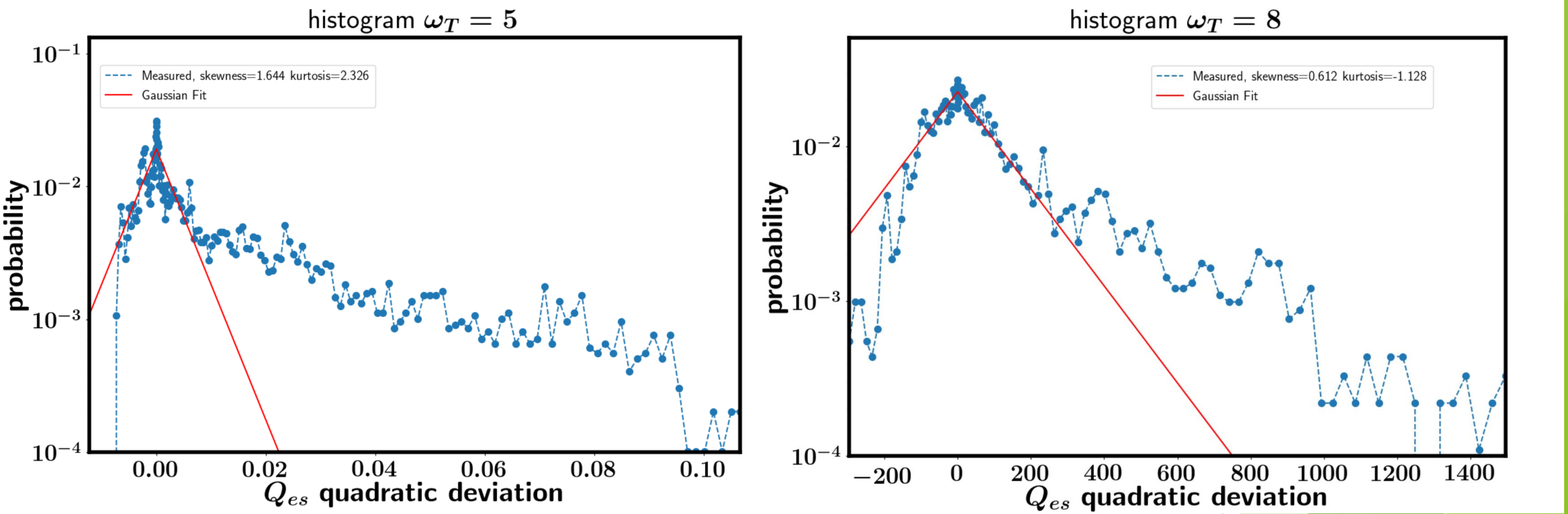
a) Zonal flow shearing rate $\omega_{E\times B} = \sum_{k_x} k_x^2 |\Phi_{ZF}|$ is larger as ω_{Ti} increases

b) $\omega_{E\times B}/\gamma$ is larger above the Dimits regime

Zonal potential spectrum $\Phi_{ZF}(k_x)$ is wider and larger as ω_{Ti} increases

- Stronger zonal flow and larger shearing rate above the Dimits regime
  → Can not be fully explained by the shearing hypothesis

Gyrokinetic Observations



- Larger skewness and excess kurtosis for Q_i^{es} within the Dimits regime
  → limited number of nonlinear interactions within the Dimits regime

Gyrokinetic Observations

- Zonal flow is strong above and within the Dimits regime
- Non-zonal interactions are stronger above the Dimits regime

→

- A theory includes strong zonal flow catalyzed interactions need
- The effect from the non-zonal interaction needs to be understood

ITG Fluid Model with Threshold Physics

The Modified Horton-Holland Model and Eigenmode Decomposition

Toroidal ITG two-field fluid model[1] with strong zonal flow

p_k : pressure in k space
ϕ_k : electrostatic potential in k space

$$\dot{p}_k + z_{11} p_k + z_{12} \phi_k = -\sum_{k'} \left[\boldsymbol{k}' \times \hat{\boldsymbol{z}} \cdot \boldsymbol{k} \right] \phi_{k'} p_{k''}$$

$$\dot{\phi}_k + z_{21} p_k + z_{22} \phi_k = \frac{1}{2\left(\delta(k_y) + k_\perp^2\right)} \sum_{k'} \left[\boldsymbol{k}' \times \hat{\boldsymbol{z}} \cdot \boldsymbol{k} \right] (k_\perp'^2 - k_\perp''^2) \phi_{k'} \phi_{k''}$$

- Model modified to match the more realistic gyrokinetic dispersion relation[2]

---

1. C. Holland et al, (2003) Nucl. Fusion 43 761
2. G. Hammett, UCLA Winter School, Center for Multiscale 403 Plasma Dynamics, Los Angeles, CA, 2007

The Modified Horton-Holland Model and Eigenmode Decomposition

$$\begin{pmatrix} p_k \\ \phi_k \end{pmatrix} = \beta_1 \mathbf{V_1^r} + \beta_2 \mathbf{V_2^r}$$

- Eigenmodes decomposed in p_k and ϕ_k
  → track the energy of the stable and unstable modes
- Keep the zonal-flow-catalyzed nonlinearities only
- C_{iFj} are the coupling coefficient: how the eigenmodes couple
- Solve the eigenmode amplitude evolution equations
  → easier to track how energy is exchanged

β_j : eigenmode amplitude, $\mathbf{V_j^r}$: right eigenvector

$$\dot{\beta}_l + i\omega_l \beta_l = \cancel{\sum_{k',k'_y \neq 0, k_y} C^{(k,k')}_{lmn} \beta'_m \beta''_n} + \sum_{k'_x} \left\{ \left[C^{(k,k')}_{lFn} v'_z \beta''_n + \cancel{C^{(k,k')}_{lPn} p'_z \beta''_n} \right] \Big|_{k'_y=0} + \left[C^{(k,k')}_{lmF} \beta'_m v''_z + \cancel{C^{(k,k')}_{lmP} \beta'_m p''_z} \right] \Big|_{k'_y=k_y} \right\}$$

$$\dot{v}_z + \nu v_z = \sum_{k'} C^{(k,k')}_{Fmn} \beta'_m \beta''_n \big|_{k_y=0}$$

$$\cancel{\dot{p}_z + \chi k_x^2 p_z = \sum_{k'} C^{(k,k')}_{Pmn} \beta'_m \beta''_n \big|_{k_y=0}}$$

Eigenmode amplitude evolution equations

$$\dot{|\beta_l|^2} - 2\,\mathrm{Im}\,\omega_l |\beta_l|^2 = 2\sum_{k'_x} \mathrm{Re} \left\{ \left[C^{(k,k')}_{lFn} \langle v'_z \beta''_n \beta_l^* \rangle \right] \Big|_{k'_y=0} + \left[C^{(k,k')}_{lmF} \langle \beta'_m v''_z \beta_l^* \rangle \right] \Big|_{k'_y=k_y} \right\}$$

$$\dot{|v_z|^2} + 2\nu |v_z|^2 = 2\sum_{k'} Re \left[C^{(k,k')}_{Fmn} \langle \beta'_m \beta''_n v_z^* \rangle \right] \big|_{k_y=0}$$

Triplet Correlation Time

- Find the lifetime of the triplet interaction with quasinormal statistics and Green's function → also known as the triplet correlation time $\tau = -i(\hat{\omega}'' + \hat{\omega}' - \hat{\omega}^*)^{-1}$, $\hat{\omega} = \omega + \Delta\omega$
- Express the third order moment in terms of the second order moments

$$\langle v_z' \beta_2'' \beta_1^* \rangle \Big|_{k_y'=0} = \sum_{n=1,2} \tau_{2F1} \left\{ (C_{2Fn}^{(k'',-k')} + C_{2nF}^{(k'',k)}) \langle \beta_n \beta_1^* \rangle \left| v_z' \right|^2 + C_{1Fn}^{(k,k')*} \langle \beta_n''^* \beta_2'' \rangle \left| v_z' \right|^2 \right\} \Big|_{k_y'=0}$$

$\tau_{ijk} = -i\left(\hat{\omega}_i'' + \hat{\omega}_j' - \hat{\omega}_k^*\right)^{-1}$, $\hat{\omega}_j = \omega_j + \Delta\omega_j$, $\Delta\omega_j$ is the nonlinear correction

$$\left|\dot{\beta}_1\right|^2 - 2\,\mathrm{Im}\,\omega_1 \left|\beta_1\right|^2 = 2\sum_{k_x'} \mathrm{Re} \Bigg\{ \left[C_{1F2}^{(k,k')} \sum_{n=1,2} \tau_{2F1} \left\{ (C_{2Fn}^{(k'',-k')} + C_{2nF}^{(k'',k)}) \langle \beta_n \beta_1^* \rangle \left| v_z' \right|^2 + C_{1Fn}^{(k,k')*} \langle \beta_n''^* \beta_2'' \rangle \left| v_z' \right|^2 \right\} \right] \Big|_{k_y'=0}$$
$$+ \left[C_{12F}^{(k,k')} \sum_{n=1,2} \tau_{F21} \left\{ (C_{2Fn}^{(k',-k)} + C_{2nF}^{(k',k)}) \langle \beta_n \beta_1^* \rangle \left| v_z'' \right|^2 + C_{1Fn}^{(k,k')*} \langle \beta_n'^* \beta_2' \rangle \left| v_z'' \right|^2 \right\} \right] \Big|_{k_y'=k_y} \Bigg\}$$

Triplet Correlation Time

- The $\Delta\omega$ in $\widehat{\omega} = \omega + \Delta\omega$ is called the eddy turnover rate
  →can break the resonance and reduce the triplet correlation time $\tau = -i(\widehat{\omega}'' + \widehat{\omega}' - \widehat{\omega}^*)^{-1}$
- $\Delta\omega \propto |\beta|^2$, stronger turbulence leads to larger $\Delta\omega$

The eddy turnover rate $\Delta\omega_i$ can be derived through renormalization, its formula is given as below

$$\Delta\omega_i = \sum_{k'} \frac{-2iC_{iFj}^{(k,k')}}{i\hat{\omega}_j'' - i\hat{\omega}_i^* + i\hat{\omega}_i'} [C_{ij}''|v_z'|^2 + C_{Fij}^{(k',k)}(|\beta_2''|^2 + \langle\beta_1''^*\beta_j''\rangle)]|_{k_y'=0}.$$

Where $\widehat{\omega}_i = \omega_i + \Delta\omega_i$, and $C_{ij}'' = C_{iFj}^{(k'',k)} + C_{ijF}^{(k'',-k')}$

Saturation Formula

Zonal flow evolution equation

$$\left[\frac{\partial}{\partial t}+2\nu\right]|v_z|^2\Big|_{k_y=0}=4\sum_{k'}\mathrm{Re}\left\{\left[C_{F12}^{(k,k'')}\tau_{12F}\left[C''_{11}e^{i\theta}+C''_{12}\kappa\right]+C_{F21}^{(k,k'')}\tau_{21F}\left[C''_{22}e^{-i\theta}+C''_{21}\right]\right]|\beta'_1|^2|v_z|^2\right\}_{k_y=0}$$

Let $\beta_2=\sqrt{\kappa}\beta_1 e^{i\theta}$, and the Markovianized solution for $|\beta'_1|^2$ is

$$|\beta'_1|^2\sim\frac{\nu}{2\sum_{k_x,k'}\mathrm{Re}\left\{C_{F12}^{(k,k'')}\left[\tau_{21F}\left(C''_{21}+\sqrt{\kappa}C''_{22}e^{-i\theta}\right)+\tau_{12F}\left(\kappa C''_{12}+\sqrt{\kappa}C''_{11}e^{i\theta}\right)\right]\right\}}$$

$$\theta=-\alpha+\sin^{-1}\left[\frac{\mathrm{Im}(\omega_1^*-\omega_2\kappa)}{\sqrt{\kappa}\,|\omega_2-\omega_1^*|}\right]\qquad \kappa\equiv\frac{|\beta_2|^2}{|\beta_1|^2}\qquad \alpha=\tan^{-1}\frac{Im\Delta\omega}{Re\Delta\omega},\ \Delta\omega=\omega_2-\omega_1^*$$

- The magnitude of the unstable mode $|\beta'_1|^2$ increases as
  1. τ decreases: shorter nonlinear interaction lifetime → stronger turbulence
  2. C_{ijk} decreases: weaker nonlinear coupling between stable and unstable modes → stronger turbulence
- Nondispersive eigenmode frequency can lead to infinite τ and $|\beta'_1|^2=0$

Saturation Formula

Ion heat flux $Q_i = -\sum_k k_y \mathrm{Im}\langle \phi_{-k} p_k \rangle$ can be expressed in terms of the eigenmodes

$$Q_i = -\sum_k k_y \left[\mathrm{Im} R_1 |\beta_1|^2 + \mathrm{Im} R_2 |\beta_2|^2 + \mathrm{Im}(R_1 + R_2) \mathrm{Re}\langle \beta_1 \beta_2^* \rangle + Re(R_1 - R_2) \mathrm{Im}\langle \beta_1 \beta_2^* \rangle \right]$$

Plug in the saturation formula for $|\beta_1|^2$

$$Q_i = \sum_{k'''} \frac{\gamma(k''')(1 + k_\perp^{2\,'''}) \boxed{(1-\kappa)} \nu}{4\varepsilon} \sum_{k_x, k'} \mathrm{Re} \left\{ C_{F12}^{(k,k'')} \left[\tau_{21F} \left(C''_{21} + \sqrt{\kappa} C''_{22} e^{-i\theta} \right) + \tau_{12F} \left(\kappa C''_{12} + \sqrt{\kappa} C''_{11} e^{i\theta} \right) \right] \right\}^{-1}$$

$$\kappa \equiv \frac{|\beta_2|^2}{|\beta_1|^2}$$

- Stable modes transfer energy back to the mean field
  → Accounting for stable modes is key in getting saturation right

---

1. Terry, P. W., P-Y. Li, M. J. Pueschel, and G. G. Whelan. *Physical Review Letters* 126, no. 2 (2021): 025004.

Saturation Formula

Modified Horton-Holland Model

- Toroidal ITG Fluid Model With Strong Zonal Flow

Eigenmode Decomposition and Eigenmode Evolution Equation

- Track How Energy is Transfer Between Eigenmodes Through Zonal Flow Catalyzed Interactions

Triplet Correlation Time τ and Coupling Coefficients C

- Quantify the Lifetime and Coupling Strength of the Nonlinearities Between Eigenmodes

Saturation Formula

- Ion Heat Flux $Q \propto (1-\kappa)/(C\tau)$, where $\kappa \equiv \frac{|\beta_2|^2}{|\beta_1|^2}$

Saturation Formula

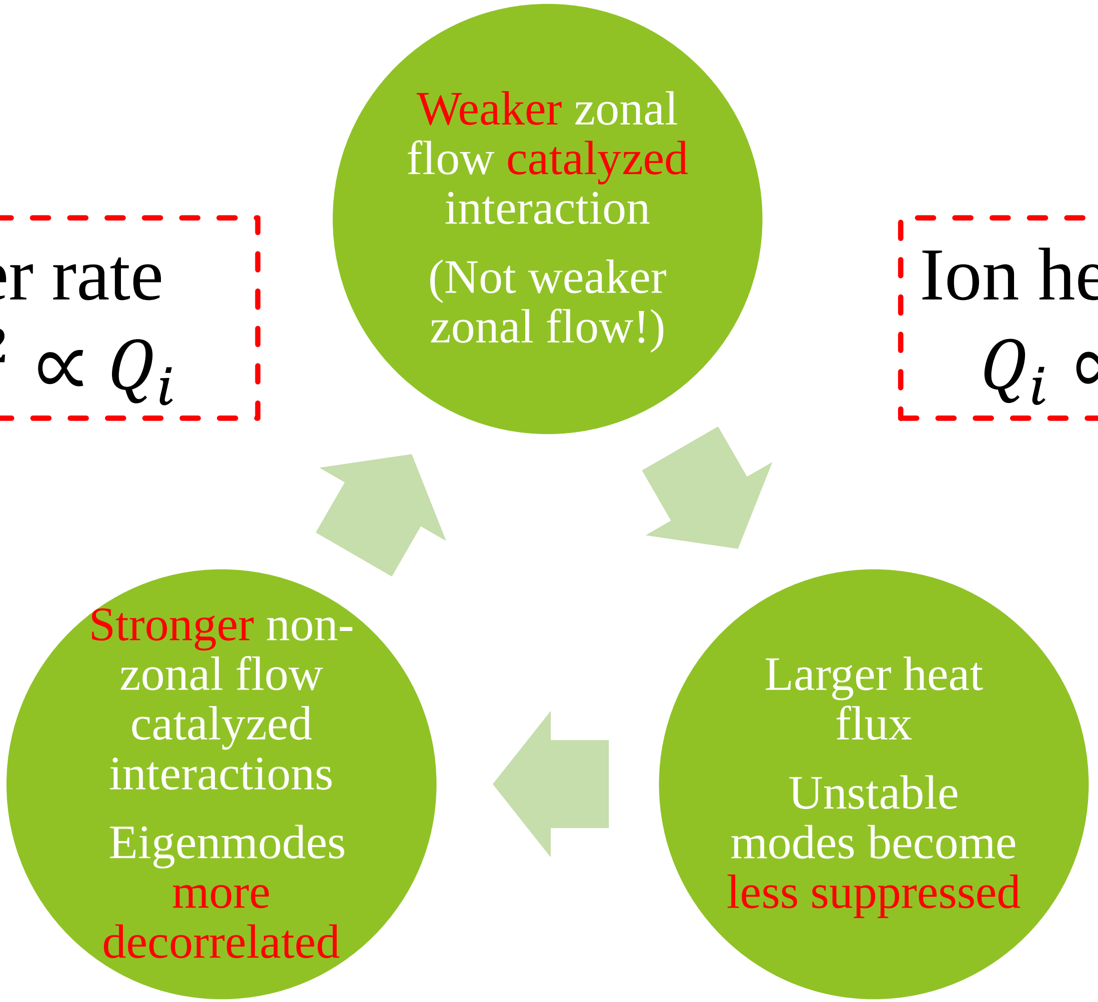
- The $\Delta\omega$ in $\hat{\omega} = \omega + \Delta\omega$ can break the resonance and reduce the triplet correlation time $\tau = -i(\hat{\omega}'' + \hat{\omega}' - \hat{\omega}^{*})^{-1}$

Eddy turnover rate

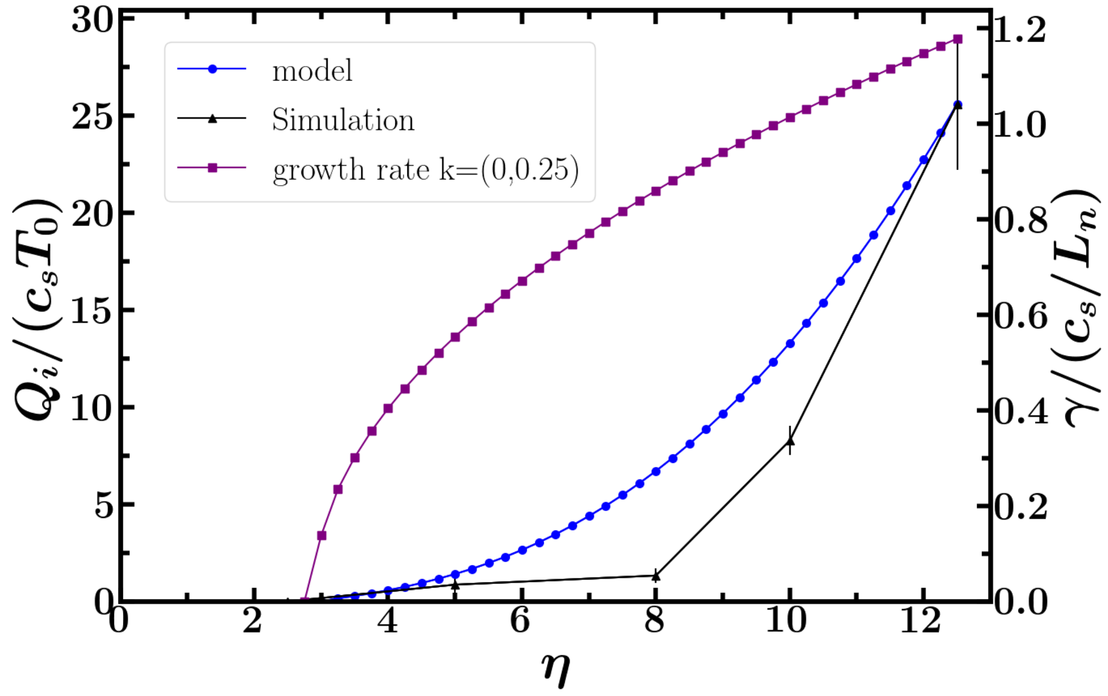
$$\Delta\omega \propto |\beta_1|^2 \propto Q_i$$

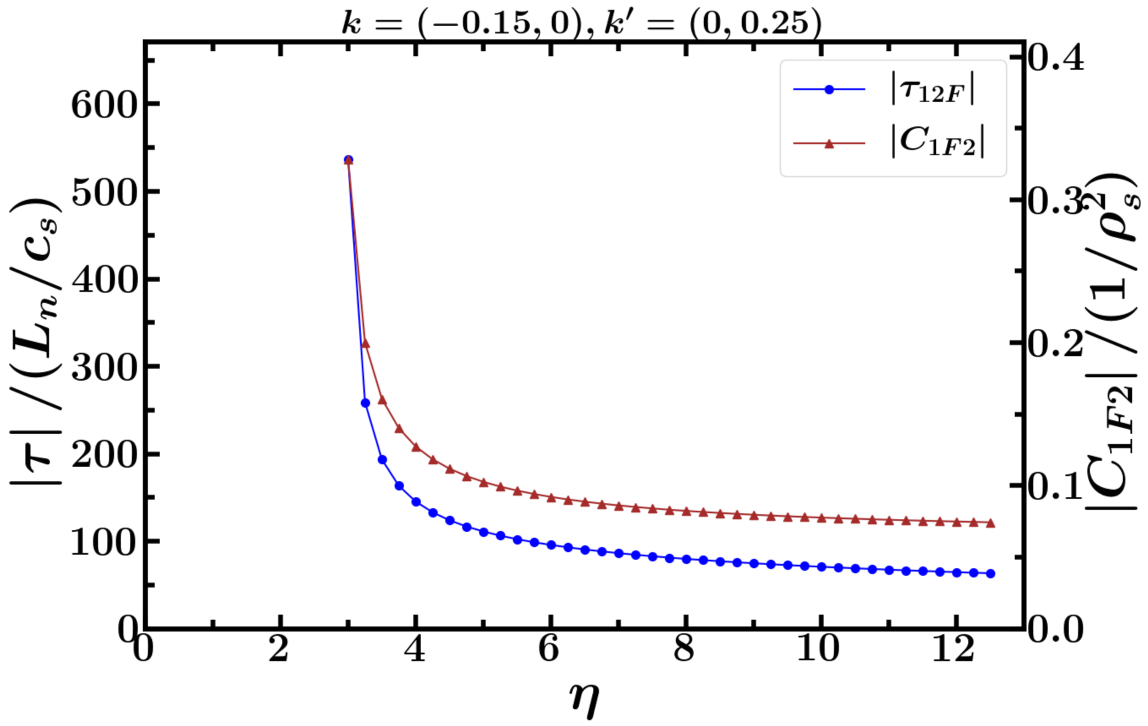
Ion heat flux

$$Q_i \propto (1-\kappa)/(C\tau)$$



Saturation Formula versus Simulation





- The formula captures the trend
- τ and C_{ijk} decrease as η (L_n/L_T) increases
  → strong resonance and nonlinear coupling when close to linear critical gradient

Analysis of the Resonance Effect in Gyrokinetics

Analysis of the Resonance Effect in Gyrokinetics

Quantities important to turbulence saturation in the ITG fluid model

1. Triplet correlation time $\tau_{lmn} = -i(\widehat{\omega}_l + \widehat{\omega}_m - \widehat{\omega}_n^*)^{-1}$
2. Stable modes

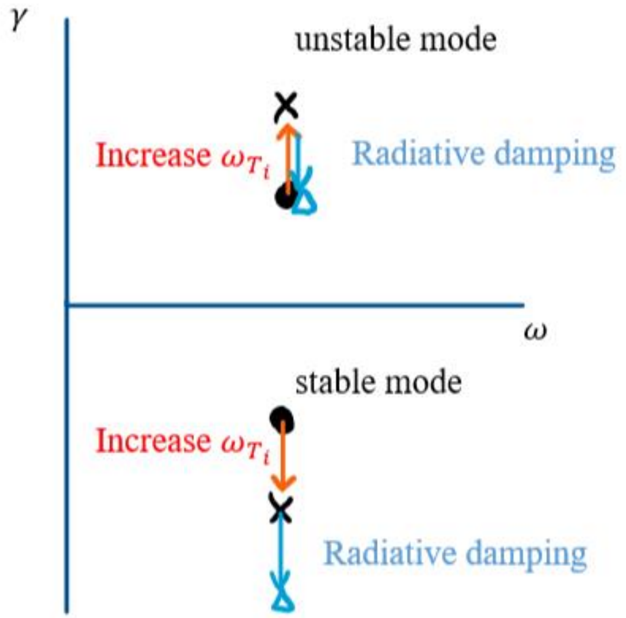
However, can we also observe similar behavior in gyrokinetics?

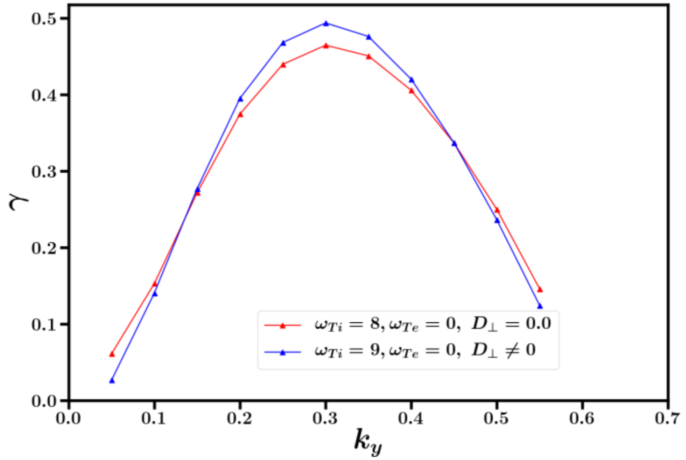
We can check

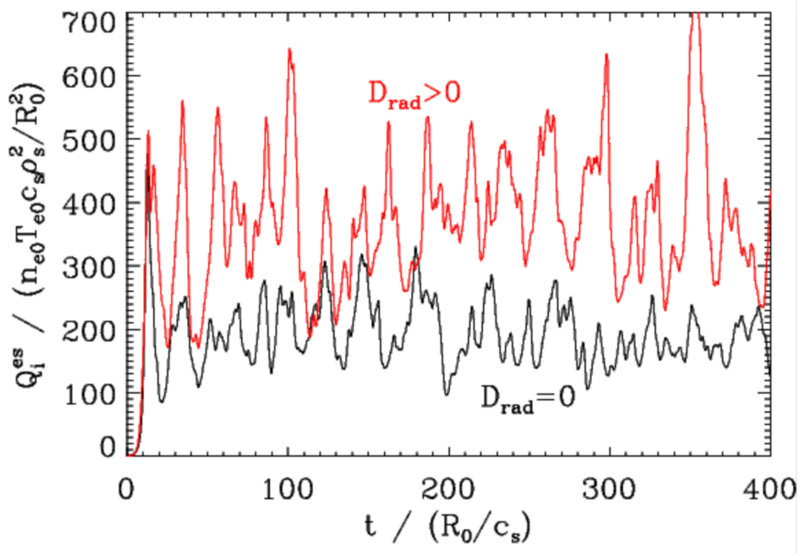
1. How the heat flux reacts to stronger damping of stable modes which breaks the symmetry
2. Cross-correlation between modes with different wavenumbers
3. Resonance-breaking numerical experiments

Simulations for 2 and 3 were done with adiabatic electrons, for 1 with kinetic electrons

Radiative Damping



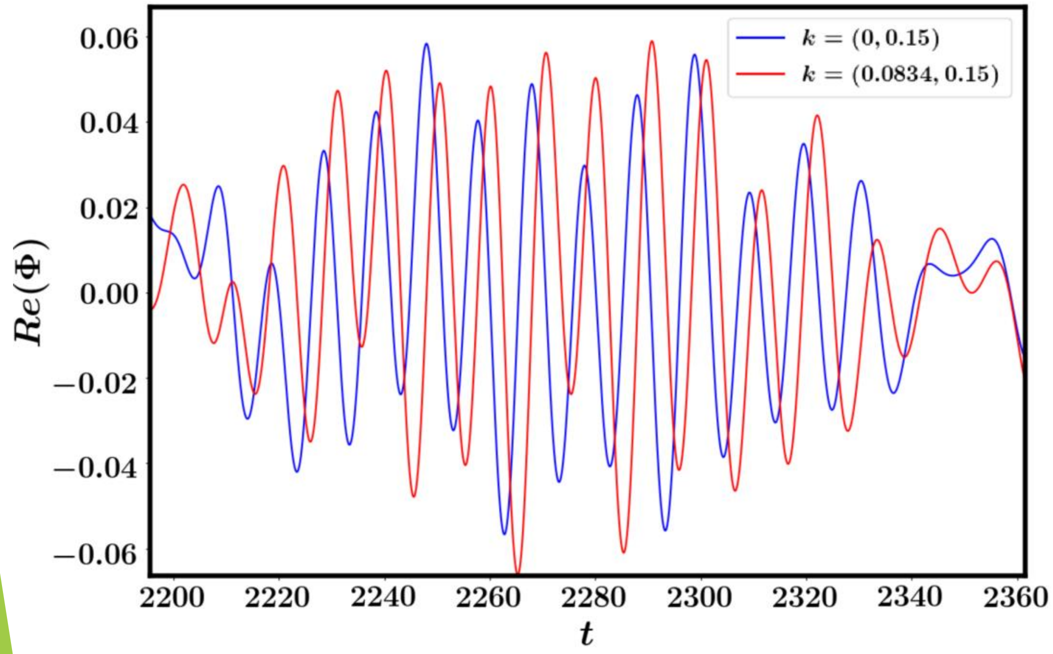




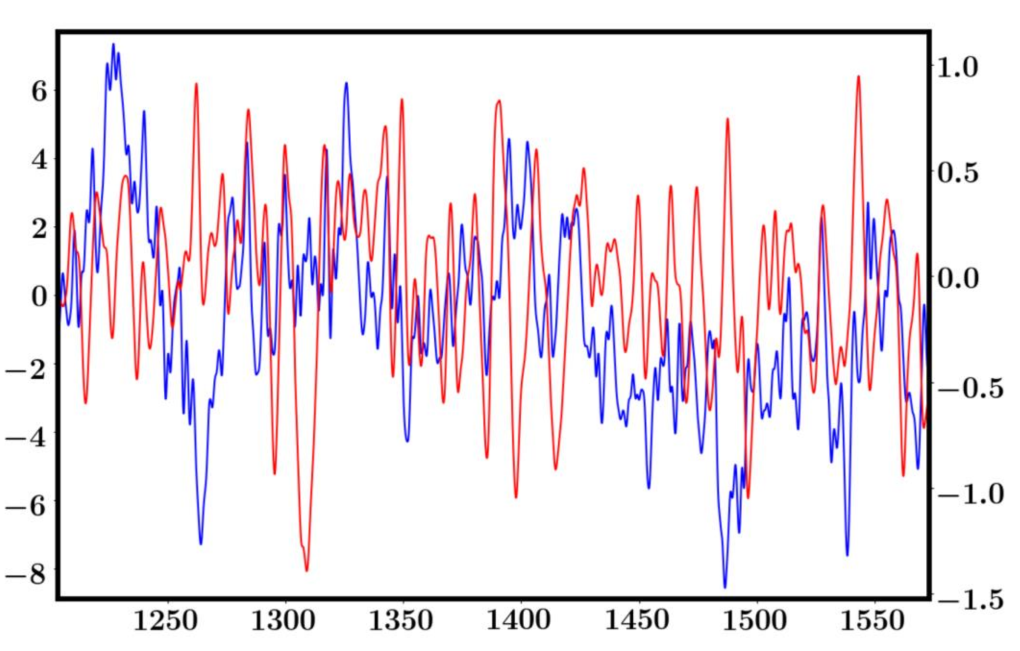
Kinetic electron CBC with $\omega_{T_e} = 0$ to isolate ITG

- More damped stable modes → larger heat flux

Cross-Correlation of Electric Potential for Different k



Time trace of $Re(\Phi_k)$ with $\omega_{Ti} = 5$

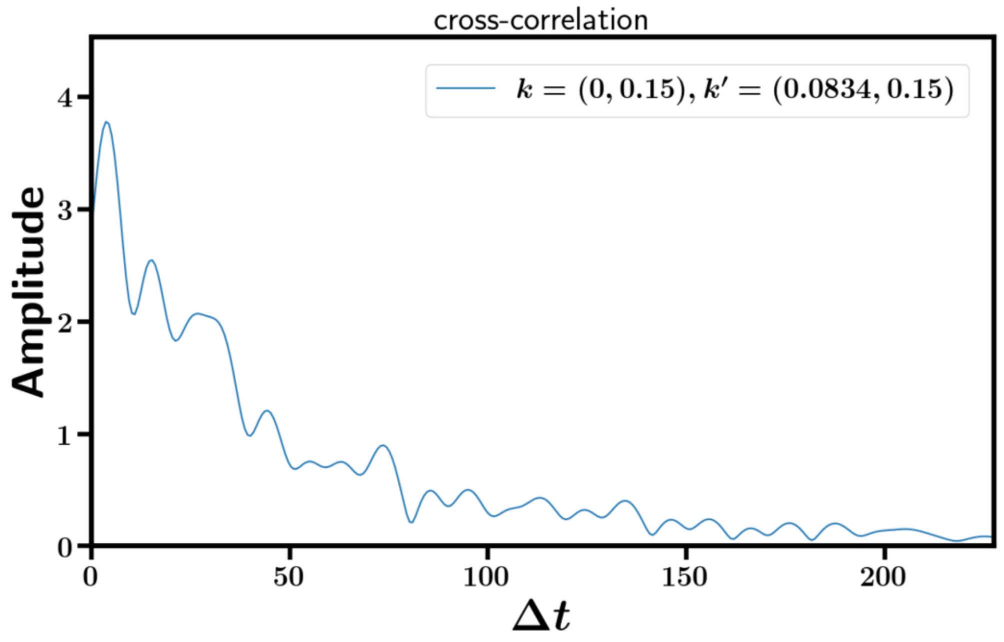


Time trace of $Re(\Phi_k)$ with $\omega_{Ti} = 8$

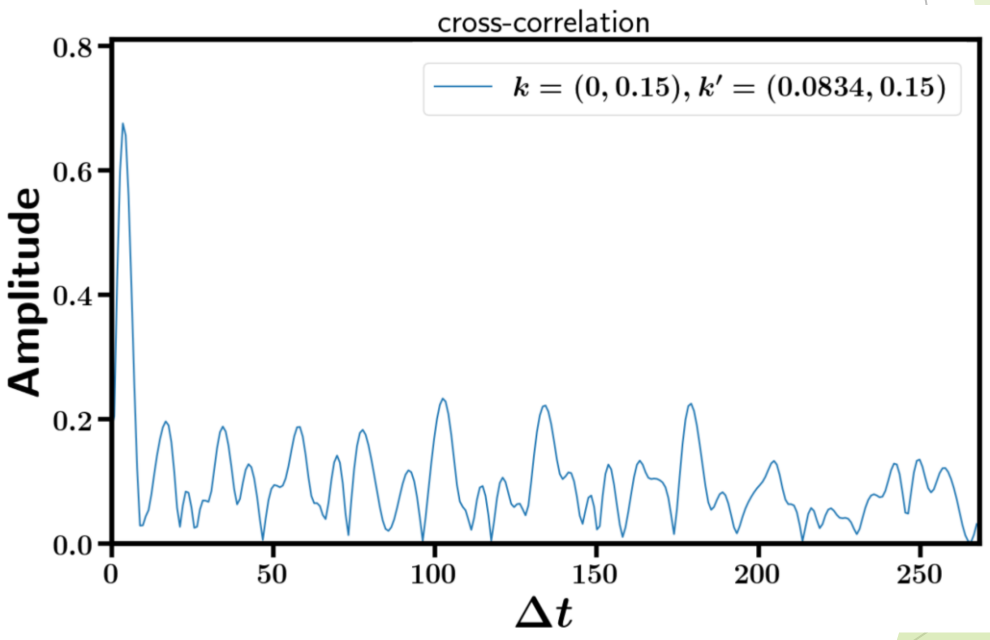
- Electric potential at different wavenumbers coupled through zonal flow shows strong coherency within the Dimits regime

Cross-Correlation of Electric Potential for Different k

cross correlations $\frac{1}{T}\int \Phi_k^*(t-\Delta t)\Phi_{k'}(t)dt$



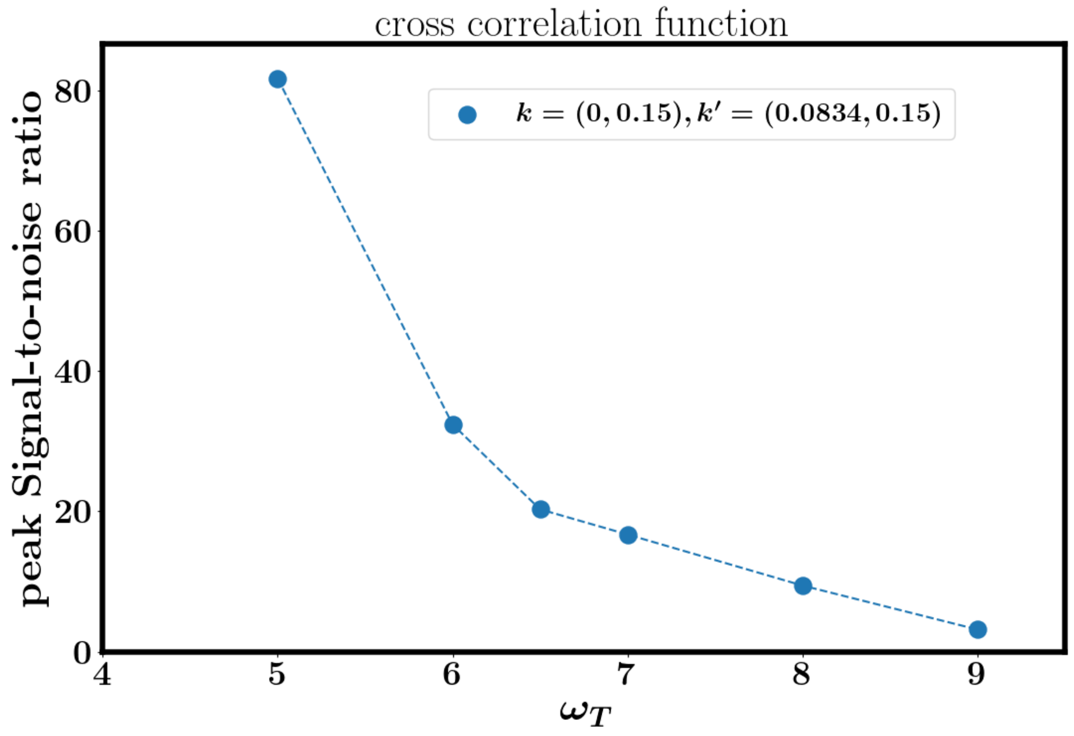
$\omega_{Ti} = 5$



$\omega_{Ti} = 8$

- Electric potential at different wavenumbers coupled through zonal flow shows strong coherency within the Dimits regime

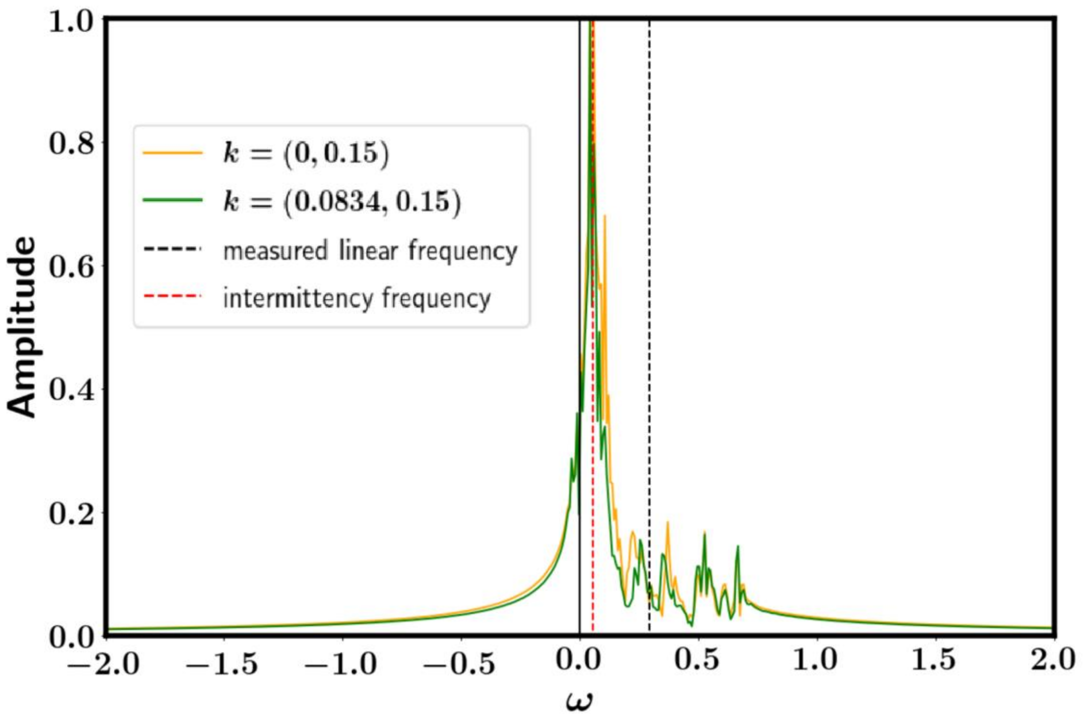
Cross-Correlation of Electric Potential for Different k



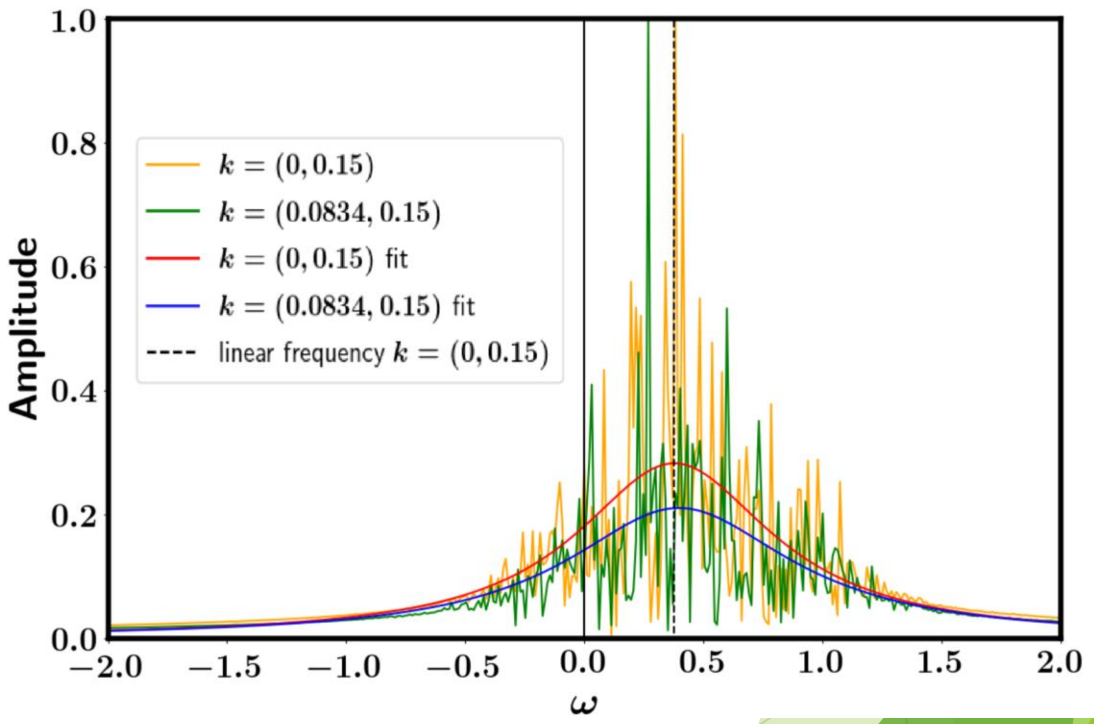
Peak-to-noise ratio, where the noise is the standard deviation of the interval when Δt is large

- Consistent with the triplet correlation time picture

Frequency spectrum comparison



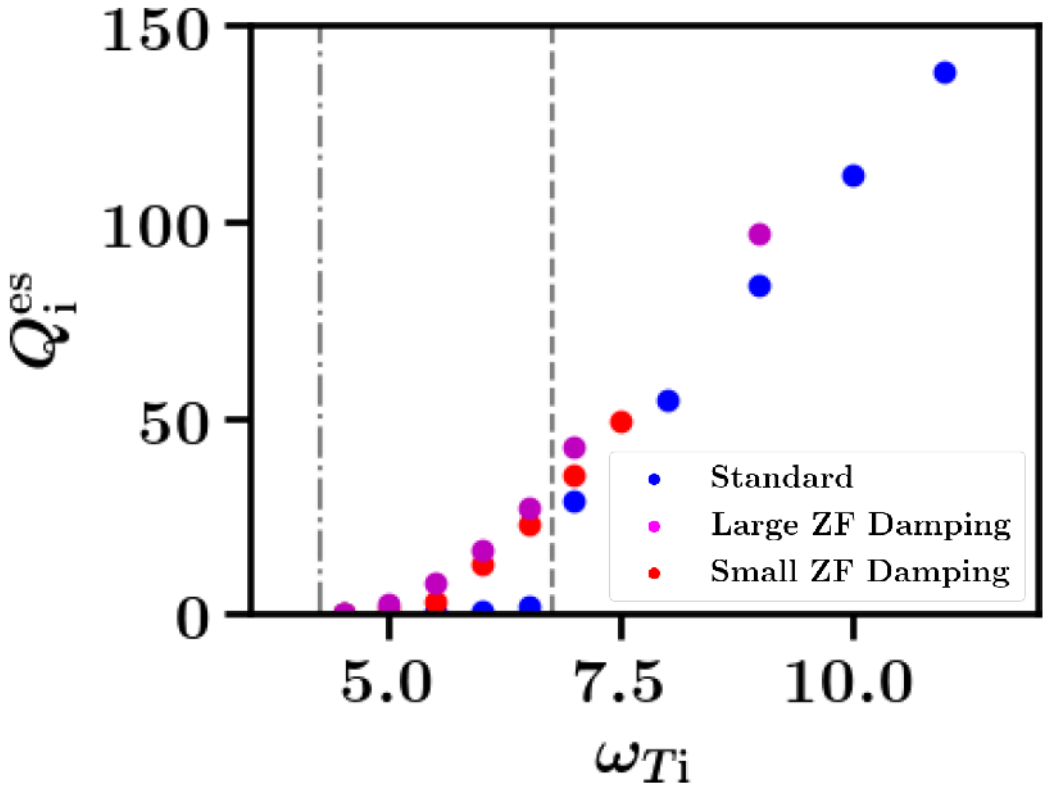
$\omega_{Ti} = 5$



$\omega_{Ti} = 8$

- The frequency spectrum within the Dimits regime are almost identical
- The frequency spectrum above the Dimits regime have large $\Delta\omega$

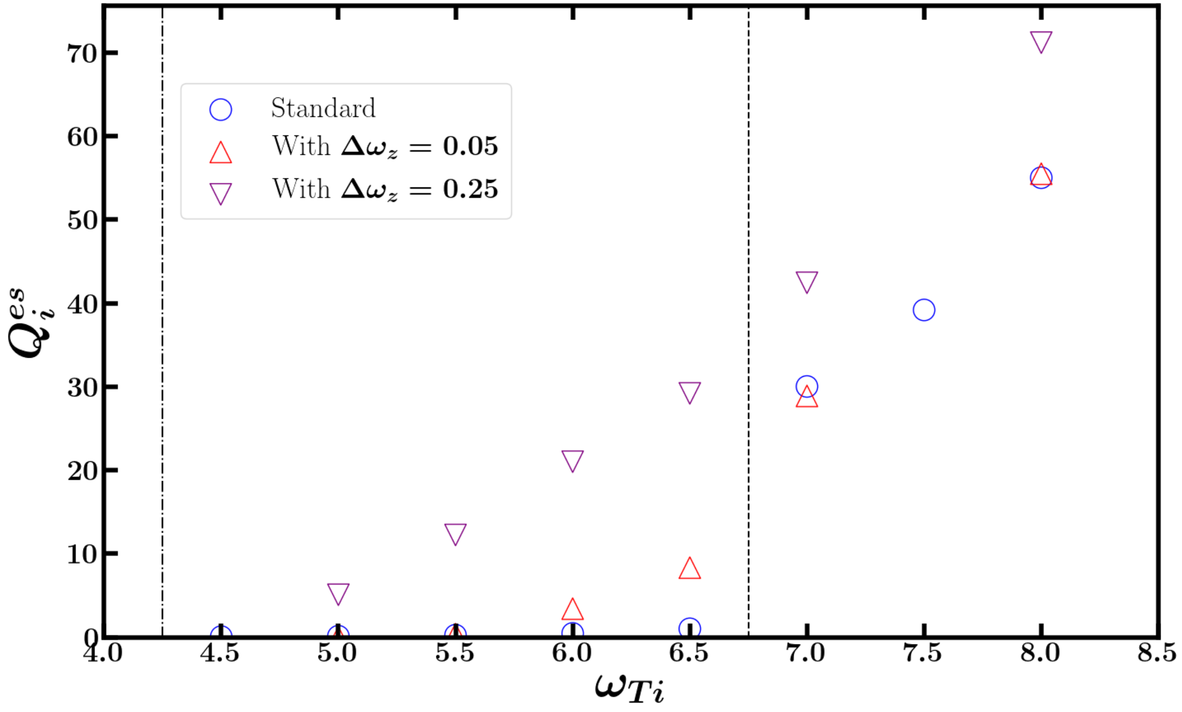
Resonance-Breaking Effect on Heat Flux

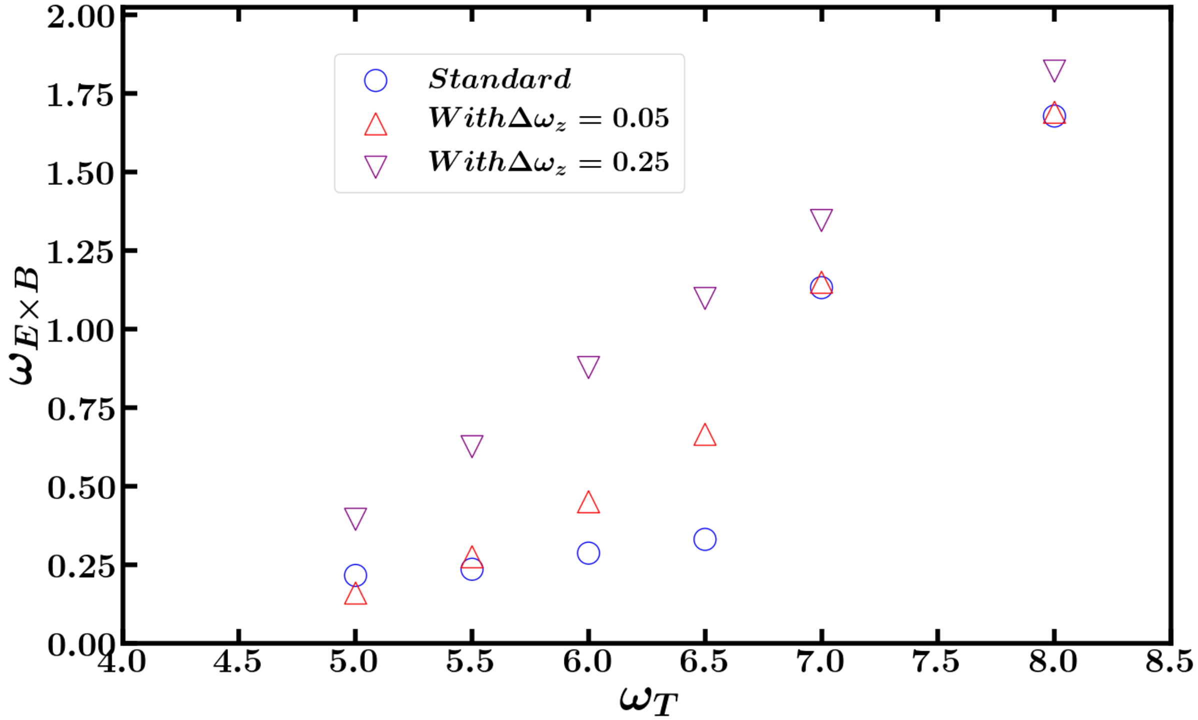


- Artificial Zonal flow damping kills the Dimits shift
- τ_{ijk} decreases significantly because $\tau_{iFk} = -i(\omega_i'' + (\omega_F' + \Delta\omega_F) - \omega_k^*)^{-1}$ becomes much smaller if originally at resonance

1. G. G. Whelan PhD Thesis

Resonance-Breaking Effect on Heat Flux





- Dimits shift can also be killed by adding artificial real frequency to zonal modes
- Shearing rate does not decrease with extra artificial real frequency

Triplet Correlation Time in Quasilinear Model Building

Triplet Correlation Time in Quasilinear Model Building

Assumptions

1. Coupling coefficients are nearly constant
2. τ is almost real

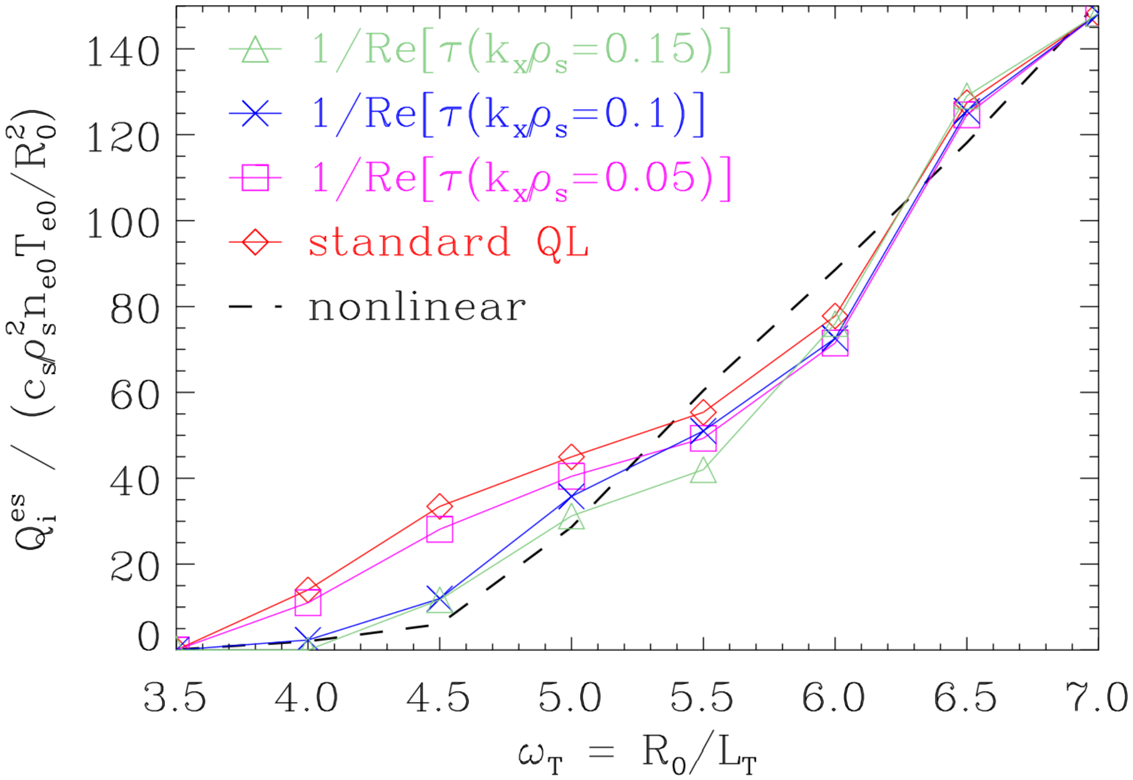
Quasilinear Heat Flux $$Q \sim \omega_T \sum_k \frac{\gamma w}{\langle k_\perp^2 \rangle Re(\tau)}$$

where γ is the growth rate, $w = \frac{Q|_{lin}}{\Phi^2|_{lin}}$ is the quasilinear weight, $\langle k_\perp^2 \rangle = \langle \frac{k_y^2 \left[1 + [g^{xy} + \hat{s}\theta_0(k_x) g^{xx}]^2\right]}{g^{xx}} \rangle$

Calculate τ

1. Zonal flow frequencies are set to 0
2. Stable mode frequencies approximated as the mirror modes of the unstable modes

ITG: Kinetic Electron CBC

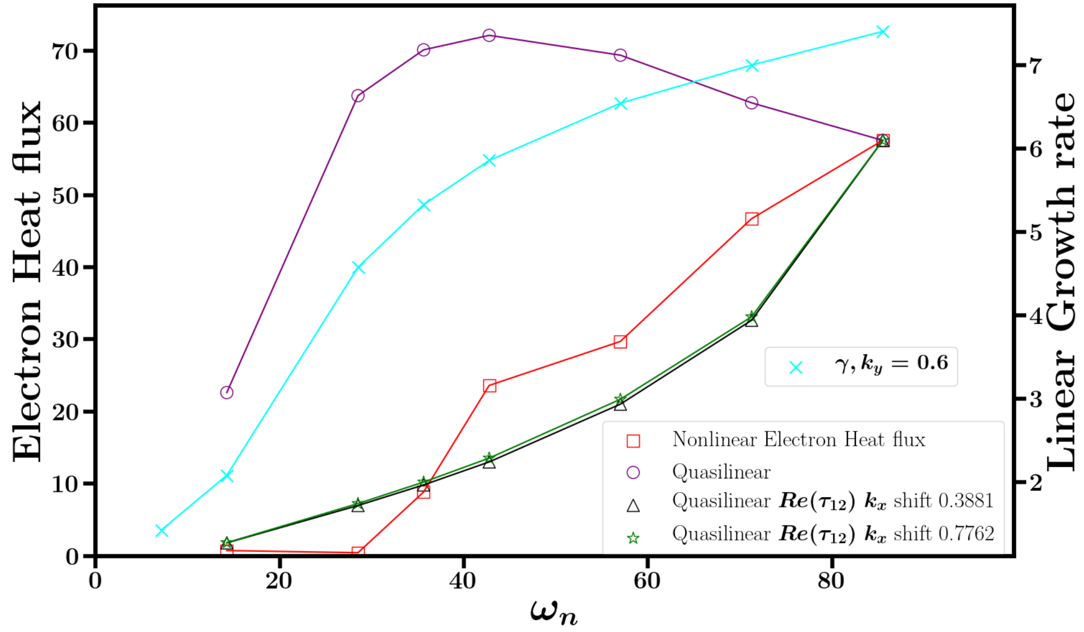


- Choosing zonal flow wavenumbers involving significant nonlinear energy transfer
  → Quasilinear model predicts Dimits shift

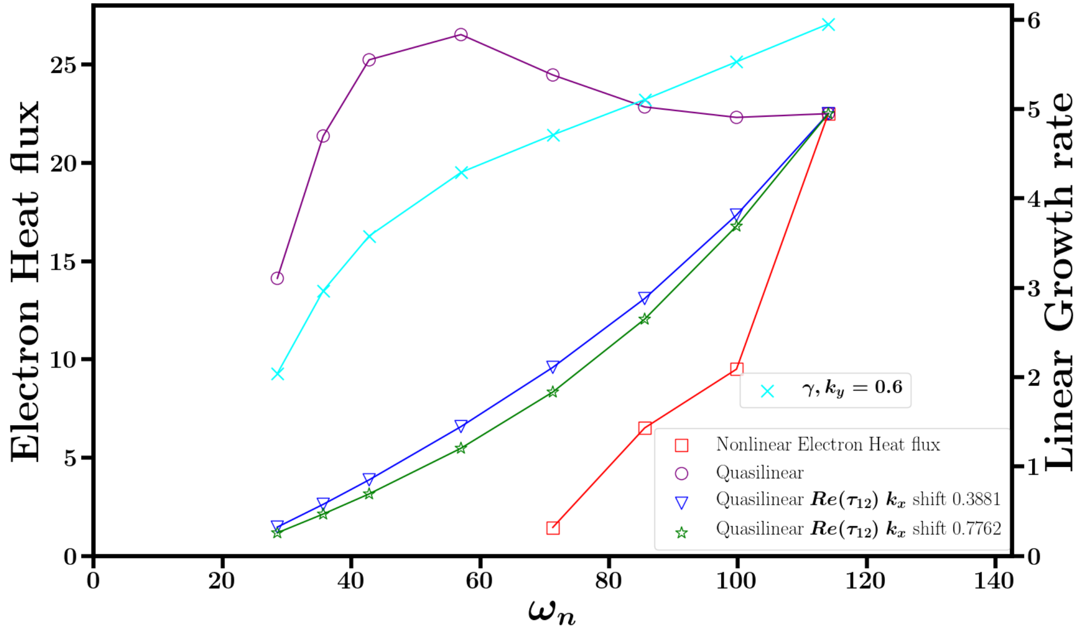
1. Pueschel, M. J., Li, P. Y., & Terry, P. W. (2021). Predicting the critical gradient of ITG turbulence in fusion plasmas. *Nuclear Fusion*.

Grad-n TEM: Madison Symmetric Torus (MST)

Grad-n TEM also shows strong zonal flow and Dimits shift[1]



Collisionless MST



Collisional MST[1,2]

- The triplet correlation time improves the quasilinear model significantly
- The collisional case violates the zero zonal flow complex frequency assumption

1. Duff, J. R., Williams, Z. R., Brower, D. L., Chapman, B. E., Ding, W. X., Pueschel, M. J., ... & Terry, P. W. (2018) *Physics of Plasmas*, *25*(1), 010701.
2. Williams, Z. R., Pueschel, M. J., Terry, P. W., & Hauff, T. (2017). Physics of Plasmas, 24(12), 122309.

Conclusions

- Saturation theory accounting for stable modes explains Dimits shift: resonant nonlinear interactions
- Quasilinear model improved with triplet correlation time τ predicts transport in Dimits regime
- Verified against nonlinear gyrokinetic simulations
- Greatly improved Dimits shift predictions in grad-n TEM turbulence
- Two more detailed paper on the way

---

1. Terry, P. W., Li, P. Y., Pueschel, M. J., & Whelan, G. G. (2021). *Physical Review Letters*, *126*(2), 025004.