NSTX-U

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Theory and computation of general force balance in non-axisymmetric tokamak equilibria

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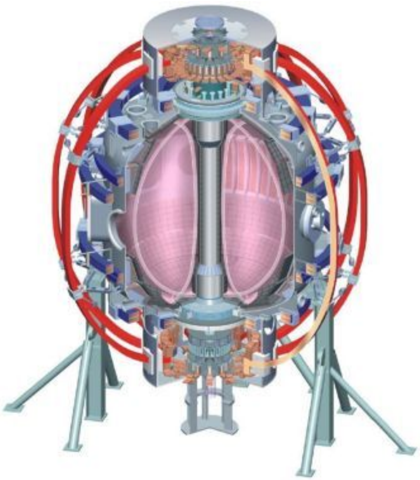


General perturbed equilibria in the state of minimum potential energy

- Minimization of perturbed potential energy with non-ideal non-axisymmetric forces provides more general form of perturbed equilibria

In zero frequency limit, perturbed potential energy with any perturbing force f_p;

$$\delta W = -\frac{1}{2}\int \vec{\xi}\cdot\left(\delta\vec{j}\times\vec{B}+\vec{j}\times\delta\vec{B}-\vec{\nabla}\delta p-\vec{f}_p\right)dx^3$$

Minimum state of dW is equivalent to general perturbed equilibrium;

$$\vec{F} = \delta\vec{j}\times\vec{B}+\vec{j}\times\delta\vec{B}-\vec{\nabla}\delta p-\vec{f}_p = 0$$

*Boozer, Phys. Plasmas **6** 831 (1999), for isotropic pressure*

*Rosenbluth and Rostocker, Phys. Fluids **2** 23 (1959), for anisotropic pressure*

Generally forces can come from kinetic anisotropic pressure, rotational force, and other sources;

$$\vec{f}_p = \vec{\nabla}\cdot\delta\overleftrightarrow{\Pi}+\delta\left(\rho\vec{v}\cdot\vec{\nabla}\vec{v}\right)+\vec{f}_a$$

- Note self-adjointness of force operator or energy principle (See additional discussion later) is necessary for determining stability boundary, but not required for problem of perturbed equlibria

External energy and toroidal torque supporting general perturbed equilibria

- Total energy of system must be conserved, and external currents must support perturbed energy in plasma

Integration by parts separates energy from plasma and external currents;

$$\delta W = -\frac{1}{2}\int_p \vec{\xi}\cdot\vec{F}dx^3 - \frac{1}{2}\int_e \delta\vec{j}\cdot\delta\vec{A}dx^3 = 0$$

- Total toroidal torque of system must also be conserved in tokamaks

$$T_\varphi = \int_p \left[\frac{\partial\vec{x}}{\partial\varphi}\cdot\vec{f}_p\right]dx^3 - \oint_e \left(\delta\vec{B}\cdot\vec{x}_\varphi\right)\left(\delta\vec{B}\cdot\vec{n}\right)da = 0$$

- Both energy and torque are needed to couple to external coil currents

Using surface current potential on a control surface $\delta\vec{j} = \vec{K}\delta(\psi-\psi_b)$, $\vec{K} = \vec{\nabla}\kappa\times\vec{\nabla}\psi$

$$-\int_p \vec{\xi}\cdot\vec{F}\,dx^3 = \oint \kappa\left(\delta\vec{B}\cdot\hat{n}\right)da$$

$$\int_p \left[\frac{\partial\vec{x}}{\partial\varphi}\cdot\vec{f}_p\right]dx^3 = \oint \frac{\partial\kappa}{\partial\varphi}\left(\delta\vec{B}\cdot\hat{n}\right)da$$

Boozer, Phys. Rev. Lett. ***86*** *5059 (2001)*

Park et al., Phys. Plasmas ***16*** *082512 (2009)*

Equivalence in theory of kinetic energy and neoclassical toroidal viscous (NTV) torque

- Kinetic potential energy includes anisotropic pressure tensor

$$\vec{f}_p = \vec{\nabla} \cdot \delta \overleftrightarrow{\Pi}$$

Anisotropic pressure tensor with non-adiabatic part of isotropic pressure;

$$\delta \overleftrightarrow{\Pi} = \delta p_\perp \overleftrightarrow{I} + (\delta p_\parallel - \delta p_\perp)\hat{b}\hat{b}$$

$$\delta W_K = \frac{1}{2}\int \vec{\xi} \cdot \left(\vec{\nabla} \cdot \delta \overleftrightarrow{\Pi}\right) dx^3$$

- Associated toroidal torque is neoclassical toroidal viscous (NTV) torque

$$T_{NTV} = \int dx^3 \left[\frac{\partial \vec{x}}{\partial \varphi} \cdot \vec{\nabla} \cdot \delta \overleftrightarrow{\Pi}\right]$$

- In tokamaks, equivalent theory has been used for kinetic energy and NTV

Anti-Hermitian dW_K, with a complex displacement for a fixed toroidal mode n is proportional to NTV;

$$T_{NTV} = 2n\,\mathrm{Im}[\delta W_k]$$

Park et al., Phys. Plasmas ***18*** *110702 (2011)*

Formulation for kinetic potential energy

- Kinetic potential energy includes anisotropic pressure tensor

$$\delta W_K = \frac{1}{2}\int \vec{\xi}\cdot\left(\vec{\nabla}\cdot\delta\vec{\Pi}\right)dx^3 = -\frac{1}{2}\int\left[(\delta p_\| - \delta p_\perp)\frac{\delta B_L}{B_0} + \delta p_\|(\vec{\nabla}\cdot\vec{\xi}_\perp)\right]dx^3$$

**On Boozer's coordinates:* $\delta W_K = \frac{1}{2}\int\left[(\delta p_\| + \delta p_\perp)\frac{\delta B_L}{B_0}\right]dx^3$

With anisotropic pressure $\delta p_\| = \int dv^3 M v_\|^2 \delta f,\ \ \delta p_\perp = \int dv^3 \frac{1}{2} M v_\perp^2 \delta f$

Orbit averaging gives $\delta W_K = \frac{1}{2M^2}\int d\psi\, d\varphi\, dE\, d\mu\left(\delta J \delta f\right)$

Variation in the action $\delta J = \sum_{\sigma\ell}\oint d\vartheta \frac{JB}{v_\| \psi_p'}\left[(2E - 3\mu B)\frac{\delta B_L}{B_0} + (2E - 2\mu B)\vec{\nabla}\cdot\vec{\xi}_\perp\right]P^\ell$

With Lagrangian variation in the field strength $\frac{\delta B_L}{B_0} = -\vec{\nabla}\cdot\vec{\xi}_\perp - \vec{\xi}_\perp\cdot\vec{\kappa}_0$

*Park et al., Phys. Plasmas **18** 110702 (2011)*

- Perturbed distribution function is calculated with orbit averaging for

$$v_\|\hat{b}\cdot\nabla\delta f + v_D^\alpha\frac{\partial\delta f}{\partial\alpha} + v_D^\psi\frac{\partial f_0}{\partial\psi} = C\left[\delta f\right]$$

Kinetic perturbed equilibria with semi-analytic formulation and effective Krook collisional operator

- With Krook collisional operator, kinetic potential energy is given by

$$\delta W_K = \frac{1}{2M^2}\sum_{\sigma\ell}\int d\psi\, d\varphi\, dE\, d\mu\, \mathrm{Re}[R]\left(\frac{\partial f}{\partial \psi}\right)(\delta J)^2$$

Resonant operator $R \equiv \dfrac{n\omega_b / e}{n(\omega_E + \omega_B) - (\ell - \sigma nq)\omega_b - i\nu_{eff}}$

*Logan et al., Phys. Plasmas **20** 122507 (2013)*

- Kinetic potential energy depends quadratically on action variation
- Action variation spatially depends on two components of displacements

Using complex representation for a fixed n; $\vec{\xi}\cdot\vec{\nabla}\psi = \sum_{mn}\Xi^{\psi}_{mn}e^{im\vartheta - in\varphi}, \quad \vec{\xi}\cdot\vec{\nabla}\alpha = \sum_{mn}\Xi^{\alpha}_{mn}e^{im\vartheta - in\varphi}$

Action matrix with displacement matrix vector; $\delta\vec{J} = \vec{W}_x\vec{\Xi}^{\partial\psi} + \vec{W}_y\vec{\Xi}^{\psi} + \vec{W}_z\vec{\Xi}^{\alpha}$ $\quad * \left(\vec{\Xi}^{\partial\psi} \equiv \partial\vec{\Xi}^{\psi}/\partial\psi\right)$

$$\delta W_K = \frac{2\pi^2}{M^2}\sum_{\sigma\ell}\int d\psi\, dE\, d\mu\, \mathrm{Re}[R]\left(\frac{\partial f}{\partial \psi_p}\right)\left(\vec{W}_X\vec{\Xi}^{\partial\psi} + \vec{W}_Y\vec{\Xi}^{\psi} + \vec{W}_Z\vec{\Xi}^{\alpha}\right)^{+}\left(\vec{W}_X\vec{\Xi}^{\partial\psi} + \vec{W}_Y\vec{\Xi}^{\psi} + \vec{W}_Z\vec{\Xi}^{\alpha}\right)$$

Modified kinetic potential energy in DCON

- With Krook collisional operator, kinetic potential energy is given by

$$\delta W_K = 2\pi^2 \int d\psi \Big[\vec{\Xi}^{\alpha+} \overleftrightarrow{A}_K \vec{\Xi}^{\alpha} + \vec{\Xi}^{\alpha+} \left(\overleftrightarrow{B}_K \vec{\Xi}^{\partial\psi} + \overleftrightarrow{C}_K \vec{\Xi}^{\psi} \right) + \left(\overleftrightarrow{B}_K \vec{\Xi}^{\partial\psi} + \overleftrightarrow{C}_K \vec{\Xi}^{\psi} \right)^{+} \vec{\Xi}^{\alpha}$$
$$+ \vec{\Xi}^{\partial\psi+} \overleftrightarrow{D}_K \vec{\Xi}^{\partial\psi} + \vec{\Xi}^{\partial\psi+} \overleftrightarrow{E}_K \vec{\Xi}^{\psi} + \vec{\Xi}^{\psi+} \overleftrightarrow{E}_K^{+} \vec{\Xi}^{\partial\psi} + \vec{\Xi}^{\psi+} \overleftrightarrow{H}_K \vec{\Xi}^{\psi} \Big]$$

- This is exactly the form of ideal potential energy in DCON, but A-H matrices are modified with kinetic effects

Glasser and Chance, Bull. Am. Phys. Soc. ***42*** *1848 (1997)*

$$\overleftrightarrow{A} = \overleftrightarrow{A}_I + \overleftrightarrow{A}_K = \overleftrightarrow{A}_I + \sum_{\sigma\ell} \int dE d\mu \, \mathrm{Re}[R] \left(\frac{\partial f}{\partial \psi} \right) \vec{W}_Z^{+} \vec{W}_Z$$

$$\overleftrightarrow{B} = \overleftrightarrow{B}_I + \overleftrightarrow{B}_K = \overleftrightarrow{B}_I + \sum_{\sigma\ell} \int dE d\mu \, \mathrm{Re}[R] \left(\frac{\partial f}{\partial \psi} \right) \vec{W}_Z^{+} \vec{W}_X$$

$$\overleftrightarrow{C} = \overleftrightarrow{C}_I + \overleftrightarrow{C}_K = \overleftrightarrow{C}_I + \sum_{\sigma\ell} \int dE d\mu \, \mathrm{Re}[R] \left(\frac{\partial f}{\partial \psi} \right) \vec{W}_Z^{+} \vec{W}_Y$$

$$\overleftrightarrow{D} = \overleftrightarrow{D}_I + \overleftrightarrow{D}_K = \overleftrightarrow{D}_I + \sum_{\sigma\ell} \int dE d\mu \, \mathrm{Re}[R] \left(\frac{\partial f}{\partial \psi} \right) \vec{W}_X^{+} \vec{W}_X$$

$$\overleftrightarrow{E} = \overleftrightarrow{E}_I + \overleftrightarrow{E}_K = \overleftrightarrow{E}_I + \sum_{\sigma\ell} \int dE d\mu \, \mathrm{Re}[R] \left(\frac{\partial f}{\partial \psi} \right) \vec{W}_Z^{+} \vec{W}_Y$$

$$\overleftrightarrow{H} = \overleftrightarrow{H}_I + \overleftrightarrow{H}_K = \overleftrightarrow{H}_I + \sum_{\sigma\ell} \int dE d\mu \, \mathrm{Re}[R] \left(\frac{\partial f}{\partial \psi} \right) \vec{W}_Z^{+} \vec{W}_Y$$

where

$$\vec{W}_X = \vec{W}_\mu \overleftrightarrow{X}$$

$$\vec{W}_Y = \vec{W}_\mu \left(3\overleftrightarrow{S} + \overleftrightarrow{Y} \right) - 2\vec{W}_E \overleftrightarrow{S}$$

$$\vec{W}_Z = \vec{W}_\mu \left(3\overleftrightarrow{S} + \overleftrightarrow{Z} \right) - 2\vec{W}_E \overleftrightarrow{T}$$

$$\vec{W}_\mu = \frac{1}{4\pi} \oint d\vartheta \frac{\mu}{\sqrt{2E - 2\mu B}} P_\sigma^\ell e^{i(m-nq)\vartheta}$$

$$\vec{W}_E = \frac{1}{4\pi} \oint d\vartheta \frac{E/B}{\sqrt{2E - 2\mu B}} P_\sigma^\ell e^{i(m-nq)\vartheta}$$

$$JB^2 \left(\vec{\xi}_\perp \cdot \vec{\kappa}_0 \right) = \sum_{mn} \left(\vec{S}\vec{\Xi}^{\psi} + \vec{T}\vec{\Xi}^{\alpha} \right) e^{im\vartheta - in\varphi}$$

$$JB^2 \left(\vec{\nabla} \cdot \vec{\xi}_\perp \right) = \sum_{mn} \left(\vec{X}\vec{\Xi}^{\partial\psi} + \vec{Y}\vec{\Xi}^{\psi} + \vec{Z}\vec{\Xi}^{\alpha} \right) e^{im\vartheta - in\varphi}$$

Modified Euler-Lagrange equation for force balance

- Minimization of ideal and kinetic potential energy by variational methods yields modified Euler-Lagrange equation (with modified F-G matrices)

$$\text{Radial displacement}: \frac{\partial}{\partial\psi}\left(\overleftrightarrow{F}\frac{\partial\vec{\Xi}^{\psi}}{\partial\psi}+\overleftrightarrow{K}\vec{\Xi}^{\psi}\right)-\left(\overleftrightarrow{K}^{+}\vec{\Xi}^{\psi}+\overleftrightarrow{G}\vec{\Xi}^{\psi}\right)=0$$

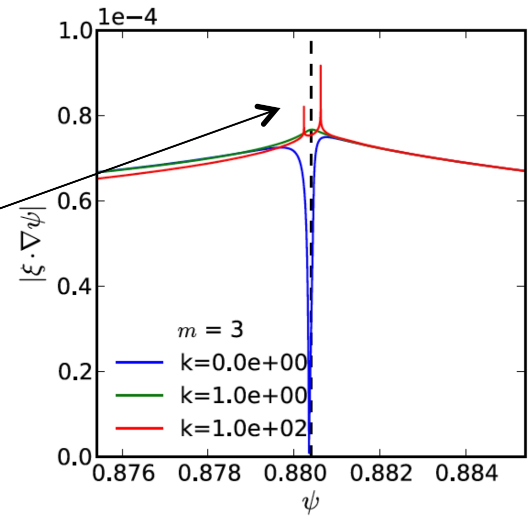
$$\text{Tangential displacement}: \vec{\Xi}^{\alpha}=-\overleftrightarrow{A}^{-1}\left(\overleftrightarrow{B}\vec{\Xi}^{\partial\psi}+\overleftrightarrow{C}\vec{\Xi}^{\psi}\right)$$

- This system of differential equation becomes singular when *det(F)* vanishes
- Ideal case: *det(F)* vanishes at the resonant surface *q=m/n*

$$\overleftrightarrow{F}_{I}=\overleftrightarrow{Q}\overleftrightarrow{\overline{F}}_{I}\overleftrightarrow{Q} \qquad *\left(\overleftrightarrow{Q}_{mm'}\equiv(m-nq)\delta_{mm'}\right)$$

- Kinetic case: *q=m/n* resonant surface can split, or disappear, or kinetically driven resonant surface may possibly occur

$$\overleftrightarrow{F}=\left(\overleftrightarrow{Q}-\overleftrightarrow{P}\right)^{+}\overleftrightarrow{F}_{K}\left(\overleftrightarrow{Q}-\overleftrightarrow{P}\right)+\overleftrightarrow{F}_{R}$$

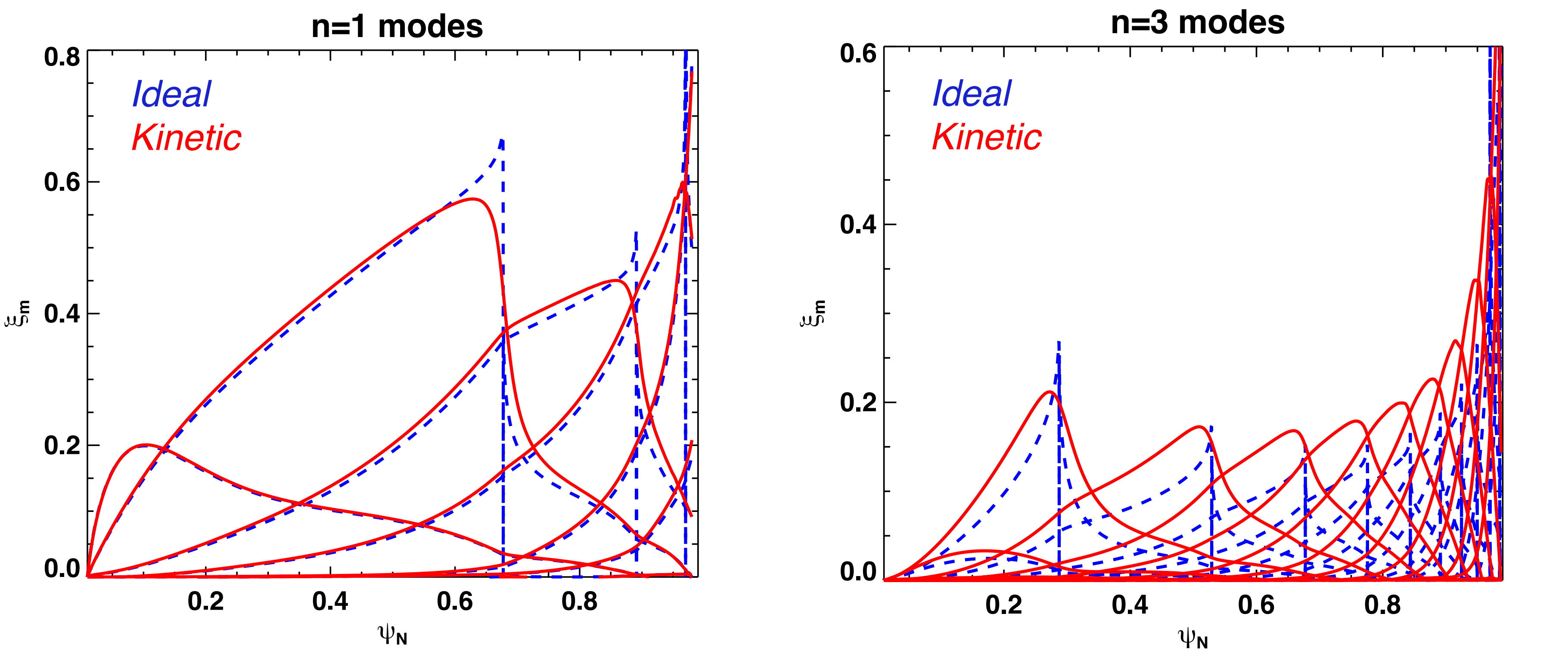


Kinetic modification of global eigenfunctions

- Kinetic potential energy is mostly stabilizing, i.e. requiring higher energy for perturbations, and modify global structure smoothly
- Kinetic modification can be critical when plasma is beyond stability limit

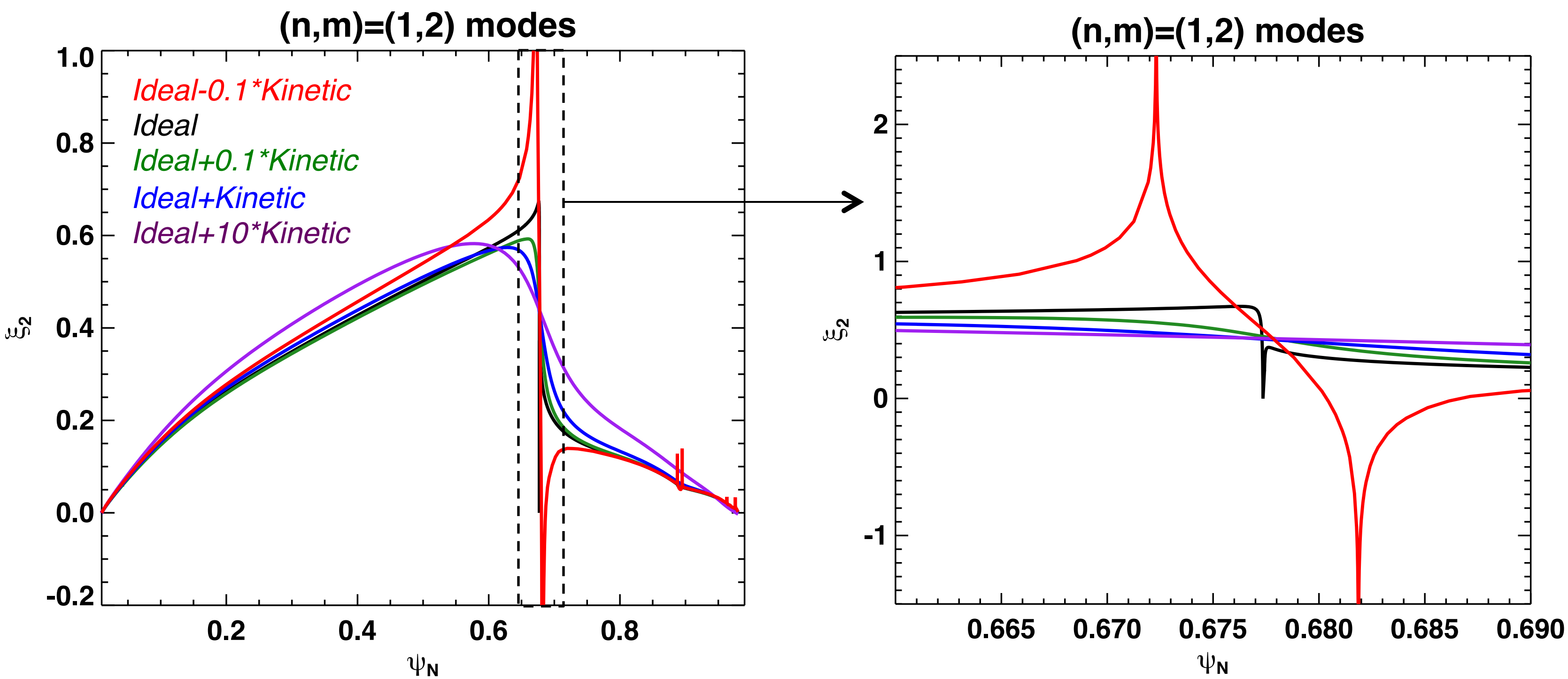
See Wang's work with MARS-K (TI.00003 on Thursday)

DIII-D high β_N~2.5 target (still below no-wall limit)



Kinetic modification of eigenfunctions near resonant surfaces

- Kinetic effects can strongly modify solution behaviors near resonance surfaces
  - Can remove ideal resonant surfaces when stabilizing
  - Can split ideal resonant surfaces when destabilizing
- Kinetic resonant surfaces are expected to have a logarithmic and integrable singularity, but it will be numerically more stable if jumped across



Construction of energy and torque matrix

- Ideal and kinetic potential energy can be integrated with Euler-Lagrange equation and can be combined with vacuum energy, yielding energy response matrix

$$\delta W_T = 2\pi^2 \int d\psi \left[\vec{\Xi}^{\partial\psi +} \vec{F} \vec{\Xi}^{\partial\psi} + \vec{\Xi}^{\partial\psi +} \vec{K} \vec{\Xi}^{\psi} + \vec{\Xi}^{\psi +} \vec{K}^{+} \vec{\Xi}^{\partial\psi} + \vec{\Xi}^{\psi +} \vec{G} \vec{\Xi}^{\psi} \right] + \delta W_V$$

$$= 2\pi^2 \int d\psi \frac{\partial}{\partial\psi} \left[\vec{\Xi}^{\psi +} \left(\vec{F} \vec{\Xi}^{\partial\psi} + \vec{K} \vec{\Xi}^{\psi} \right) \right] + \delta W_V = \vec{\Xi}_b^{\psi +} \left(\vec{W}_P + \vec{W}_V \right) \vec{\Xi}_b^{\psi} = \vec{\Phi}^{+} \left(\vec{W}_P^{\Phi} + \vec{W}_V^{\Phi} \right) \vec{\Phi}$$

*Φ : Normal magnetic field

- Toroidal torque produced by kinetic perturbed equilibrium can also be represented by torque response matrix using solutions of Euler-Lagrange equation

$$T_\varphi = \vec{\Xi}_b^{\psi +} \vec{T}_P \vec{\Xi}_b^{\psi} = \vec{\Phi}^{+} \vec{T}_P^{\Phi} \vec{\Phi}$$

$$\vec{T}_P = 4n\pi^2 \int d\psi \left[\vec{T}_{\alpha\psi}^{+} (\psi, \psi_b) \vec{A}_T \vec{T}_{\alpha\psi} (\psi, \psi_b) + \vec{T}_{\alpha\psi}^{+} (\psi, \psi_b) \vec{B}_T \vec{T}_{\partial\psi,\psi} (\psi, \psi_b) + \cdots \right]$$

$$\vec{A}_T = \sum_{\sigma\ell} \int dE d\mu \, \mathrm{Im}[R] \left(\frac{\partial f}{\partial\psi} \right) \vec{W}_Z^{+} \vec{W}_Z$$

$$\vec{B}_T = \sum_{\sigma\ell} \int dE d\mu \, \mathrm{Im}[R] \left(\frac{\partial f}{\partial\psi} \right) \vec{W}_Z^{+} \vec{W}_X$$

$$\ldots$$

$\vec{T}_{\alpha\psi}^{+} (\psi, \psi_b)$: Kernel detemining $\vec{\Xi}^{\alpha} (\psi)$ given a $\vec{\Xi}^{\psi} (\psi_b)$ using solutions of Euler-Lagrange equation

$$\ldots$$

Construction of plasma response matrix for free boundary perturbed equilibria

- Plasma response can be determined using energy and torque response matrix

$$\text{Plasma inductance : } \overleftrightarrow{\Lambda} = 2\left(\overleftrightarrow{W}_P^{\Phi} + \overleftrightarrow{W}_V^{\Phi}\right) + \frac{i}{n}\overleftrightarrow{T}_P^{\Phi}$$

$$\text{Plasma response (Permeability) : } \overleftrightarrow{P} = \overleftrightarrow{\Lambda}\overleftrightarrow{L}_p^{-1}$$

$$\text{External field } \vec{\Phi}^x \text{ to Total field } \vec{\Phi} : \ \vec{\Phi} = \overleftrightarrow{P}\vec{\Phi}^x$$

- Plasma response to rotating field can also be determined by combining eddy currents at the wall (e.g. VALEN3D code can provide required matrices)

$$\text{External field } \vec{\Phi}_w^x \text{ to Total field } \vec{\Phi}_w \text{ at the wall : } \vec{\Phi}_w = \overleftrightarrow{S}\vec{\Phi}_w^x$$

$$\overleftrightarrow{S} \equiv \left(\overleftrightarrow{\Theta} + in\omega\overleftrightarrow{L}_w\overleftrightarrow{R}_w\right)^{-1}$$

$$\overleftrightarrow{\Theta} \equiv \left(\overleftrightarrow{L}_w\overleftrightarrow{M}_{pw}^{-1}\right)\overleftrightarrow{P}^{-1}\left(\left(\overleftrightarrow{1} - \overleftrightarrow{C}\right)\overleftrightarrow{P}^{-1} + \overleftrightarrow{C}\right)^{-1}\left(\overleftrightarrow{L}_w\overleftrightarrow{M}_{pw}^{-1}\right)^{-1}$$

$$\overleftrightarrow{C} \equiv \overleftrightarrow{M}_{pw}\overleftrightarrow{L}_w^{-1}\overleftrightarrow{M}_{wp}\overleftrightarrow{L}_p$$

Wall resistivity : $\overleftrightarrow{R}_w$

Plasma surface inductance : $\overleftrightarrow{L}_P$

Wall surface inductance : $\overleftrightarrow{L}_w$

Plasma-Wall coupling : $\overleftrightarrow{M}_{pw}$

*Park et al., Phys. Plasmas **16** 082512 (2009)*

Kinetic perturbed equilibria with first-principle code

- Kinetic energy and toroidal torque can be computed by first-principle code, such as POCA particle simulation code based on Boozer formulation
- Computations have shown quadratic dependency on variation in the field strength

$$\text{Assume code matrix } \overleftrightarrow{W}^{C}(\psi) = \overrightarrow{\left(\frac{\delta B_L}{B_0}\right)}^{+} \overleftrightarrow{W}_K^{C}(\psi) \overrightarrow{\left(\frac{\delta B_L}{B_0}\right)} \text{ and } \left(\overleftrightarrow{W}_K^{C}\right)^{+} = \overleftrightarrow{W}_K^{C}$$

- Matrix can be constructed using computational results, from diagonal to off-diagonal bands in sequence
  - *Individual Fourier components giving diagonal band*
  - *2-component applications giving the first off-diagonal band since the diagonal band is known*
  - *k-component applications giving the (k-1)-th off-diagonal band repeatedly*

- In Boozer coordinates, A-H matrices for energy and torque can be calculated with

$$\delta W_K = \int d\psi \left[\overleftrightarrow{X}\vec{\Xi}^{\partial\psi} + (\overleftrightarrow{Y} + \overleftrightarrow{S})\vec{\Xi}^{\psi} + (\overleftrightarrow{Z} + \overleftrightarrow{T})\vec{\Xi}^{\alpha} \right]^{+} \overleftrightarrow{W}_K^{C}(\psi) \left[\overleftrightarrow{X}\vec{\Xi}^{\partial\psi} + (\overleftrightarrow{Y} + \overleftrightarrow{S})\vec{\Xi}^{\psi} + (\overleftrightarrow{Z} + \overleftrightarrow{T})\vec{\Xi}^{\alpha} \right]$$

- Code matrix is diagonal-dominant and strongly banded, enabling truncation with only a few bands and thus efficient computation

See Kim's work with POCA (BP8.00027 in this session)

Perturbed equilibria with arbitrary force

- An arbitrary small force in the plasma changes potential energy

$$\delta W = -\frac{1}{2}\int \vec{\xi}\cdot\left(\delta\vec{j}\times\vec{B}+\vec{j}\times\delta\vec{B}-\vec{\nabla}\delta p-\vec{f}_a\left(\psi,\vartheta,\varphi\right)\right)dx^3$$

*Boozer et al., Phys. Plasmas **13** 102501 (2006)*

- Parallel force balance gives

$$\vec{f}_a\cdot\vec{B}_0 = \vec{B}_0\cdot\vec{\nabla}\left(2\vec{\xi}\cdot\vec{\nabla}p+\gamma\vec{\nabla}\cdot\vec{\xi}\right)$$

- In complex representation, parallel force balance in general leads to

$$\frac{1}{2}\int d\vartheta\int d\varphi\vec{f}_a\cdot\vec{\xi} = \overleftrightarrow{X}\vec{\Xi}^{\partial\psi}+\overleftrightarrow{Y}\vec{\Xi}^{\psi}+\overleftrightarrow{Z}\vec{\Xi}^{\alpha}$$

- Minimization of potential energy yields an inhomogeneous E-L equation

$$\frac{\partial}{\partial\psi}\left(\overleftrightarrow{F}\frac{\partial\vec{\Xi}^{\psi}}{\partial\psi}+\overleftrightarrow{K}\vec{\Xi}^{\psi}\right)-\left(\overleftrightarrow{K}^{+}\vec{\Xi}^{\psi}+\overleftrightarrow{G}\vec{\Xi}^{\psi}\right)=\vec{S}-\frac{\partial\overleftrightarrow{R}}{\partial\psi}$$

$$\vec{\Xi}^{\alpha} = -\overleftrightarrow{A}^{-1}\left(\overleftrightarrow{B}\vec{\Xi}^{\partial\psi}+\overleftrightarrow{C}\vec{\Xi}^{\psi}+\overleftrightarrow{Z}\right)$$

where

$$\overleftrightarrow{R} = \overleftrightarrow{X}-\overleftrightarrow{B}^{+}\overleftrightarrow{A}^{-1}\overleftrightarrow{Z}$$

$$\vec{S} = \overleftrightarrow{Y}-\overleftrightarrow{C}^{+}\overleftrightarrow{A}^{-1}\overleftrightarrow{Z}$$

- With a particular solution, Hermitian part of inductance matrix can be calculated by virtual casing and finding surface currents driving perturbation

*Park et al., Phys. Plasmas **14** 052110 (2007)*

On kinetic energy principle in low frequency and collisionless limit

- In low frequency limit, kinetic energy principle provides necessary and sufficient condition for stability

$$\delta W_{RR} = \delta W_I - \frac{1}{2}\int \vec{\xi}^* \cdot \left(\vec{\nabla}\cdot\overleftrightarrow{\Pi}\right) dx^3$$

With collisionless trapped particles, lower and upper bound of stability;

$$\delta W_I \le \delta W_{RR} \le \delta W_{CGL}$$

*Rosenbluth and Rostocker, Phys. Fluids **2** 23 (1959)*

- Self-adjointness with kinetic energy operator is manifest by

$$\delta W_{RR} = \delta W_I + \frac{1}{2M^2}\int d\psi_p\, d\varphi\, dE\, d\mu \sum_{\ell} \frac{n\omega_b / e}{\ell\omega_b - n(\omega_E + \omega_B)} \frac{\partial f_0}{\partial \psi_p} \delta J^* \delta J$$

- Kinetic energy principle relies on 'collisionless' condition, in which imaginary term in kinetic energy seemingly disappears

*Kruskal and Oberman, Phys. Fluids **1** 275 (1958)*

For extended kinetic energy principle, see *Taylor and Hastie, Phys. Fluids **8** 323 (1965)*
*Antonsen and Lee, Phys. Fluids **25** 132 (1982)*
*Van Dam and Rosenbluth, Phys. Fluids **25** 1349 (1982)*

Finite energy dissipation in collisionless limit

- However, imaginary part of kinetic energy is finite in collisionless limit

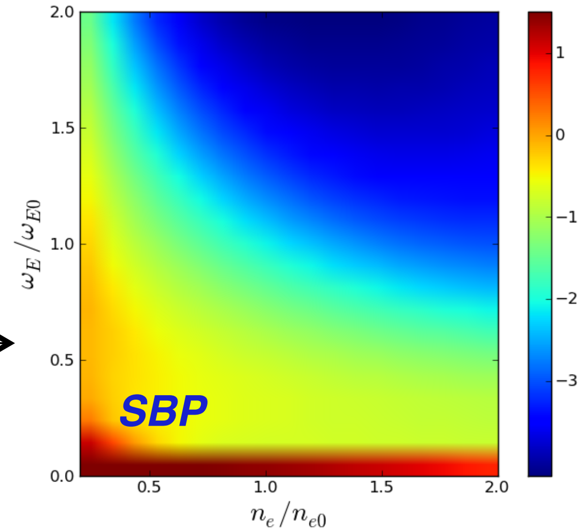
$$\lim_{\nu\to 0}\frac{1}{\ell\omega_b - n(\omega_E+\omega_B)+i\nu_{eff}} = i\pi\delta\left(\ell\omega_b - n(\omega_E+\omega_B)\right)$$

- Contributions from these resonant particles are in fact most important

$\omega_E+\omega_B \to 0$ *Superbanana plateau resonance (SBP)*
Shaing et al., Plasma Phys. Control. Fusion ***51*** *035009 (2009)*

$\ell\omega_b - n\omega_E \to 0$ *Bounce-harmonic resonance (BH)*
Park et al., Phys. Rev. Lett. ***102*** *065002 (2009)*

Example with NSTX-U target with PENT Im(dW) increases in low density and low collisional limit and saturates to plateau



- In truly collisionless limit, perturbation theory fails and plasma enters superbanana regime, which predicts no imaginary term : This limit is mostly unrealistic

- Kinetic energy principle should be revisited for realistic application

Summary and Discussion

- Theory and numerical implementation of kinetic perturbed equilibria were successfully formulated and preliminary tests show expected modifications
  - Kinetic singular surfaces are logarithmic and integrable, but better be found in advance to cross the surfaces with (continuous) jump conditions
- Kinetic perturbed equilibrium solutions, with semi-analytic formulation, will be under benchmark with MARS-K without fluid rotation
- Plasma response matrix can be constructed from energy and torque matrix, which will allow free boundary perturbed equilibrium
- Arbitrary force can be included by solving inhomogeneous Euler-Lagrange equation, and response matrix can be determined by virtual casing method

- Torque consistent perturbed equilibrium : Presented treatment is consistent with anisotropic pressure tensor, but not consistent with torque inside plasma
- Stability determination with kinetic energy principle : Imaginary contributions exist by resonance even in collisionless limit, like Landau damping, and accordingly kinetic energy principle should be revisited
- However, slowly varying instability, such as RWM, can be calculated by coupling energy and torque to external currents at the wall, by ignoring inertia energy